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Discussion on:

On the Theory of Continuous-Time Recursive Utility

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Studying the connection between

- **Recursive utility:** $V(c_0, c_1, ..) = W(c_0, \mu[V(c_1, c_2, \ldots)])$

  Epstein-Zin (1989)

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  cont.time: stochastic differential utility Duffie-Epstein (1992)

  preferences formalized via a differential equation

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- **Time-global problem formulation**

  Issue of **Time-Consistency**

  $\rightarrow$ optimize not among admissible, but only among consistent strategies!

  (e.g. mean-variance problem without precommitment)

  Basak-Chabakauri (2010), Björk-Murgoci (2010)

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Result: The resulting optimal strategies *sometimes* coincide

(Example: Black-Scholes market)

Model Assumptions

$$dB(t) = B(t)\, r\, dt$$

$$dS(t) = S(t)\Big[(r + \lambda(t, Y(t)))\, dt + \sigma(t, Y(t))\, dW(t)\Big]$$

$$dY(t) = \alpha(t, Y(t))\, dt + \beta(t, Y(t))\left(\rho\, dW(t) + \sqrt{1-\rho^2}\, d\overline{W}(t)\right)$$

Wealth process $X(t)$: invest πX in S and $(1-\pi)X$ in B,

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Classical problem: value function

γ..risk/time preference

$$V(t, x, y) = \sup_{c,\pi} E_{t,x,y}\left[\int_t^n e^{-\delta(s-t)}\, \frac{1}{1-\gamma}\, c^{1-\gamma}(s)\, ds\right]$$

Linearity: local $\rightarrow$ global

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New problem: value function θ..elasticity of intertemporal substitution

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certainty equivalent on c

Non-Linear: look only for optimal control among consistent ones

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More flexibility, conceptually: analytical advantages over recursive utility

Bellman-type equation with *additional terms*, which cancel if

- $\theta = 1$: time-additive utility
- Separation condition and $\beta = 0$

Questions

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  What is the 'price' of such a restriction?
- Could one replace power utility by a distance to prespecified consumption stream?

Final Remarks

- Related problem in insurance:

  Maximize (utility of) expected discounted dividend payments until ruin

$$\max_{c} E\left(\int_0^{\tau} e^{-\delta s} U(c_s) ds\right), \qquad \max_{C} E\left[U\left(\int_0^{\tau} e^{-\delta s} dC_s\right)\right]$$

Diffusion: Hubalek-Schachermayer (2004), Zigo-Grandits-Hubalek-Schachermayer (2007)

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  Can the same utility function be used for all time horizons?

  Time Consistency of Valuations (rather than Actions)

Musiela-Zariphopoulou (2010), ElKaroui-M'Rad (2010)

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- Risk measures:

  Coherence vs. Time Consistency