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Convex Risk Measures Beyond Bounded Risks, or The Canonical Model Space for Law-Invariant Convex Risk Measures is L^1

Damir Filipović and Gregor Svindland

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Convex Risk Measures Beyond Bounded Risks, or The Canonical Model Space for Law-invariant Convex Risk Measures is L^1 *

Damir Filipović[†] Gregor Svindland[‡]

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Abstract

In this paper we provide a rigorous toolkit for extending convex risk measures from L^∞ to L^p , for $p \geq 1$. Our main result is a one-to-one correspondence between law-invariant convex risk measures on L^∞ and L^1 . This proves that the canonical model space for the predominant class of law-invariant convex risk measures is L^1 . Some significant counterexamples illustrate the many pitfalls with convex risk measures and their extensions to L^p .

Key words: convex risk measures, extensions to L^p , law-invariant convex functions

1 Introduction

Convex risk measures are best known on L^∞ . Indeed, Artzner et al. [2] introduced the seminal axioms of coherence, which then were further generalised to the convex case by Föllmer and Schied [12] and Frittelli and Rosazza-Gianin [14], on L^∞ . However, there is a growing mathematical finance literature dealing with convex risk measures beyond L^∞ , see e.g. [3, 4, 7, 16, 17, 19]. This extended approach is vital since important risk models, such as normal distributed random variables, are not contained in L^∞ .

In some of the above mentioned articles, the model space is chosen such that some preselected risk measure remains finite valued. In contrast, we believe that

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[†]Filipović is supported by WWTF (Vienna Science and Technology Fund). Address: Vienna Institute of Finance, University of Vienna, and Vienna University of Economics and Business Administration, Heiligenstädter Strasse 46-48, A-1190 Wien, Austria. Email: damir.filipovic@vif.ac.at

[‡]Svindland gratefully acknowledges financial support from Munich Re Grant for doctoral students. Address: Mathematics Institute, University of Munich, Theresienstrasse 39, D-80333 München, Germany. Email: svindla@mathematik.uni-muenchen.de

the model space should be maximal possible to capture the universe of financial risks from the outset. On the other hand, from a computational point of view, the model space should be standard and endowed with a topological structure supporting convex duality. In sum, we propose the model space L^p for $p \geq 1$.

The aim of this paper is threefold. First, the last ten years of risk measure theory has been mainly focused on L^∞ . This requires—and we provide—a rigorous toolkit for extending convex risk measures from L^∞ to L^p . Second, we establish a one-to-one correspondence between law-invariant convex risk measures on L^∞ and L^1 . This proves that the canonical model space for the predominant class of law-invariant convex risk measures is L^1 . Third, we provide some significant counter-examples which illustrate the pitfalls with convex risk measures and their extensions to L^p . Notwithstanding the above mentioned literature, we believe that such elaboration is needed. In fact, our paper can be seen as a completion of the interplay of convex risk measures on and beyond L^∞ .

The structure of the paper is as follows. In section 2 we recall some fundamental properties of convex risk measures on L^p and conclude with a remarkable dichotomy: a convex risk measure ρ on L^p is either continuous on L^p or its domain has empty interior.

Section 3 contains our main results. Point of departure is an arbitrary given function f on L^∞ . We then discuss existence and uniqueness of closed convex extensions of f to L^p . It turns out that such extensions do not always exist (section 5.3), and—if they exist—are not unique in general (section 5.2). We define and show that the L^p -closure \bar{f}^p of f is the greatest closed convex function on L^p majorised by f on L^∞ (theorem 3.2). Moreover, if a closed convex extension of f to L^p exists, then \bar{f}^p is the greatest of such extensions (theorem 3.7). As an advice against using value at risk (VaR), we show that there exists no non-trivial closed convex function majorised by VaR (section 5.5). Our main result (theorem 3.8) states that there is a one-to-one correspondence between law-invariant closed convex functions on L^∞ and L^1 . In particular, for any law-invariant closed convex function f on L^∞ , the L^1 -closure \bar{f}^1 is the unique law-invariant closed convex extension to L^1 . We conclude that the canonical model space for law-invariant convex risk measures is L^1 . The proof of this theorem is given in sections 6–7.

In section 4, we discuss an alternative “sup inf”-extension of convex risk measures from L^∞ to L^p , which was proposed by Krätschmer [17]. However, it turns out that this approach has pitfalls and is actually the source for some of the above mentioned (counter-)examples.

For the sake of readability of the paper, we collected all examples, which illustrate the pitfalls with convex risk measures and their extensions, in section 5.

Finally, section 8 contains the proof of theorem 2.3.

2 Convex Risk Measures on L^p

Throughout, we fix a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms and $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is separable (in fact, this assumption is only needed in

theorem 3.8 and sections 5–7 below). All equalities and inequalities between random variables are understood in the \mathbb{P} -almost sure (a.s.) sense. The topological dual space of $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$, for $p \in [1, \infty]$, is denoted by L^{p*} , the positive order cone by L_+^p and its polar cone by L_-^{p*} . It is well known that $L^{\infty*} \supset L^1$ can be identified with ba , the space of all bounded finitely additive signed measures on (Ω, \mathcal{F}) which vanish on \mathbb{P} -null sets. With some facilitating abuse of notation, we shall write $(X, Z) \mapsto E[XZ]$ for the dual pairing on (L^p, L^{p*}) also for the case $p = \infty$.

We suppose the reader is familiar with standard terminology and basic duality theory for convex functions as outlined in [18] or [8]. We recall that a function $f : L^p \rightarrow [-\infty, \infty]$ is

- (i) *convex* if its *epigraph* $\text{epi } f := \{(X, y) \mid f(X) \leq y\}$ is a convex subset of $L^p \times \mathbb{R}$,
- (ii) *proper* if $f > -\infty$ and its *domain* $\text{dom } f := \{f < \infty\} \neq \emptyset$,
- (iii) *closed* if either $f \equiv -\infty$ or $f \equiv \infty$ or f is proper and *lower semi-continuous* (*l.s.c.*),
- (iv) *law-invariant* if $f(X) = f(Y)$ for all identically distributed $X \sim Y$.

The *conjugate* $f^*(Z) = \sup_{X \in L^p} (E[XZ] - f(X))$ of f is a closed convex function on L^{p*} . The Fenchel–Moreau theorem (proposition 4.1 in [8]) states that $f^{**} = f$ if and only if f is closed convex. The *indicator function* $\delta(\cdot \mid C)$ of a set C is defined as zero on C and ∞ elsewhere. We denote by \overline{C}^p the closure of a set C in L^p .

Definition 2.1. A convex function $\rho : L^p \rightarrow (-\infty, \infty]$ is called convex risk measure if it is

- (i) *cash-invariant*: $\rho(0) \in \mathbb{R}$ and $\rho(X + m) = \rho(X) - m$ for all $m \in \mathbb{R}$,
- (ii) *monotone*: $\rho(X) \leq \rho(Y)$ for $X \geq Y$.

We denote by $\mathcal{A}_\rho := \{\rho \leq 0\}$ the acceptance set of ρ .

A positively homogeneous convex risk measure ρ is called coherent.

Remark 2.2. Example 5.1 below shows that convex risk measures are not closed in general.

Convex risk measures on L^p and more general model spaces have been studied, among others, in [3, 4, 16, 19]. The next theorem summarises some fundamental properties that will be used in the sequel. For the sake of completeness, we provide a proof in section 8.

Theorem 2.3. Let ρ be a convex risk measure on L^p .

- (i) $\text{int dom } \rho \neq \emptyset$ if and only if ρ is continuous on L^p .
- (ii) \mathcal{A}_ρ is convex with $\mathcal{A}_\rho + L_+^p \subset \mathcal{A}_\rho$, $\mathbb{R} \cap \mathcal{A}_\rho \neq \emptyset$, and $\overline{\mathcal{A}_\rho}^p \neq L^p$.

(iii) For any $\mathcal{A} \subset L^p$ satisfying properties (ii) instead of \mathcal{A}_ρ ,

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}$$

defines a convex risk measure on L^p . Moreover, if \mathcal{A} is closed, then $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$ and $\rho_{\mathcal{A}}$ is closed.

(iv) $\text{dom } \rho^* \subset \mathcal{P}^{p*} := \{Z \in L_-^{p*} \mid E[1Z] = -1\}$ and for all $Z \in \mathcal{P}^{p*}$ we have $\rho^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[ZX]$.

Part (i) yields a remarkable dichotomy for convex risk measures ρ on L^p : either ρ is continuous on L^p or $\text{int dom } \rho = \emptyset$.

3 L^p -closures

This section contains our main results, which are valid not only for convex risk measures. We therefore fix $p \in [1, \infty]$ and some function $f : L^\infty \rightarrow [-\infty, \infty]$. Its conjugate

$$f^*(Z) = \sup_{X \in L^\infty} (E[ZX] - f(X))$$

is a closed convex function on $L^{\infty*}$, and hence on L^{p*} . The following is thus well defined.

Definition 3.1. *The L^p -closure of f is defined as*

$$\bar{f}^p(X) := \sup_{Z \in L^{p*}} (E[ZX] - f^*(Z)), \quad X \in L^p. \quad (3.1)$$

We next prove some fundamental properties of \bar{f}^p .

Theorem 3.2. (i) \bar{f}^p is the greatest closed convex function on L^p majorised by f on L^∞ .

(ii) $(\bar{f}^p)^* = f^*|_{L^{p*}}$.

(iii) \bar{f}^p is proper if and only if f is proper and $\text{dom } f^* \cap L^{p*} \neq \emptyset$.

(iv) If either f is finite or \bar{f}^p is proper then $\text{epi } \bar{f}^p = \overline{\text{co epi } f^p}$, where the right hand side denotes the $L^p \times \mathbb{R}$ -closure of the convex hull of $\text{epi } f$.

Proof. By construction, \bar{f}^p is a closed convex function on L^p with

$$\bar{f}^p \leq f \text{ on } L^\infty. \quad (3.2)$$

Now let g be any closed convex function on L^p with $g \leq f$ on L^∞ . Then, for all $Z \in L^{p*}$,

$$g^*(Z) = \sup_{X \in L^p} E[XZ] - g(X) \geq \sup_{X \in L^\infty} E[XZ] - f(X) = f^*(Z). \quad (3.3)$$

Hence,

$$g(X) = g^{**}(X) \leq \sup_{Z \in L^{p*}} E[XZ] - f^*(Z) = \bar{f}^p(X),$$

and (i) is proved.

Now let $Z \in L^{p*}$. By definition we obtain

$$(\bar{f}^p)^*(Z) = \sup_{X \in L^p} (E[XZ] - \sup_{Y \in L^{p*}} (E[XY] - f^*(Y))) \leq f^*(Z).$$

On the other hand, from (3.3) we infer $(\bar{f}^p)^*(Z) \geq f^*(Z)$. This proves (ii).

Property (iii) is obvious.

As for (iv), inequality (3.2) implies $\overline{\text{epi } f} \subset \text{epi } \bar{f}^p$ on $L^\infty \times \mathbb{R}$. By convexity and closedness of $\text{epi } \bar{f}^p$ we thus have $\overline{\text{co epi } f^p} \subset \text{epi } \bar{f}^p$. To show the converse inclusion, we note that

$$g(X) = \inf\{a \mid (X, a) \in \overline{\text{co epi } f^p}\}$$

defines a l.s.c. convex function on L^p with $f \geq g$ on L^∞ and $g \geq \bar{f}^p$. Thus, if either f is finite or \bar{f}^p is proper then g is closed. In view of the first part of the lemma, we conclude that $g = \bar{f}^p$ and thus $\overline{\text{co epi } f^p} = \text{epi } \bar{f}^p$. Thus the theorem is proved. \square

Remark 3.3. Note that, for $p \leq r$, even if \bar{f}^p is proper, we do not necessarily have that $\bar{f}^p|_{L^r} = \bar{f}^r$, see example 5.6 below.

Moreover, theorem 3.2(iv) does not hold without requiring that f be finite or \bar{f}^p be proper. Indeed, example 5.7 below shows that $\overline{\text{co epi } f^p} \subsetneq \text{epi } \bar{f}^p$ is possible.

The next corollary shows that monotonicity and cash-invariance of convex risk measures is preserved under L^p -closure.

Corollary 3.4. Let ρ be a convex risk measure on L^∞ . Then $\bar{\rho}^p$ is a closed convex risk measure on L^p if and only if $\overline{\mathcal{A}_\rho^p} \neq L^p$. In either case, $\bar{\rho}^p = \rho_{\overline{\mathcal{A}_\rho^p}}$ and $\mathcal{A}_{\bar{\rho}^p} = \overline{\mathcal{A}_\rho^p}$.

Proof. Since any convex risk measure ρ on L^∞ is finite (due to monotonicity and cash-invariance), we infer from theorem 3.2(iv) that

$$\overline{\text{epi } \rho^p} = \text{epi } \bar{\rho}^p, \quad (3.4)$$

$$\overline{\mathcal{A}_\rho^p} = \{X \in L^p \mid (X, 0) \in \overline{\text{epi } \rho^p}\} = \{X \in L^p \mid (X, 0) \in \text{epi } \bar{\rho}^p\} = \mathcal{A}_{\bar{\rho}^p}. \quad (3.5)$$

Hence, if $\bar{\rho}^p$ is a closed convex risk measure on L^p , then $\overline{\mathcal{A}_\rho^p}$ must have the properties stated in theorem 2.3(ii), in particular $\overline{\mathcal{A}_\rho^p} \neq L^p$.

Conversely, suppose that $\overline{\mathcal{A}_\rho^p} \neq L^p$. We claim that $\overline{\mathcal{A}_\rho^p}$ satisfies the conditions of theorem 2.3(ii). Firstly, convexity and $\overline{\mathcal{A}_\rho^p} \cap \mathbb{R} \neq \emptyset$ are obvious, and secondly we have $\overline{\mathcal{A}_\rho^p} \neq L^p$ by assumption. In order to verify the yet missing condition, let $X \in \overline{\mathcal{A}_\rho^p}$ and $Y \in L_+^p$. Choose $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}_\rho$ converging to X . As $Y \wedge n \in L_+^\infty$ for all $n \in \mathbb{N}$, we have $Y_n := Y \wedge n + X_n \in \mathcal{A}_\rho$ for all

$n \in \mathbb{N}$. Since Y_n converges to $X + Y$ w.r.t. the $\|\cdot\|_p$ -norm, we conclude that $(X + Y) \in \overline{\mathcal{A}_\rho^p}$. Consequently, according to theorem 2.3(iii), $\rho_{\overline{\mathcal{A}_\rho^p}}$ is a closed convex risk measure on L^p . As $\text{epi } \rho_{\overline{\mathcal{A}_\rho^p}} = \overline{\text{epi } \rho^p}$, we infer from (3.4) and (3.5) that $\overline{\rho^p} = \rho_{\overline{\mathcal{A}_\rho^p}}$ and $\mathcal{A}_{\overline{\rho^p}} = \overline{\mathcal{A}_\rho^p}$. \square

Definition 3.5. A function $g : L^p \rightarrow (-\infty, \infty]$ is called an extension of f to L^p if $g = f$ on L^∞ .

In the following we elaborate on the existence and uniqueness of closed convex extensions.

Let us first discuss uniqueness. For illustration, consider a closed convex function g on L^p . Obviously, $\overline{g|_{L^\infty}}$ is a closed convex extension of $g|_{L^\infty}$ to L^p . That is, $g = \overline{g|_{L^\infty}}$ on L^∞ . However, example 5.3 below illustrates that this equality may fail on L^p . Hence uniqueness does not hold in general. In fact, an immediate consequence of theorem 3.2(ii) is the following:

Corollary 3.6. Let g be a closed convex function on L^p . Then $g = \overline{g|_{L^\infty}}$ if and only if $g^* = (g|_{L^\infty})^*$ on L^{p^*} .

Examples 5.4–5.6 below show convex risk measures on L^∞ which admit no closed convex extension to L^p . We now give necessary and sufficient conditions for the existence of closed convex extensions and illustrate the particular role of the L^p -closure.

Theorem 3.7. The following properties are equivalent:

- (i) There exists a closed convex extension of f to L^p .
- (ii) $\overline{f^p}$ is an extension of f to L^p .
- (iii) f is convex and $\sigma(L^\infty, L^{p^*})$ -closed.

In either case, $\overline{f^p}$ is the greatest closed convex extension of f to L^p .

Proof. (i) \Rightarrow (ii): let g be a closed convex extension of f to L^p . Theorem 3.2(i) implies $g \leq \overline{f^p}$ and $f = g \leq \overline{f^p} \leq f$ on L^∞ . This proves (ii) and also the last statement of the theorem.

(ii) \Rightarrow (iii): this is obvious since $\overline{f^p}|_{L^\infty}$ is convex and $\sigma(L^\infty, L^{p^*})$ -l.s.c.

(iii) \Rightarrow (ii): the Fenchel–Moreau theorem yields $f(X) = \sup_{Z \in L^{p^*}} (E[XZ] - f^*(Z)) = \overline{f^p}(X)$. \square

If we restrict to law-invariant closed convex functions, then existence and uniqueness of closed convex extensions always holds. This is our main result, the proof of which is given in sections 6 and 7 below.

Theorem 3.8. For any law-invariant closed convex function f on L^∞ and $p \in [1, \infty)$, the L^p -closure $\overline{f^p}$ equals $\overline{f^1}|_{L^p}$ and is the unique law-invariant closed convex extension of f to L^p . In particular, f is $\sigma(L^\infty, L^\infty)$ -closed.

Hence there exists a one-to-one correspondence between law-invariant closed convex risk measures on L^∞ and L^1 . In this sense, we may conclude that the canonical model space for law-invariant convex risk measures is L^1 .

4 An Alternative Extension?

Let ρ be a convex risk measure on L^∞ . The following map seems to be an alternative extension of ρ to L^p :

$$\hat{\rho}(X) := \sup_{m \in \mathbb{N}} \inf_{n \in \mathbb{N}} \rho(X^+ \wedge n - X^- \wedge m), \quad X \in L^p. \quad (4.6)$$

This formula has been suggested by Krättschmer [17]. Clearly, $\hat{\rho} = \rho$ on L^∞ . However, we demonstrate in examples 5.4–5.6 that $\hat{\rho}$ is no closed convex risk measure on L^p in general, even if $\bar{\rho}^p$ is a closed convex risk measure (see example 5.6). We also remark that in case $p \in [1, \infty)$ there are closed convex risk measures on L^p which do not admit a representation of type (4.6). This means that plugging a closed convex risk measure ρ on L^p into the formula (4.6), we may have that $\hat{\rho} \neq \rho$. This is illustrated in example 5.3. Trivially, if ρ is continuous, then $\rho = \hat{\rho}$.

Corollary 4.1. *Suppose that $\hat{\rho}$ is closed and convex on L^p . Then $\hat{\rho} = \bar{\rho}^p$.*

Proof. Since $\hat{\rho}$ is a closed convex extension of ρ , theorem 3.7 states that $\hat{\rho} \leq \bar{\rho}^p$ and equality holds on L^∞ . Monotonicity and l.s.c. of $\bar{\rho}^p$ (corollary 3.4) yield the converse inequality, for $X \in L^p$,

$$\begin{aligned} \hat{\rho}(X) &= \sup_{m \in \mathbb{N}} \liminf_{n \rightarrow \infty} \bar{\rho}^p(X^+ \wedge n - X^- \wedge m) \geq \sup_{m \in \mathbb{N}} \bar{\rho}^p(X^+ - X^- \wedge m) \\ &\geq \liminf_{m \rightarrow \infty} \bar{\rho}^p(X^+ - X^- \wedge m) \geq \bar{\rho}^p(X). \end{aligned}$$

□

Remark 4.2. *If ρ is law-invariant and continuous from below (i.e. for every bounded sequence $(X_n)_{n \in \mathbb{N}} \subset L^\infty$ such that $X_n \nearrow X$ we have $\rho(X_n) \searrow \rho(X)$) then we know that $\hat{\rho} = \rho$. This is proved in a forthcoming paper [11].*

However, it is still an open problem whether $\hat{\rho}$ is closed and convex on L^p if ρ is closed convex on L^p in general.

5 Examples

For the sake of readability of the paper, we collect all examples, which illustrate the pitfalls with convex risk measures and their extensions, in this section.

5.1 Non-closed convex risk measures

The first example of this section shows that there are non-closed convex risk measures on L^p , $p \in [1, \infty)$.

Example 5.1. For $p \in [1, \infty)$, consider

$$\rho : L^p \rightarrow (-\infty, \infty], \quad \rho(X) = -E[X] + \delta(X^- | L^\infty).$$

This law-invariant coherent risk measure assigns to an endowment X the value ∞ in case the possible losses are not bounded, and $E[-X]$ else. Clearly, the acceptance set \mathcal{A}_ρ is not closed, so ρ is not closed. For $Z \in \mathcal{P}^{p*}$ we have that

$$\begin{aligned} \rho^*(Z) &= \sup_{X \in L^p} E[XZ] - \rho(X) = \sup_{X \in \{Y \in L^p \mid Y^- \in L^\infty\}} E[X(Z+1)] \\ &\geq \sup_{k \in \mathbb{R}} E[k(Z+1)1_{\{Z > -1\}}]. \end{aligned}$$

Hence $\rho^* = \delta(\cdot \mid \{-1\}) = (-E)^*$, and thus $\bar{\rho}^p = -E$.

5.2 Non-uniqueness of closed convex extensions

The following lemma will be used to show that closed convex extensions of convex risk measures, if they exist, are not unique in general.

Lemma 5.2. *Let $T \in L^1$ satisfy $\text{essinf } T < 0$ and $\text{esssup } T = \infty$. Denote by \mathcal{A} the closed convex cone generated by T and L_+^1 . Then $\rho_{\mathcal{A}}$ is a closed coherent risk measure on L^1 such that $\rho_{\mathcal{A}} \neq -\text{essinf}$ and $\rho_{\mathcal{A}}|_{L^\infty} = -\text{essinf}|_{L^\infty}$.*

Proof. Theorem 2.3 implies that $\rho_{\mathcal{A}}$ is a closed coherent risk measure on L^1 such that $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$. Moreover, $\rho_{\mathcal{A}} \neq -\text{essinf}$ on L^1 since $\rho_{\mathcal{A}}(T) \leq 0 < -\text{essinf } T$ by construction.

We claim that

$$\mathcal{A} \cap L^\infty = L_+^\infty, \quad (5.7)$$

implying that $\rho_{\mathcal{A}}|_{L^\infty} = -\text{essinf}|_{L^\infty}$. As for the proof of (5.7), note that $\mathcal{A} = \bar{A}^{-1}$ where

$$A = \{tX \mid X \in B, t \geq 0\} \quad \text{and} \quad B = \text{conv}(T, L_+^1) + L_+^1.$$

The inclusion $L_+^\infty \subset \mathcal{A} \cap L^\infty$ follows from construction. To show the converse, $L^\infty \setminus L_+^\infty \subset L^\infty \setminus (\mathcal{A} \cap L^\infty)$, we choose any $S \in L^\infty$ such that $\mathbb{P}(S < 0) > 0$. Since $S \notin L_+^1$ is bounded whereas T is unbounded from above, S cannot be an element of the convex hull $\text{conv}(T, L_+^1)$, and neither of its monotone hull B , because any convex combination in $\text{conv}(T, L_+^1)$ is either \mathbb{P} -a.s. positive or unbounded from above, so it cannot be dominated by S . But then, clearly $S \notin A$ too. Now suppose $S \in \mathcal{A} \setminus A$. Then there would be a sequence $(S_n)_{n \in \mathbb{N}} \subset A$ converging to S in L^1 . By monotonicity of A we may assume that $S_n \geq S$ for all $n \in \mathbb{N}$ (otherwise $\hat{S}_n := S_n \vee S \in A$ will do), and shifting to a subsequence if necessary, we may assume that $S_n \rightarrow S$ \mathbb{P} -a.s. Clearly, there is some $N_0 \in \mathbb{N}$ such that $\mathbb{P}(S_n < 0) > 0$ for all $n \geq N_0$. By construction of A there are $t_n \geq 0$, $\alpha_n \in (0, 1]$ and $X_n, Z_n \in L_+^1$ such that $S_n = t_n(\alpha_n T + (1 - \alpha_n)X_n + Z_n)$. Thus $\{S_n < 0\} \subset \{T < 0\}$, and consequently we have \mathbb{P} -a.s. that

$$\{S < 0\} = \bigcup_{k \in \mathbb{N}} \bigcap_{m \geq k} \{S_m < 0\} \subset \{T < 0\}.$$

Let $\epsilon := E[-S1_{\{S < 0\}}] > 0$ and $\delta := E[-T1_{\{s < 0\}}] > 0$. Choose $N_1 \geq N_0$ such that for all $n \geq N_1$: $\|S - S_n\|_1 = E[S_n - S] \leq \frac{\epsilon}{2}$. Then, for $n \geq N_1$ we have:

$$\begin{aligned} \frac{\epsilon}{2} \geq E[S_n - S] &\geq E[(S_n - S)1_{\{S < 0\}}] \\ &= E[(t_n \alpha_n T + t_n(1 - \alpha_n)X_n + t_n Z_n - S)1_{\{S < 0\}}] \\ &\geq t_n \alpha_n E[T1_{\{S < 0\}}] + E[-S1_{\{S < 0\}}] \\ &= -t_n \alpha_n \delta + \epsilon. \end{aligned}$$

Consequently, $t_n \alpha_n \geq \frac{\epsilon}{2\delta} =: r > 0$, and thus $S_n \geq rT$, for all $n \geq N_1$. But this contradicts the boundedness of S . Hence $S \notin \mathcal{A}$ and (5.7) is proved. \square

Example 5.3. Let T and $\rho_{\mathcal{A}}$ be as in lemma 5.2, and let $\hat{\rho}_{\mathcal{A}}$ be the function defined by (4.6). It is easily verified that $\hat{\rho}_{\mathcal{A}} = -\text{essinf} = \overline{\rho_{\mathcal{A}}|_{L^\infty}}^{-1}$ on L^1 . Hence both $\hat{\rho}_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ are closed convex risk measures that extend $-\text{essinf}$ to L^1 . This shows that such extensions are not unique in general.

5.3 Non-extendable convex risk measures

The next two examples show convex risk measure on L^∞ which cannot be extended to L^p .

Example 5.4. Let $p \in [1, \infty)$ and $Z \in \mathcal{P}^{\infty*} \setminus \mathcal{P}^{p*}$. Define $\rho(X) = E[XZ]$. Then ρ is a convex risk measure on L^∞ with L^p -closure $\overline{\rho}^p \equiv -\infty$. Hence, we know by theorem 3.7 that there exists no closed convex extension of ρ to L^p . In particular $\hat{\rho}$ as defined in (4.6) cannot be closed convex on L^p .

Example 5.5. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1], \mathcal{B}(0, 1], \lambda)$ where λ denotes the Lebesgue measure restricted to the Borel- σ -algebra $\mathcal{B}(0, 1]$, and let $A_n := (0, \frac{1}{2^n}]$, $n \in \mathbb{N}$. Moreover, let $\mathbb{P}_n(\cdot) := \mathbb{P}(\cdot | A_n)$, and we denote by $\text{essinf}_{\mathbb{P}_n}(X)$ the essential infimum of a random variable X under the measure \mathbb{P}_n . Define

$$\rho(X) := \lim_{n \rightarrow \infty} -\text{essinf}_{\mathbb{P}_n}(X), \quad X \in L^0.$$

In fact, the function $\rho_\infty := \rho|_{L^\infty}$ is a coherent risk measure on L^∞ . However, it is easily verified that for $p \in [1, \infty)$ there are $X \in L^p$ such that $\rho(X) = -\infty$, so ρ fails to be proper on L^p . Moreover, $\overline{\mathcal{A}_{\rho_\infty}}^p = L^p$. The domain of ρ_∞^* is a subset of $(L^\infty)^* \setminus L^1$, because for any $Z \in \mathcal{P}^{\infty*} \cap L^1$ we have

$$\rho_\infty^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[XZ] \geq \sup_{k, n \in \mathbb{N}} E[-k1_{A_n^c} Z] = \sup_{k, n \in \mathbb{N}} k(1 + E[Z1_{A_n}]) = \infty.$$

That is condition (iii) of theorem 3.2 is not satisfied. Hence, $\overline{\rho_\infty}^p = -\infty$, and, according to theorem 3.7, ρ_∞ is a coherent risk measure on L^∞ which admits no closed convex extension to L^p , $p \in [1, \infty)$.

Next we illustrate by a simple example that we cannot expect the L^p -closure to be an extension in general, even if it is a closed convex function on L^p .

Example 5.6. Let $\rho(X) := \max\{E[-X], E[ZX], E[\tilde{Z}X]\}$, $X \in L^\infty$, for some $Z \in \mathcal{P}^{2*} \setminus L^\infty$ and $\tilde{Z} \in \mathcal{P}^{\infty*} \setminus L^1$. It is easily verified that ρ is a coherent risk measure on L^∞ and that $\text{dom } \rho^*$ is the convex hull $\text{co}\{-1, Z, \tilde{Z}\}$. We have $\bar{\rho}^p(X) = \max\{E[-X], E[ZX]\}$ for all $p \in [2, \infty)$, but $\bar{\rho}^1 = E[-X]$. Clearly, $\bar{\rho}^1 \neq \bar{\rho}^2$ on L^2 and $\bar{\rho}^p \neq \rho$ on L^∞ for all $p \in [1, \infty)$. Hence, $\bar{\rho}^p$ is no extension of ρ although $\bar{\rho}^p$ is a continuous coherent risk measure on L^p . Moreover, we observe that L^p -closures of ρ for different p may differ.

As regards (4.6), theorem 3.7 implies that $\hat{\rho}$ (which is proper due to $\hat{\rho}(X) \geq E[-X]$, and an extension of ρ by definition) cannot be closed convex, because otherwise $\bar{\rho}^p$ would be an extension. In particular, we have that $\bar{\rho}^p \neq \hat{\rho}$ for every $p \in [1, \infty)$.

5.4 A counter-example related to theorem 3.2

The next example shows that the requirements in theorem 3.2(iv) cannot be dropped in general.

Example 5.7. Recall example 5.5 and define

$$f : L^\infty \rightarrow (-\infty, \infty], \quad X \mapsto \rho(X) + \delta(X \mid \mathcal{C})$$

where $\mathcal{C} := \{X \in L^1 \mid X \geq 0 \text{ a.s. on } [1/2, 1]\}$. Clearly, $\mathcal{C} \cap L^p$ is convex and closed for every $p \in [1, \infty]$. Hence, f is a closed convex function on L^∞ . Next we prove by similar arguments as in example 5.5 that $\text{dom } f^* \cap L^1 = \emptyset$ implying that $\bar{f}^p = -\infty$ for all $p \in [1, \infty)$. To this end, note that $\rho(k1_{[0,1/2]}) = -k$ for all $k \in \mathbb{R}$. Consequently, for any $Z \in L^{\infty*}$ we have

$$f^*(Z) \geq \sup_{k \in \mathbb{R}} k(E[Z1_{[0,1/2]}] + 1),$$

so either $E[Z1_{[0,1/2]}] = -1$ or $Z \notin \text{dom } f^*$. Now let $Z \in L^1$ such that $E[Z1_{[0,1/2]}] = -1$. Then $\mathbb{P}(\{Z < 0\} \cap [0, 1/2]) > 0$, and since

$$X_{k,n} := -k1_{A_n^c}1_{[0,1/2]}1_{\{Z < 0\}} \in \mathcal{C}$$

satisfy $\rho(X_{k,n}) = 0$ for all $k, n \in \mathbb{N}$, we obtain

$$f^*(Z) \geq \sup_{k,n \in \mathbb{N}} E[ZX_{k,n}] \geq \sup_{k \in \mathbb{N}} -kE[Z1_{\{Z < 0\}}1_{[0,1/2]}] = \infty.$$

Hence, $\bar{f}^p = -\infty$ for all $p \in [1, \infty)$ which is equivalent to $\text{epi } \bar{f}^p = L^p \times \mathbb{R}$. However, it is easily verified that $\overline{\text{epi } f^p} = (\mathcal{C} \cap L^p) \times \mathbb{R} \subsetneq L^p \times \mathbb{R}$.

5.5 Value at risk cannot be fixed

The value at risk at some level $\alpha \in (0, 1)$ of a random variable X is defined as

$$\text{VaR}(X) = -q_\alpha(X)$$

where q_α denotes the upper α -quantile of X . Despite its non-convexity, VaR is still the industry standard risk measure. In view of its predominance in real applications, it would thus be desirable to replace VaR by the greatest closed convex risk measure majorised by VaR, i.e. its L^p -closure. Unfortunately, it turns out that, for any $p \in [1, \infty]$,

$$\overline{\text{VaR}}^p = -\infty. \quad (5.8)$$

Indeed, VaR is a continuous cash-invariant and monotone function on L^∞ . Hence $\text{dom}(\text{VaR}^*) \subset \mathcal{P}^{\infty*}$ (see e.g. Lemma 3.2 in [9]). Now let $Z \in \mathcal{P}^{\infty*}$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, there exists some $A \in \mathcal{F}$ with $E[Z1_A] < 0$ and $0 < \mathbb{P}[A] < \alpha$ (consider any finite partition $\Omega = \cup A_i$ with $0 < \mathbb{P}[A_i] < \alpha$). Then $\text{VaR}(-k1_A) = 0$ for all $k \in \mathbb{N}$, and hence

$$\text{VaR}^*(Z) \geq \sup_k (E[-k1_A Z] - \text{VaR}(-k1_A)) = \sup_k -kE[Z1_A] = \infty.$$

We conclude that $\overline{\text{VaR}^*} = \infty$, which proves (5.8).

6 Law-Invariant Convex Functions on L^p

In the following we collect and prove results on law-invariant convex functions which will play a fundamental role in the proof of theorem 3.8 which is stated in section 7. Throughout this section we let $p \in [1, \infty]$ and $q := \frac{p}{p-1}$ where $\frac{1}{0} := \infty$ and $\frac{\infty}{\infty-1} := 1$.

One of the main ingredients for proving theorem 3.8 is lemma 6.2 below. In fact, this lemma is an extension of results by Jouini, Schachermayer, and Touzi in [15].

Let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra. As in [15] we define the conditional expectation on $L^{\infty*}$ as a function $E[\cdot | \mathcal{G}] : L^{\infty*} \rightarrow L^{\infty*}$ where $E[\mu | \mathcal{G}]$ is given by

$$E[E[\mu | \mathcal{G}]X] := E[\mu E[X | \mathcal{G}]] \quad \forall X \in L^\infty.$$

Clearly, this definition is consistent with the ordinary conditional expectation in case $\mu \in L^1 \subset L^{\infty*}$.

Remark 6.1. *If $\mathcal{G} = \sigma(A_1, \dots, A_n)$ is finite, then $E[\mu | \mathcal{G}] \in L^\infty$. In order to verify this, note that for all $X \in L^\infty$ we have*

$$E[E[\mu | \mathcal{G}]X] = E[\mu E[X | \mathcal{G}]] = \sum_{i=1}^n E[X1_{A_i}] \frac{\mu(A_i)}{\mathbb{P}(A_i)}.$$

Hence, $E[\mu | \mathcal{G}] = \sum_{i=1}^n \frac{\mu(A_i)}{\mathbb{P}(A_i)} 1_{A_i} \in L^\infty$.

Note that (L^p, L^r) is a dual pair for every $r \in [q, \infty]$.

Lemma 6.2. *A law-invariant convex function $f : L^p \rightarrow [-\infty, \infty]$ is closed (w.r.t. $\sigma(L^p, L^{p*})$) if and only if it is closed w.r.t. any $\sigma(L^p, L^r)$ -topology for every $r \in [q, \infty]$.*

For the proof of lemma 6.2 we will need the following result.

Lemma 6.3. *Let $D \subset L^p$ be a $\|\cdot\|_p$ -closed convex law-invariant set. Then D is $\sigma(L^p, L^r)$ -closed for every $r \in [q, \infty]$.*

Proof. 0. If $D = \emptyset$, the assertion is obvious. For the remainder of this proof, we assume that $D \neq \emptyset$.

1. According to lemma 4.2 in [15], for all $Y \in D$ and all sub- σ -algebras $\mathcal{G} \subset \mathcal{F}$ we have $E[Y | \mathcal{G}] \in D$.

2. Let $(X_i)_{i \in I}$ be a net in D converging to some $X \in L^p$ in the $\sigma(L^p, L^r)$ -topological sense, i.e. $E[ZX_i] \rightarrow E[ZX]$ for all $Z \in L^r$. Then, in view of remark 6.1, if \mathcal{G} is finite, we have $E[E[\mu | \mathcal{G}]X_i] \rightarrow E[E[\mu | \mathcal{G}]X]$ for all $\mu \in L^{p^*}$. But by definition this equals $E[\mu E[X_i | \mathcal{G}]] \rightarrow E[\mu E[X | \mathcal{G}]]$ for all $\mu \in L^{p^*}$. In other words, the net $(E[X_i | \mathcal{G}])_{i \in I}$ converges to $E[X | \mathcal{G}]$ in the $\sigma(L^p, L^{p^*})$ -topology. Since, according to 1., $E[X_i | \mathcal{G}] \in D$ for all $i \in I$, we conclude that $E[X | \mathcal{G}] \in D$, because D is closed and convex and thus $\sigma(L^p, L^{p^*})$ -closed. Hence, $E[X | \mathcal{G}] \in D$ for all finite sub- σ -algebras $\mathcal{G} \subset \mathcal{F}$. Recalling that we can approximate X in $(L^p, \|\cdot\|_p)$ by a sequence of conditional expectations $(E[X | \mathcal{G}_n])_{n \in \mathbb{N}}$ in which the \mathcal{G}_n s are all finite, we conclude by means of the norm-closedness of D that $X \in D$. Thus D is $\sigma(L^p, L^r)$ -closed. \square

Proof of lemma 6.2. Let f be closed. Then, for every $k \in \mathbb{R}$ the level sets $\{X \in L^p | f(X) \leq k\}$ are $\|\cdot\|_p$ -closed, convex, and law-invariant. Hence, lemma 6.3 yields the $\sigma(L^p, L^r)$ -closedness of the level sets, i.e. f is closed with respect to the $\sigma(L^p, L^r)$ -topology. The converse implication is trivial. \square

Besides lemma 6.2, the proof of theorem 3.8 draws heavily on lemmas 6.4 and 6.6 below. It will require some working with quantiles and some further analysis of law-invariant convex functions in order to deriving these lemmas.

For the proof of lemma 6.4 we will need the following fact.

(F1) *For $X \in L^p$ and $Z \in L^q$ we have that*

$$\int_0^1 q_X(s)q_Z(s) ds = \sup_{\hat{X} \sim X} E[\hat{X}Z] = \sup_{\hat{Z} \sim Z} E[X\hat{Z}].$$

The relation (F1) is stated and proved in Föllmer/Schied [13] lemma 4.55 for the case $X \in L^1$ and $Z \in L^\infty$. This result in turn can be easily extended to $X \in L^p$ and $Z \in L^q$ by suitable approximation, so we omit a proof here.

Lemma 6.4. *Let $f : L^p \rightarrow [-\infty, \infty]$ be a closed convex function. Then the following conditions are equivalent:*

- (i) *f is law-invariant.*
- (ii) *f is $\sigma(L^p, L^q)$ -closed and f^* (resp. $f^*|_{L^1}$ if $p = \infty$) is law-invariant.*

Moreover, if either holds, then:

$$f^*(Z) = \sup_{X \in L^p} \int_0^1 q_X(s)q_Z(s) ds - f(X), \quad Z \in L^q,$$

and

$$f(X) = \sup_{Z \in L^q} \int_0^1 q_X(s)q_Z(s) ds - f^*(Z), \quad X \in L^p.$$

Proof. (i) \Rightarrow (ii): the first statement is trivial if $p \in [1, \infty)$ and proved in lemma 6.2 if $p = \infty$. Moreover, for any $Z \in L^q$ we gather from (F1) that

$$\begin{aligned} f^*(Z) &= \sup_{X \in L^p} E[XZ] - f(X) = \sup_{X \in L^p} \left(\sup_{\hat{X} \sim X} E[\hat{X}Z] \right) - f(X) \\ &= \sup_{X \in L^p} \int_0^1 q_X(s)q_Z(s) ds - f(X) \end{aligned}$$

in which the latter expression depends on the law of Z only.

(ii) \Rightarrow (i): again by (F1)

$$\begin{aligned} f(X) &= f^{**}(X) = \sup_{Z \in L^q} \left(\sup_{\hat{Z} \sim Z} E[X\hat{Z}] \right) - f^*(Z) \\ &= \sup_{Z \in L^q} \int_0^1 q_X(s)q_Z(s) ds - f^*(Z) \end{aligned}$$

for all $X \in L^p$. Hence, f is law-invariant. \square

Next we introduce two orders on L^1 which are well-known from utility theory. To this end, recall that a utility function is a strictly concave and strictly increasing function $u : \mathbb{R} \rightarrow \mathbb{R}$.

Definition 6.5. For any two $X, Y \in L^1$ we define

(i) the concave order:

$$X \succeq_c Y \Leftrightarrow E[u(X)] \geq E[u(Y)] \text{ for all concave functions } u : \mathbb{R} \rightarrow \mathbb{R},$$

(ii) the second order stochastic order:

$$X \succeq Y \Leftrightarrow E[u(X)] \geq E[u(Y)] \text{ for all utility functions } u.$$

A function $f : L^p \rightarrow [-\infty, \infty]$ is said to be \succeq_c (\succeq)-monotone if $f(X) \leq f(Y)$ whenever $X \succeq_c Y$ ($X \succeq Y$).

Clearly, both $X \geq Y$ and $X \succeq_c Y$ imply $X \succeq Y$. However, \geq and \succeq_c are not related in general.

The subsequent lemma 6.6 is a main result in Dana [6] where she proves it for $p \in \{1, \infty\}$. In the proof of theorem 3.8 we need this result for general $p \in [1, \infty]$, so we provide a self-contained proof below. However, this proof builds on the following two facts which in turn are proved in Dana [6]. Let $X, Y \in L^1$, then

(F2) $X \succeq Y$ if and only if

$$\int_0^1 q_X(s)g(s) ds \leq \int_0^1 q_Y(s)g(s) ds$$

for all increasing $g : (0, 1) \rightarrow (-\infty, 0]$ such that both integrals exist.

(F3) $X \succeq_c Y$ if and only if

$$\int_0^1 q_X(s)g(s) ds \leq \int_0^1 q_Y(s)g(s) ds$$

for all increasing $g : (0, 1) \rightarrow \mathbb{R}$ such that both integrals exist.

Lemma 6.6. Let $f : L^p \rightarrow [-\infty, \infty]$ be a closed convex function. Equivalent are:

(i) f is law-invariant.

(ii) f is \succeq_c -monotone.

Moreover, if in addition f is monotone, then (i) and (ii) are equivalent to

(iii) f is \succeq -monotone.

In particular, if either of the conditions (i), (ii) or (iii) holds, then

(iv) $f(E[X | \mathcal{G}]) \leq f(X)$ for all $X \in L^p$ and all sub- σ -algebras $\mathcal{G} \subset \mathcal{F}$.

Proof. (i) \Rightarrow (ii): let $X \succeq_c Y$. Then by lemma 6.4 and (F3):

$$\begin{aligned} f(X) &= \sup_{Z \in L^q} \int_0^\infty q_X(s)q_Z(s) ds - f^*(Z) \\ &\leq \sup_{Z \in L^q} \int_0^\infty q_Y(s)q_Z(s) ds - f^*(Z) = f(Y). \end{aligned}$$

(ii) \Rightarrow (i): conversely, suppose that f is \succeq_c -monotone and let $X \sim Y$. Trivially, $X \succeq_c Y$ and $Y \succeq_c X$, so $f(X) = f(Y)$.

(i) \Rightarrow (iii): let $X \succeq Y$. Since f is monotone, we have that $\text{dom } f^* \cap L^q \subset L^q_-$ (see [9] lemma 3.2). Hence, by lemma 6.4 and (F2),

$$\begin{aligned} f(X) &= \sup_{Z \in L^q_-} \int_0^1 q_X(s)q_Z(s) ds - f^*(Z) \\ &\leq \sup_{Z \in L^q_-} \int_0^1 q_Y(s)q_Z(s) ds - f^*(Z) = f(Y). \end{aligned}$$

(iii) \Rightarrow (i): if $X \sim Y$, then $X \succeq Y$ and $Y \succeq X$. Thus, $f(X) = f(Y)$.

(iv): for any concave function $u : \mathbb{R} \rightarrow \mathbb{R}$ Jensen's inequality yields

$$E[u(E[X | \mathcal{G}])] \geq E[E[u(X) | \mathcal{G}]] = E[u(X)].$$

Hence, $E[X | \mathcal{G}] \succeq_c X$. Now apply (ii). □

Remark 6.7. Note that the proof of lemma 6.6 relies on lemma 6.2, and thus on lemma 4.2 in [15], only in case $p = \infty$. We recall that lemma 4.2 in [15] states that if $\emptyset \neq D \subset L^p$ is a convex law-invariant and $\|\cdot\|_p$ -closed set, then $E[X | \mathcal{G}] \in D$ for all $X \in D$ and all sub- σ -algebras $\mathcal{G} \subset \mathcal{F}$. Indeed, for every such set D , the indicator function $\delta(\cdot | D)$ is a law-invariant closed convex function. Therefore, according to lemma 6.6(iv), $\delta(E[X | \mathcal{G}] | D) \leq \delta(X | D)$. This implies $E[X | \mathcal{G}] \in D$ whenever $X \in D$. Hence, we have derived an alternative proof of lemma 4.2 in [15] for the cases $p \in [1, \infty)$ (but clearly not for $p = \infty$).

7 Proof of Theorem 3.8

According to lemma 6.2 any law-invariant closed convex function $f : L^\infty \rightarrow [-\infty, \infty]$ is $\sigma(L^\infty, L^\infty)$ -closed. Hence, by theorem 3.7 \bar{f}^p is a closed convex extension of f to L^p . Theorem 3.2(ii) and lemma 6.4 yield the law-invariance of \bar{f}^p . In order to prove that \bar{f}^p is the unique law-invariant closed convex extension of f to L^p , let g be any such extension. For every $X \in L^p$ and all $m \in \mathbb{N}$ there exists a finite partition A_1^m, \dots, A_n^m of Ω such that the L^p -distance between X and the simple random variable $X_m := E[X | \sigma(A_1^m, \dots, A_n^m)] \in L^\infty$ is less than $1/m$. On the one hand, lemma 6.6(iv) implies that $g(X_m) \leq g(X)$ for all $m \in \mathbb{N}$. On the other hand, by l.s.c. of g , we know that $g(X) \leq \liminf_{m \rightarrow \infty} g(X_m)$. Hence, $g(X) = \lim_{m \rightarrow \infty} g(X_m)$. Since the latter observation in particular holds for \bar{f}^p , we obtain

$$g(X) = \lim_{m \rightarrow \infty} g(X_m) = \lim_{m \rightarrow \infty} f(X_m) = \lim_{m \rightarrow \infty} \bar{f}^p(X_m) = \bar{f}^p(X)$$

and uniqueness is proved. Finally, it immediately follows that $\bar{f}^1|_{L^p} = \bar{f}^p$. \square

8 Proof of Theorem 2.3

(i): clearly, if ρ is continuous, then e.g. $0 \in \text{int dom } \rho$. In order to prove the converse implication note that according to [19] proposition 3.1 ρ is continuous on $\text{int dom } \rho$. Hence, it suffices to prove that $\text{int dom } \rho = \text{dom } \rho = L^p$. We show this by means of contradiction. Suppose for the moment that there is a $\tilde{X} \in L^p$ such that $\rho(\tilde{X}) = \infty$. Since the interior of the convex set $\text{dom } \rho$ is non-empty by assumption and $\tilde{X} \notin \text{dom } \rho$, an appropriate version of the Hahn-Banach separating hyperplane theorem ensures the existence of a nontrivial $Z \in L^{p*}$ such that

$$\sup_{Y \in \text{dom } \rho} E[Z Y] \leq E[Z \tilde{X}].$$

Since $L^\infty \subset \text{dom } \rho$ we obtain

$$tE[Z X] \leq E[Z \tilde{X}] \quad \text{for all } X \in L^\infty \text{ and } t \in \mathbb{R},$$

and thus $E[X Z] = 0$ for all $X \in L^\infty$. As L^∞ is dense in L^p , we infer that Z must be trivial. But this is a contradiction. Therefore, $\text{dom } \rho = L^p$.

(ii): we only prove $\overline{\mathcal{A}}_\rho^p \neq L^p$, the other three properties are obvious. Suppose we had $\overline{\mathcal{A}}_\rho^p = L^p$. Then for every $n \in \mathbb{N}$ there is a $X_n \in \mathcal{A}_\rho$ such that

$$\|X_n - (-n)\|_p = \|X_n + n\|_p \leq \frac{1}{2^n}.$$

By monotonicity we may assume that $X_n + n \geq 0$. Then, the sequence $Y_n := \sum_{k=1}^n X_k + k$, $n \in \mathbb{N}$, converges to

$$Y := \left(\sum_{k=1}^{\infty} X_k + k \right) \in L^p, \quad \|Y\|_p \leq 1,$$

and $Y \geq Y_n \geq X_n + n$ for all $n \in \mathbb{N}$. Thus, by monotonicity, cash-invariance, and $X_n \in \mathcal{A}_\rho$ we infer

$$\rho(Y) \leq \rho(X_n + n) \leq \rho(X_n) - n \leq -n \quad \text{for all } n \in \mathbb{N}.$$

Consequently $\rho(Y) = -\infty$. But this is a contradiction to the properness of ρ . Hence, $\overline{\mathcal{A}}_\rho^p \subsetneq L^p$.

(iii): it is easily verified that $\rho_{\mathcal{A}}$ is a convex, cash-invariant, and monotone function such that $\rho_{\mathcal{A}}(0) < \infty$. It remains to prove that $\rho_{\mathcal{A}}$ is proper. In doing so, it suffices to show that $\rho_{\overline{\mathcal{A}}^p}$ is proper because $\rho_{\overline{\mathcal{A}}^p} \leq \rho_{\mathcal{A}}$. Observe that $\overline{\mathcal{A}}^p$ satisfies properties (ii) because \mathcal{A} does. Hence, $\rho_{\overline{\mathcal{A}}^p}$ is convex, cash-invariant, and monotone too, and $\rho_{\overline{\mathcal{A}}^p}(0) < \infty$. If we had $\rho_{\overline{\mathcal{A}}^p}(0) = -\infty$, then it follows that $\mathbb{R} \subset \overline{\mathcal{A}}^p$, and thus $L^\infty \subset \overline{\mathcal{A}}^p$, so actually $\overline{\mathcal{A}}^p = L^p$ which is a contradiction to the assumption $\overline{\mathcal{A}}^p \neq L^p$. Consequently, $\rho_{\overline{\mathcal{A}}^p}(0) > -\infty$. Clearly, $\overline{\mathcal{A}}^p = \mathcal{A}_{\rho_{\overline{\mathcal{A}}^p}}$, so $\rho_{\overline{\mathcal{A}}^p}$ is l.s.c. Since any l.s.c. convex function which assumes the value $-\infty$ cannot take any finite value (see [8] proposition 2.4.), and since $\rho_{\overline{\mathcal{A}}^p}(0) \in \mathbb{R}$, we conclude that $\rho_{\overline{\mathcal{A}}^p} > -\infty$, i.e. $\rho_{\overline{\mathcal{A}}^p}$ is proper.

(iv): for a proof of $\text{dom } \rho^* \subset \mathcal{P}^{p*}$, please consult [9] lemma 3.2. The stated representation of $\rho^*(Z)$ for any $Z \in \mathcal{P}^{p*}$ is easily verified in view of $E[1Z] = -1$ and cash-invariance of ρ . \square

References

- [1] Aliprantis, C.D. and Border K.C. (1999), *Infinite Dimensional Analysis*, Springer.
- [2] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999), Coherent measures of risk, *Mathematical Finance*, 9, 203–228.
- [3] Biagini, S. and Frittelli, M. (2006), On Continuity Properties and Dual Representation of Convex and Monotone Functionals on Frechet Lattices, Working Paper.
- [4] Cheridito, P. and Li, T. (2007) Risk Measures on Orlicz Hearts, preprint.

- [5] Cherny, A.S. and Madan, D.B. (2006), CAPM, Rewards, and Empirical Asset Pricing with Coherent Risk, SSRN Working Paper.
- [6] Dana, R-A. (2005), A Representation Result for Concave Schur Concave Functions, *Mathematical Finance*, 15, 613–634.
- [7] Delbaen, F. (2000), *Coherent Risk Measures*, Cattedra Galileiana. Scuola Normale Superiore di Pisa.
- [8] Ekeland, I. and Témam, R. (1999), *Convex Analysis and Variational Problems, Chapter I*, SIAM.
- [9] Filipović, D. and Kupper, M. (2007), Monotone and Cash-Invariant Convex Functions and Hulls, *Insurance: Mathematics and Economics*, 41, 1–16.
- [10] Filipović, D. and Svindland, G. (2007), Optimal Capital and Risk Allocations for Law-Invariant Convex Cash-Invariant Functions, preprint.
- [11] Filipović, D. and Svindland, G. (2008), Subgradients of law-invariant convex risk measures, work in progress.
- [12] Föllmer, H. and Schied, A. (2002), Convex measures of risk and trading constraints, *Finance and Stochastics*, 6, 429–447.
- [13] Föllmer, H. and Schied, A. (2002), *Stochastic Finance, An Introduction in Discrete Time*, de Gruyter Studies in Mathematics 27.
- [14] Frittelli, M. and Rosazza Gianin, E. (2002), Putting order in risk measures, *Journal of Banking and Finance*, 26, 1473–1486.
- [15] Jouini, E., Schachermayer, W. and Touzi, N. (2006), Law invariant risk measures have the Fatou property, *Advances in Mathematical Economics*, Vol. 9, pp. 49-71.
- [16] Kaina, M. and Rüschendorf, L. (2007), On Convex Risk Measures on L^p , Working Paper.
- [17] Krätschmer, V. (2007), On σ -additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model, SFB 649 Discussion Paper 2007-010, Humboldt University Berlin.
- [18] Rockafellar, R.T. (1997), *Convex Analysis*, Princeton University Press.
- [19] Ruszczyński, A. and Shapiro, A. (2006), Optimization of Convex Risk Functions, *Math. Oper. Res.*, 31, 433–451.