On the Theory of

Continuous-Time Recursive Utility

Mogens Steffensen

Swissquote Conference on Asset Management

Lausanne, 21.10.11

Outline:

- Quadratic and collective objectives
- Further comments on consistency
- Recursive utility
- Putting the pieces together

Quadratic and collective objectives

Portfolio selection

$$\begin{aligned} dB(t) &= rB(t)\,dt \\ dS(t) &= \alpha S(t)\,dt + \sigma S(t)\,dW(t) \end{aligned}$$

$$dX(t) = (r + \pi(t)(a - r))\,X(t)\,dt + \pi(t)\,\sigma X(t)\,dW(t)$$

Hamilton-Jacobi-Bellman equation for the continuum of problems

$$V(t,x) = \sup_{\pi} E_{t,x}\left[u(X(T))\right]$$

$$\begin{aligned} V_t &= \inf_{\pi}\left[-(r + \pi(\alpha - r))\,xV_x - \frac{1}{2}\sigma^2\pi^2x^2V_{xx}\right] \\ V(T,x) &= u(x) \end{aligned}$$

Mean-variance optimization $\sim$ mean-variance utility $\sim$ quadratic optimization

$$\inf_{\pi : E[X(T)] \geq k} Var\left[X\left(T\right)\right] \sim \sup_{\pi} \left(E\left[X\left(T\right)\right] - \frac{\gamma}{2} Var\left[X\left(T\right)\right]\right)$$

How should they be embedded in a continuum of problems?

a) Classical but problematic:

$$V\left(t,x\right) = \sup_{\pi : E[X(T)] = k} E_{t,x}\left[X\left(T\right) - \frac{\gamma}{2}\left(X\left(T\right) - k\right)^2\right]$$

b) 2010 and onwards: (Basak and Chabakauri, Björk and Murgoci, Kryger and Steffensen)

$$\begin{aligned} V\left(t,x\right) &= \sup_{\pi}\left(E_{t,x}\left[X\left(T\right)\right] - \frac{\gamma}{2} Var_{t,x}\left[X\left(T\right)\right]\right) \\ &\sim \sup_{\pi}\left(aE_{t,x}\left[X\left(T\right)\right] + bE_{t,x}\left[X^2\left(T\right)\right] + c\left(E_{t,x}\left[X\left(T\right)\right]\right)^2\right) \end{aligned}$$

Kryger and Steffensen: Simple market, general objective including: mean-variance, mean-std, endogenous habit formation, collective utility (getting back to the sup)

$$V(t,x) = \sup_{\pi} \left(f \left(\underbrace{E_{t,x}\left[g\left(X\left(T\right)\right)\right]}_{G}, \underbrace{E_{t,x}\left[h\left(X\left(T\right)\right)\right]}_{H} \right) \right)$$

Pseudo-Bellman equation

$$V_t = \inf_{\pi} \left[-\left(r + \pi\left(\alpha - r\right)\right) x V_x - \frac{1}{2}\sigma^2\pi^2x^2\left(V_{xx} - U\right) \right]$$

$$U = f_{GG}G_x^2 + 2 f_{GH} G_x H_x + f_{HH} H_x^2$$

Collective objectives

$$\sup_{\pi} \left\{ u_1^{-1} \left(E_{t,x} \left[u_1 \left(X \left(T \right) \right) \right] \right) + u_2^{-1} \left(E_{t,x} \left[u_2 \left(X \left(T \right) \right) \right] \right) \right\}$$

Nice mathematics - interesting economics!

Further comments on consistency

Consider a two period model:

linear objective - **sup** over all strategies	$t = 0$	$t = 1$
The $t = 0$ strategy	**50%**	60%
The $t = 1$ strategy		**60%**

non-linear objective - **sup** over all strategies	$t = 0$	$t = 1$
The $t = 0$ strategy	**40%**	50%
The $t = 1$ strategy		**45%**

non-linear objective - **sup** over consistent strategies	$t = 0$	$t = 1$
The $t = 0$ strategy	**42%**	45%
The $t = 1$ strategy		**45%**

$$\text{a) linear objective} \quad : \quad V(t,x) = \sup_{\pi} E_{t,x}\left[u\left(X\left(T\right)\right)\right]$$
$$\text{b) non-linear objective} \quad : \quad V(t,x) = \sup_{\pi} f\left(E_{t,x}\left[u\left(X\left(T\right)\right)\right]\right)$$

The solution to the linear problems a) is time-consistent

The solution to the non-linear problem b) is time-inconsistent **if we choose among all strategies**

But we can **restrict the set of strategies to include only time-consistent strategies**

Then we find a solution to b) which is, per construction, time-consistent

Of course, the value function is smaller than if there were no restriction!

Recursive utility

Wealth dynamics

$$dX(t) = (r + \pi(t)(a - r)) X(t) dt + \pi(t) \sigma X(t) dW(t) - c(t) dt$$

Classical value function based on time-additive utility

$$V(t,x) = \sup_{c,\pi} E_{t,x}\left[\int_t^n e^{-\delta(s-t)} \frac{1}{1-\gamma} c^{1-\gamma}(s) ds\right]$$

But γ is more than risk aversion: The deterministic problem ($\pi = 0$)

$$V(t,x) = \sup_{c}\left[\int_t^n e^{-\delta(s-t)} \frac{1}{1-\gamma} c^{1-\gamma}(s) ds\right]$$

is well-posed and has a nice explicit solution. So, γ is also related to Elasticity of Intertemporal Subtitution

Recursive Utility Classic (power utility):

$$V(t,x) = \underbrace{W}_{\text{aggregator}} \left(c(t), \underbrace{\left((1-\gamma) E_{t,x} \left[V(t+\Delta, X(t+\Delta)) \right] \right)^{\frac{1}{1-\gamma}}}_{\text{certainty equivalent of value function}} \right)$$

$$W = \left[c^{1-\phi}(t) + e^{-\delta\Delta} \left(\underbrace{\left((1-\gamma) E_{t,x} \left[V(t+\Delta, X(t+\Delta)) \right] \right)^{\frac{1}{1-\gamma}}}_{\text{certainty equivalent of value function}} \right)^{1-\phi} \right]^{\frac{1-\gamma}{1-\phi}}$$

Complicating problems with differentiability of the certainty equivalent and the aggregator!

Duffie and Epstein (1992), Kraft and Seifried (2010), Kraft, Seifried and Steffensen (2010)

Putting the pieces together

$$V(t,x) = \sup_{\pi,c} \left[\int_t^n e^{-\delta(s-t)} \left(\underbrace{\left((1-\gamma) E_{t,x} \left[\frac{1}{1-\gamma} c^{1-\gamma}(s) \right] \right)^{\frac{1}{1-\gamma}}}_{\text{certainty equivalent of consumption}} \right)^{1-\phi} ds \right]$$

Since the value function is messed up by a series of non-linearities, we have to think carefully about consistency! We look for a solution in the set of consistent strategies!

This explains why Duffie and Epstein could prove consistency AND why 'the global formulation' has not been proposed before! (?)

One more time:

$$V(t,x) = \sup_{\pi,c} \left[\int_t^n e^{-\delta(s-t)} \left(\underbrace{\left((1-\gamma) E_{t,x} \left[\frac{1}{1-\gamma} c^{1-\gamma}(s) \right] \right)^{\frac{1}{1-\gamma}}}_{\text{certainty equivalent of consumption}} \right)^{1-\phi} ds \right]$$

Recognize time-additive utility as special case ($\phi = \gamma$)

$$\begin{aligned} V(t,x) &\sim \sup_{\pi,c} \left[\int_t^n e^{-\delta(s-t)} E_{t,x} \left[\frac{1}{1-\gamma} c^{1-\gamma}(s) \right] ds \right] \\ &= \sup_{\pi,c} E_{t,x} \left[\int_t^n e^{-\delta(s-t)} \left[\frac{1}{1-\gamma} c^{1-\gamma}(s) \right] ds \right] \end{aligned}$$

The good news: For a simple (Black-Sholes) financial market the solution to the local and the global formulations coincide:

The value function is essentially equivalent with the solution to

$$
\begin{aligned}
V_t &= \inf_{c,\pi}\left[-f\left(c,V\right)-\left(\left(r+\pi\left(\alpha-r\right)\right)x-c\right)V_x-\frac{1}{2}\pi^2\sigma^2x^2V_{xx}+U\right] \\
f\left(c,V\right) &= \delta\theta V\left(\left(\frac{c}{\left(\left(1-\gamma\right)V\right)^{\frac{1}{1-\gamma}}}\right)^{1-\phi}-1\right) \\
\theta &= \frac{1-\gamma}{1-\phi}
\end{aligned}
$$

This classical continuous-time aggregator $f\left(c,V\right)$ goes with $U=0$

The bad (?) news: For a general one-factor model the solution to the local and the global formulations do not coincide: This classical continuous-time aggregator $f(c, V)$ does not go with $U = 0$. That *disturbs* (at least) Chacko and Viceira (2005) and Kraft, Seifried and Steffensen (2010)

EITHER one of us is doing something wrong

OR we are just solving different problems

IN CASE we can discuss who solved the 'right' problem