REGULARITY OF FINITE-DIMENSIONAL REALIZATIONS FOR EVOLUTION EQUATIONS

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Abstract. We show that a continuous local semiflow of C^k-maps on a finite-dimensional C^k-manifold M can be embedded into a local C^k-flow on M under some weak (necessary) assumptions. This result is applied to an open problem in [2]. We prove that finite-dimensional realizations for interest rate models are highly regular objects, namely given by submanifolds M of $D(A^\infty)$, where A is the generator of a strongly continuous semigroup.

1. Introduction

Let $k \geq 1$ be given. We consider a Banach space X and a *continuous local semiflow Fl of C^k-maps* on it, i.e.

i) There is $\varepsilon > 0$ and $V \subset X$ open with $Fl : [0,\varepsilon[\times V \to X$ a continuous map.
ii) $Fl(0,x) = x$ and $Fl(s, Fl(t,x)) = Fl(s+t,x)$ for $s,t,s+t \in [0,\varepsilon[$ and $x, Fl(t,x) \in V$.
iii) The map $Fl_t : V \to X$ is C^k for $t \in [0,\varepsilon[$.

Continuous local semiflows of C^k-maps appear naturally as mild solutions of nonlinear evolution equations (see the appendix). The continuous local semiflow Fl is called C^k if $Fl : [0,\varepsilon[\times V \to X$ is C^k.

We assume that we are given a finite-dimensional C^k-submanifold M of X such that M is *locally invariant* for Fl, i.e. for every $x \in M \cap V$ there is $\delta_x \in]0,\varepsilon[$ such that $Fl(t,x) \in M$ for $0 \leq t \leq \delta_x$. In this case Fl restricts in a small open neighborhood of any point $x \in M \cap V$ to a continuous local semiflow of C^k-maps on M (defined as above), see Lemma 1.3 below.

We show that the restriction of Fl to M is jointly C^k and can in particular be embedded in a local C^k-flow on M. We shall apply classical methods from [8] developed to solve the fifth Hilbert problem. Nevertheless we have to face the difficulty that Fl is only a continuous *local semi*flow. We can prove the result under a weak assumption, which will always be satisfied for our applications.

The problem arises in several contexts, for example recently in interest rate theory, see [2, 3].

We first cite the classical results from Dean Montgomery and Leo Zippin [8] and draw the simple conclusion:

Theorem 1.1. *Let M be a finite-dimensional C^k-manifold and $Fl : \mathbb{R} \times M \to M$ a continuous flow of C^k-maps on M, then Fl is a C^k-flow on M.*

Example 1.2. *Let S be a strongly continuous group on a Banach space X and assume that M is a locally S-invariant finite-dimensional C^k-submanifold of X.*

Date: November 2, 2001 (first draft); December 20, 2001 (this draft).
We thank K. David Elworthy for bringing this interesting problem to our attention.

Then $M \subset D(A^k)$, where A denotes the infinitesimal generator of S, and the restriction of A to M is a C^{k-1}-vector field on M.

The paper is organized as follows. In Section 2 we prove the extension of Theorem 1.1 for continuous local semiflows. In Section 3 we apply this result to a problem that arises in connection with finite-dimensional realizations for interest rate models, as it has been announced in [2]. The appendix contains a regularity result for the dependence of solutions to evolution equations on the initial point.

We end this section by the announced lemma. Let M be locally invariant for Fl, as above.

Lemma 1.3. *For every $x \in M \cap V$ there exists an open neighborhood V' in X and $\varepsilon' > 0$ such that $Fl(t,y) \in M$ for all $(t,y) \in [0,\varepsilon'[\times (V' \cap M)$.*

Proof. Take a submanifold chart $u : U \subset X \to X$ with $u(U \cap M) = \{0\} \times W \subset \{0\} \times \mathbb{R}^k$, where $U \subset \overline{U} \subset V$ is open around x and $W \subset \mathbb{R}^k$ is open around 0. We may assume that u has a continuous extension on $\overline{U}$ with $u(\overline{U \cap M}) = u(\overline{U} \cap M) = \{0\} \times \overline{W}$.

For $y \in U$ define the lifetime in $U \cap M$

$$T(y) := \sup\{0 < t < \varepsilon \mid Fl(s,y) \in U \cap M,\ \forall 0 \le s < t\}.$$

By continuity of Fl we have $Fl(T(y),y) \in \overline{U \cap M} \setminus (U \cap M)$ if $T(y) < \varepsilon$. We claim that there exists an open neighborhood $V' \subset U$ of x in X and $\varepsilon' > 0$ such that $T(y) \ge \varepsilon'$ for all $y \in V'$. Indeed, otherwise we could find a sequence (x_n) in $U \cap M$ with $x_n \to x$ and $\varepsilon > T(x_n) \to 0$. But this means that $u(Fl(T(x_n),x_n)) \in \{0\} \times (\overline{W} \setminus W)$ converges to $u(Fl(0,x)) = u(x) \in \{0\} \times W$, a contradiction. Whence the claim, and the lemma follows. □

2. The classical proof revisited

Since we are treating local questions (differentiability) we can – without any restriction – assume that $f : [0,\varepsilon[\times V \to \mathbb{R}^n$ is a given continuous local semiflow, where V is an open ball in $\mathbb{R}^n$. We shall always assume in this section that f is continuous and $f(t,.)$ is C^k for all $t \in [0,\varepsilon[$, for some $k \ge 1$. Furthermore for any $x \in V$ there is $\varepsilon_x > 0$ such that $D_x f(t,x)$ is invertible for $0 \le t \le \varepsilon_x$ ($D_x f$ is the derivative with respect to x). We write $f(t,x) = (f_1(t,x),...,f_n(t,x))$ for $(t,x) \in [0,\varepsilon[\times V$.

Lemma 2.1. *The mapping $(t,x) \mapsto D_x f(t,x)$ is continuous.*

Proof. For the proof we proceed from the Baire category theorem and Lemma 2 of [8] on p. 198. We then obtain the following result:

Let Z be any compact space, V the open set in $\mathbb{R}^n$ and let $F : Z \times V \to \mathbb{R}$ be a continuous real valued function, such that $F(g,.)$ is C^1 for any $g \in Z$. Given $a \in V$ and $1 \le i \le n$, the set of points g_0 such that $\frac{\partial}{\partial x_i} F$ is continuous at (g_0,a) is dense in Z, even more, the set where it is false is of first category.

Let now $a \in V$ be fixed, then the set of points $t_0 \in [0,\varepsilon[$ such that $f_{ij} := \frac{\partial}{\partial x_i} f_j$ is continuous at (t_0,a), for all $1 \le i,j \le n$, is everywhere dense in $[0,\varepsilon[$. We shall denote this set by I_a. In addition the determinant $\det(f_{ij})$ is continuous at these points, too. We want to show now that for fixed $a \in V$ the mappings f_{ij} are continuous at $(0,a)$. Notice that the determinant at any point of continuity (t_0,a), with $t_0 \in I_a$ small enough, is bounded away from zero in a neighborhood.

We fix $a \in V$, then for $t_0 \in [0, \varepsilon[$

$$f(t_0 + h, a + y) = f(t_0, f_1(h, a + y), ..., f_n(h, a + y))$$

for $h \geq 0$ and $y \in \mathbb{R}^n$, both sufficiently small, hence

$$D_x f(t_0 + h, a + y) = D_x f(t_0, f(h, a + y)) \cdot D_x f(h, a + y).$$

There is $t_0 \in I_a$ such that $D_x f(t_0, z)$ is invertible in a neighborhood of a, hence

$$D_x f(t_0, f(h, a + y))^{-1} \cdot D_x f(t_0 + h, a + y) = D_x f(h, a + y)$$

and therefore

$$id = \lim_{h \downarrow 0, y \to 0} D_x f(t_0, f(h, a + y))^{-1} \cdot D_x f(t_0 + h, a + y) = \lim_{h \downarrow 0, y \to 0} D_x f(h, a + y)$$

by continuity of $D_x f$ at (t_0, a), continuity of f in both variables and the continuity of the inversion of matrices. So $0 \in I_a$ for all $a \in V$.

Now we can conclude for arbitrary $t \in]0, \varepsilon[$ in the following way:

$$D_x f(t + h, a + y) = D_x f(t, f(h, a + y)) \cdot D_x f(h, a + y)$$

for $h \geq 0$ and $y \in \mathbb{R}^n$ sufficiently small, hence by continuity at $(0, a)$

$$\lim_{h \downarrow 0, y \to 0} D_x f(t + h, a + y) = \lim_{h \downarrow 0, y \to 0} D_x f(t, f(h, a + y)) \cdot D_x f(h, a + y) = D_x f(t, a).$$

For left continuity we apply

$$D_x f(t, a + y) = D_x f(h, f(t - h, a + y)) \cdot D_x f(t - h, a + y)$$

for $h \geq 0$ and $y \in \mathbb{R}^n$ sufficiently small, hence by continuity of $D_x f$ at $(0, a)$ and $(0, f(t, a))$, the continuity of $D_x f$ in the second variable and the existence of the inverse for small h

$$\lim_{h \downarrow 0, y \to 0} D_x f(t - h, a + y) = \lim_{h \downarrow 0, y \to 0} D_x f(h, f(t - h, a + y))^{-1} \cdot D_x f(t, a + y) = D_x f(t, a).$$

Consequently the desired assertion holds. □

In the next step we shall show that there is a derivative at 0.

Lemma 2.2. *The right-hand derivative* $\frac{d}{dt} f(t, x)|_{t=0}$ *exists for* $x \in V$*, and for small* $h \geq 0$ *we have the formula*

$$f(h, x) - x = \int_0^h D_x f(t, x) dt \cdot (\frac{d}{dt} f(0, x)).$$

Moreover, $\frac{d}{dt} f(t, .)|_{t=0} : V \to \mathbb{R}^n$ *is continuous.*

Proof. We may differentiate with respect to x under the integral sign by Lemma 2.1 und uniform convergence, so

$$T(h, x) := \int_0^h f(t, x) dt$$

$$D_x T(h, x) := \int_0^h D_x f(t, x) dt.$$

By the mean value theorem we obtain

$$T(h, y) - T(h, x) = D_x T(h, \widetilde{x})(y - x),$$

where $\widetilde{x} \in [x, y]$. Now we take $y = f(p, x)$, then

$$\begin{aligned} T(h,y) - T(h,x) &= \int_p^{h+p} f(t,x)dt - \int_0^h f(t,x)dt \\ &= \int_h^{h+p} f(t,x)dt - \int_0^p f(t,x)dt, \end{aligned}$$

which finally yields

$$\frac{1}{p}(\int_0^p f(t+h,x)dt - \int_0^p f(t,x)dt) = D_x T(h, \widetilde{x})[\frac{1}{p}(f(p,x) - x)].$$

This equation can be solved by joint continuity of $(h, z) \mapsto \frac{1}{h}\int_0^h D_x f(t, z)dt$: we obtain for small h and a compact set in x that the expression is in a small neighborhood of the identity matrix. So inversion leads to the desired result and then to the given formula.

The formula asserts again by inversion, that the derivative is continuous with respect to x. □

By the semigroup-property and the chain rule, the result of Lemma 2.2 can be extended for $0 < t < \varepsilon$, and the derivative of $f(\cdot, x)$ exists for all $t \in [0, \varepsilon[$. Indeed,

$$\begin{aligned} \lim_{p\downarrow 0} \frac{f(t+p,x) - f(t,x)}{p} &= \lim_{p\downarrow 0} \frac{f(p, f(t,x)) - f(t,x)}{p} = \frac{d}{dt} f(0, f(t,x)) \\ \lim_{p\downarrow 0} \frac{f(t,x) - f(t-p,x)}{p} &= \lim_{p\downarrow 0} \frac{f(p, f(t-p,x)) - f(t-p,x)}{p} \\ &= \lim_{p\downarrow 0} \frac{1}{p}(\int_0^p D_x f(t, f(t-p,x))dt) \frac{d}{dt} f(0, f(t-p,x)) \\ &= \frac{d}{dt} f(0, f(t,x)) \end{aligned}$$

by Lemmas 2.1 and 2.2. Consequently for small $h \geq 0$

$$\frac{d}{dt} f(t,x) = \frac{d}{dt} f(0, f(t,x)) = (\int_0^h D_x f(s, f(t,x))ds)^{-1} \cdot (f(h,x) - x). \tag{2.1}$$

In particular $(t, x) \mapsto \frac{d}{dt} f(t, x)$ is continuous in both variables on the whole domain of definition.

Lemma 2.3. *The semiflow f is C^k in both variables.*

Proof. If $f(t, .)$ is C^k for $t \in [0, \varepsilon[$, then the r-jet

$$(f(t,x_0), D_x f(t,x_0) \cdot x_1, ..., D_x^r f(t,x_0) \cdot x_1 \cdot ... \cdot x_r)$$

for $(t, x_0, x_1, ..., x_r) \in [0, \varepsilon[\times V \times \mathbb{R}^n \times ... \times \mathbb{R}^n$ is a local semiflow of C^{k-r}-maps, for $0 \leq r \leq k-1$. For $r = 1$, the 1-jet is a continuous local semiflow of C^{k-1}-maps, by Lemma 2.1. Assume that for $r < k$ the r-jet is a continuous local semiflow then, by Lemma 2.1 again, the $(r+1)$-jet is continuous. By induction

$$(t, x) \mapsto D_x^r f(t, x)$$

is continuous in both variables for $0 \leq r \leq k$.

If we apply the above results to the r-jet for $r < k$, we conclude by equation (2.1) that $D_x^r f(t, x)$ can be $(k-r)$ times differentiated with respect to the t-variable, and these derivatives are continuous. Hence f is C^k in both variables. □

Theorem 2.4. *Let $k \geq 1$ be given and let Fl be a local semiflow on a finite-dimensional C^k-manifold M, which satisfies the following conditions:*

i) *The semiflow $Fl : [0, \varepsilon[\times U \to M$ is continuous with $U \subset M$ open.*
ii) *The mapping $Fl(t, .)$ is C^k.*
iii) *For fixed $x \in U$ there exists $\varepsilon_x > 0$ such that $T_x Fl(t, .)$ is invertible for $0 \leq t \leq \varepsilon_x$.*

Then Fl is C^k and for any $x \in U$ there is a local C^k-flow $\tilde{Fl} :]-\delta, \delta[\times V \to M$ with $V \subset U$ open around x and $\delta \leq \varepsilon$ such that $Fl(y, t) = \tilde{Fl}(y, t)$ for $y \in V$ and $0 \leq t \leq \delta$.

Proof. By the previous lemmas the map $Fl : [0, \varepsilon[\times U \to M$ is a C^k-semiflow. We fix $x \in U$, then there is $0 < \delta < \varepsilon$ and $x \in W \subset U$ open such that $(t, x) \mapsto (t, Fl(t, x))$ is C^k-invertible for $(t, x) \in [0, \delta[\times W$ (by the inverse function theorem on manifolds with boundary, see [6]) and such that $\cap_{t \in [0,\delta[} Fl(t, W)$ contains an open neighborhood V of x. This can be done since the size of the domain, where the inverse in x is defined, can be estimated by the operator norm of $D_x Fl(t, .)$, which depends continuously on t. Therefore we can define $\tilde{Fl}(-t, y) := \{Fl(t, .)\}^{-1}(y)$ for $t \in [0, \delta[$ and $y \in V$. Since this is the unique solution in z of the C^k-equation $Fl(t, z) = y$, we obtain a C^k-map $\tilde{Fl}$. The flow property holds by uniqueness. □

Remark 2.5. *Remark that for evolutions (which correspond in the differentiable case to time-dependent vector fields) we can pass to the extended phase space and apply the results thereon.*

3. Applications

The following application has been announced in [2] in connection with finite-dimensional realizations for stochastic models of the interest rates. Let X be a Banach space, S a strongly continuous semigroup on X with infinitesimal generator $A : D(A) \to X$, and let $P : X \to X$ be a locally Lipschitz map. For $x \in D(A)$ we write

$$\mu(x) := Ax + P(x).$$

Proposition A.2 yields the existence of a continuous, local semiflow $Fl^\mu : \mathbb{R}_{\geq 0} \times X \to X$ of mild solutions to the evolution equation

$$\frac{d}{dt} x(t) = \mu(x(t)). \tag{3.1}$$

That is, for every $x_0 \in X$ there exists a neighborhood U of x_0 in X and $T > 0$ such that $Fl^\mu \in C([0, T] \times U, X)$ and

$$Fl^\mu(t, x) = S_t x + \int_0^t S_{t-s} P(Fl^\mu(s, x)) ds, \quad \forall (t, x) \in [0, T] \times U. \tag{3.2}$$

Now let $k \geq 1$, and M be a finite-dimensional C^k-submanifold in X that is locally invariant for Fl^μ. Hence, by Lemma 1.3, $x_0 \in M$ implies $Fl^\mu(t, x) \in M$ for all $(t, x) \in [0, T] \times (U \cap M)$, for some open neighborhood U of x_0 in X and $T > 0$. It is shown in [1] that necessarily $M \subset D(A)$ and

$$\mu(x) \in T_x M, \quad \forall x \in M. \tag{3.3}$$

We now can strengthen this result.

Theorem 3.1. *Suppose*

$$P \in \bigcap_{r=0}^{k} C^{k-r}(X, D(A^r)). \tag{3.4}$$

Then $M \subset D(A^k)$ and $\mu|_M$ is a C^{k-1}-vector field on M.

Proof. Let $x_0 \in M$, and U, T as above. Hence $Fl^\mu(t,x) \in M$ for all $(t,x) \in [0,T] \times (U \cap M)$. From (3.4) we have $P \in C^k(X)$, hence Theorem A.3 applies and we may assume that $Fl^\mu(t,\cdot)$ is C^k on U for all $t \in [0,T]$. Now let $x \in U \cap M$. We claim that there exists $\varepsilon_x > 0$ such that

$$D_x Fl^\mu(t,x) : T_x M \to T_{Fl^\mu(t,x)} M \text{ is invertible for } 0 \le t \le \varepsilon_x. \tag{3.5}$$

Indeed, (3.2) implies that

$$D_x Fl^\mu(t,x) = S_t + O(t), \tag{3.6}$$

where $O(t) \to 0$ in $L(X)$ (the space of bounded linear operators on X), for $t \to 0$. Since $T_x M$ is finite-dimensional, it is easy to see that

$$\sup_{y \in T_x M,\, \|y\| \le 1} \|S_t y - y\| \to 0 \quad \text{for } t \to 0.$$

Hence there exists $\delta_x > 0$ such that S_t restricted to $T_x M$ is injective, and hence invertible, for all $t \in [0, \delta_x]$. This together with (3.6) yields the claim (3.5).

Therefore Theorem 2.4 applies and $Fl^\mu : [0,T] \times (U \cap M) \to M$ is C^k. In particular, since $\mu(x) = \partial_t Fl^\mu(0,x)$, $\mu|_M$ is a C^{k-1}-vector field on M, and $Fl^\mu(\cdot, x_0)$ is C^k on $[0,T]$.

From (3.2) we have

$$S_t x_0 = Fl^\mu(t,x_0) - \int_0^t S_{t-s} P(Fl^\mu(s,x_0))\, ds, \quad t \in [0,T].$$

By (3.4), the integral on the right is C^k in $t \in [0,T]$. Indeed, we obtain inductively by dominated convergence

$$\partial_t^r \int_0^t S_{t-s} P(Fl^\mu(s,x_0))\, ds = \partial_t^{r-1} P(Fl^\mu(t,x_0)) + \partial_t^{r-2} AP(Fl^\mu(t,x_0)) \\ + \cdots + A^{r-1} P(Fl^\mu(t,x_0)) + \int_0^t S_{t-s} A^r P(Fl^\mu(s,x_0))\, ds,$$

for $r \le k$. We conclude that $S_t x_0$ is C^k in $t \in [0,T]$. But this means that $x_0 \in D(A^k)$ and the theorem is proved. □

We now consider a setup that is given in [2]. Let W be a connected open set in X, $d \ge 1$ and $\sigma = (\sigma_1, \dots, \sigma_d)$ such that

- **(A1):** P and σ_i are Banach maps from X into $D(A^\infty)$, for $1 \le i \le d$.
- **(A2):** $\mu, \sigma_1, \dots, \sigma_d$ are pointwise linearly independent on $W \cap D(A^\infty)$.

For the definition of a Banach map see [2, 4]. The Banach map principle ([4, Theorem 5.6.3]) yields that each σ_i generates a local flow Fl^{σ_i} on X with the following property: for every $x_0 \in X$ there exists an open neighborhood V of x_0 in X and $T > 0$ such that

$$Fl^{\sigma_i} \in C^\infty(]-T,T[\times V, X) \quad \text{and} \quad Fl^{\sigma_i} \in C^\infty(]-T,T[\times V', D(A^\infty)),$$

where $V' := V \cap D(A^\infty)$ is considered as an open set in $D(A^\infty)$, and $Fl^{\sigma_i}(\cdot, x)$ is the unique solution of

$$\frac{d}{dt}x(t) = \sigma_i(x(t)), \quad x(0) = x, \qquad (t,x) \in]-T,T[\times V.$$

Local invariance for Fl^{σ_i} is defined as for Fl^μ above.

Theorem 3.2. *Let $M \subset W$ be a $(d+1)$-dimensional C^∞-submanifold of X. If M is locally invariant for Fl^μ, $Fl^{\sigma_1}, \dots, Fl^{\sigma_d}$ then M is a C^∞-submanifold of $D(A^\infty)$.*

Proof. Theorem 3.1 implies that $M \subset D(A^\infty)$ and $\mu|_M$ is a C^∞-vector field on M. It is shown in [2, Section 2] that, by **(A1)**, $Fl^\mu : \mathbb{R}_{\geq 0} \times D(A^\infty) \to D(A^\infty)$ is locally C^∞. Let $x_0 \in M$. As in the proof of [2, Theorem 3.9] it follows, by **(A2)**, that

$$\alpha(u, x_0) := Fl^{\sigma_1}_{u_1} \circ \cdots \circ Fl^{\sigma_d}_{u_d} \circ Fl^\mu_{u_{d+1}}(x_0) : U \to D(A^\infty),$$

where U is an open neighborhood of 0 in $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$, is a C^∞-parametrization of a $(d+1)$-dimensional submanifold with boundary N of $D(A^\infty)$, and $x_0 \in N \subset M$.

The parametrization α is also smooth with respect to the topology of X since $D(A^\infty) \hookrightarrow X$ is a smooth map, and it is an immersion since the tangent maps have full rank at x_0. We then take a prolongation of Fl^μ to the negative time axis, which exists uniquely on M by smoothness of $\mu|_M$. Since the rank is full, we obtain a smooth parametrization (with respect to X) of the $(d+1)$-dimensional submanifold M of X by the extension of α to the negative time axis, denoted by $\beta : U' \to X$, where $U' \subset \mathbb{R}^{d+1}$ is open around 0.

The map β is in particular a local homeomorphism into the smooth submanifold $M \subset X$. But β takes values in $D(A^\infty)$ and is a smooth map with respect to this locally convex structure, too, since the prolongation of Fl^μ is also a smooth curve in $D(A^\infty)$. Therefore β is locally one-to-one to an open subset of M and a smooth immersion by construction, so it defines the unique structure of a smooth submanifold of $D(A^\infty)$ for M.

Hence M is a C^∞-submanifold of $D(A^\infty)$. □

Appendix A. Regular Dependence on the Initial Point

Let X be a Banach space, S a strongly continuous semigroup on X with infinitesimal generator A, and $P : \mathbb{R}_{\geq 0} \times X \to X$ a continuous map. In this section we shall provide the basic existence, uniqueness and regularity results for the evolution equation

$$\frac{d}{dt}x(t) = Ax(t) + P(t, x(t)). \tag{A.1}$$

We first recall a classical existence and uniqueness result (see [7, Theorem 1.2, Chapter 6]).

Theorem A.1. *Let $T > 0$. Suppose $P : [0,T] \times X \to X$ is uniformly Lipschitz continuous (with constant C) on X. Then for every $x \in X$ there exists a unique mild solution $x(t)$, $t \in [0,T]$, to (A.1) with $x(0) = x$. If $x(t)$ and $y(t)$ are two mild solutions of (A.1) with $x(0) = x$ and $y(0) = y$ then*

$$\sup_{t\in[0,T]} \|x(t) - y(t)\| \leq Me^{MCT}\|x - y\|, \tag{A.2}$$

where

$$M := \sup_{t\in[0,T]} \|S_t\|. \tag{A.3}$$

There is an immediate local version of Theorem A.1. We say that $P : \mathbb{R}_{\geq 0} \times X \to X$ is *locally Lipschitz continuous on* X if for every $T \geq 0$ and $K \geq 0$ there exists $C = C(T, K)$ such that

$$\|P(t,x) - P(t,y)\| \leq C\|x - y\|$$

for all $t \in [0,T]$, and $x, y \in X$ with $\|x\| \leq k$ and $\|y\| \leq k$.

Proposition A.2. *Suppose $P : \mathbb{R}_{\geq 0} \times X \to X$ is locally Lipschitz continuous on X. Let $x_0 \in X$. Then there exist a neighborhood U of x_0 and $T > 0$ such that, for every $x \in U$, equation (A.1) has a unique mild solution $x(t)$, $t \in [0,T]$, with $x(0) = x$. If $x(t)$ and $y(t)$ are two mild solutions of* (A.1) *with $x(0) = x \in U$ and $y(0) = y \in U$ then* (A.2) *holds, for M as in* (A.3) *and some $C = C(T, U)$.*

Proof. Set $K := 2\|x_0\|$ and fix $T' > 0$. Define

$$\tilde{P}(t,x) := \begin{cases} P(t,x), & \text{if } \|x\| \leq K, \\ P(t, Kx/\|x\|), & \text{if } \|x\| > K. \end{cases}$$

Then $\tilde{P} : [0,T'] \times X \to X$ is uniformly Lipschitz continuous on X with constant $C = C(T', K)$. Hence Theorem A.1 yields existence and uniqueness of mild solutions for equation (A.1) where P is replace by $\tilde{P}$. By (A.2) there exists $0 < T \leq T'$ and a neighborhood U of x_0 such that $\sup_{t \in [0,T]} \|x(t)\| \leq K$ for every mild solution $x(t)$ with $x(0) \in U$. It is now easy to see that T and U satisfy the assertions of the proposition. □

Here is the announced regularity result.

Theorem A.3. *Let $k \geq 1$. Suppose $P : \mathbb{R}_{\geq 0} \times X \to X$ is C^k in x and $D_x^r P$ is coninuous on $\mathbb{R}_{\geq 0} \times X$, for all $r \leq k$. Let $x_0 \in X$. Then there exists an open neighborhood U of x_0 and $T > 0$, and a map $F \in C([0,T] \times U, H)$ such that, for every $x \in U$, $F(\cdot, x)$ is the unique mild solution of (A.1) with $F(0,x) = x$. Moreover $F(t, \cdot) \in C^k(U, X)$ for all $t \in [0,T]$.*

Proof. By assumption, $D_x^r P$ is locally Lipschitz continuous on X, for all $r \leq k$, and Proposition A.2 yields the existence of U, T and $F \in C([0,T] \times U, H)$ such that $F(\cdot, x) \in C([0,T], H)$ is the unique mild solution of (A.1) with $F(0,x) = x$, for all $x \in U$. It remains to show regularity of $F(t, \cdot)$.

Let $x \in U$ and $y \in X$. The candidate, say $\psi(t,x,y)$, for the Gateaux directional derivative $D_x F(t,x)y$ is given by the linear evolution equation

$$\begin{aligned} \frac{d}{dt}\psi(t,x,y) &= A\psi(t,x,y) + D_x P(t, F(t,x))\psi(t,x,y) \\ \psi(0,x,y) &= y. \end{aligned} \tag{A.4}$$

Since $C_1 = C_1(x) := \sup_{t \in [0,T]} \|D_x P(t, F(t,x))\| < \infty$, Theorem A.1 yields the existence of a unique mild solution $\psi(\cdot, x, y) \in C([0,T], X)$ to (A.4), and by (A.2)

$$\sup_{t \in [0,T]} \|\psi(t,x,y)\| \leq M e^{M C_1 T} \|y\|. \tag{A.5}$$

Now let $t \in [0,T]$ and (x_n) be a sequence in U converging to x. We claim that

$$\sup_{y \in X,\, \|y\| \leq 1} \|\psi(t, x_n, y) - \psi(t,x,y)\| \to 0, \quad \text{for } n \to \infty. \tag{A.6}$$

Indeed, $\Delta_n(t) := \psi(t, x_n, y) - \psi(t, x, y)$ satisfies

$$\Delta_n(t) = \int_0^t S_{t-s} \left(D_x P(s, F(s, x_n))\psi(s, x_n, y) - D_x P(s, F(s, x))\psi(s, x, y) \right) ds.$$

Hence

$$\|\Delta_n(t)\| \le MC_2 \int_0^t \|\Delta_n(s)\| \, ds + M^2 C_3 e^{M(C_0+C_1)T} \|y\| \|x_n - x\|,$$

where C_0 and C_3 are local Lipschitz constants of P and $D_x P$, respectively, and $C_2 := \sup_n \sup_{s\in[0,T]} \|D_x P(s, F(s, x_n))\|$. By Gronwall's inequality

$$\|\Delta_n(t)\| \le M^2 C_3 e^{M(C_0+C_1+C_2)T} \|y\| \|x_n - x\|,$$

whence (A.6).

Next, we claim that

$$D_x F(t, x)y = \psi(t, x, y). \tag{A.7}$$

Let $\varepsilon_0 > 0$ be such that $x + \varepsilon y \in U$ for all $\varepsilon \in [0, \varepsilon_0]$. For such ε we write $\delta(t, \varepsilon) := F(t, x + \varepsilon y) - F(t, x) - \varepsilon\psi(t, x, y)$, and obtain

$$\begin{aligned} \delta(t, \varepsilon) &= \int_0^t S_{t-s}(P(s, F(s, x + \varepsilon y)) - P(s, F(s, x))) \, ds \\ &\quad - \varepsilon \int_0^t S_{t-s} D_x P(s, F(s, x))\psi(s, x, y) \, ds \\ &= \int_0^t S_{t-s} \left(D_x P(s, F(s, x))\delta(s, \varepsilon) + \Delta(s, \varepsilon) \right) ds, \end{aligned}$$

where

$$\Delta(s, \varepsilon) := P(s, F(s, x+\varepsilon y)) - P(s, F(s, x)) - D_x P(s, F(s, x))(F(s, x+\varepsilon y) - F(s, x)).$$

By regularity of P and in view of (A.2) there exists $C_4 = C_4(T, U)$ such that

$$\sup_{t\in[0,T]} \|\Delta(t, \varepsilon)\| \le C_4 \varepsilon.$$

Hence, writing $C_5 := \sup_{t\in[0,T]} \|D_x P(t, F(t, x))\|$,

$$\|\delta(t, \varepsilon)\| \le C_5 MT \int_0^t \|\delta(s, \varepsilon)\| \, ds + C_4 MT\varepsilon,$$

and by Gronwall's inequality $\lim_{\varepsilon\to0} \|\delta(t, \varepsilon)\| = 0$, whence (A.7).

By (A.7) it follows that $D_x F(t, x)y$ is well defined for all $x \in U$ and $y \in X$, and by (A.6) the mapping $D_x F(t, \cdot) : U \mapsto L(X)$ is continuous, hence $F(t, \cdot) \in C^1(U, X)$ for all $t \in [0, T]$.

Higher order regularity is shown by induction of the above argument. We only sketch the case C^2. Let $x \in U$ and $y_1, y_2 \in X$, and write $\psi_2(x, y_1, y_2)$ for the candidate of $D_x^2 F(t, x)(y_1, y_2)$, which solves the inhomogeneous linear evolution equation

$$\begin{aligned} \frac{d}{dt}\psi_2(t, x, y_1, y_2) &= A\psi_2(t, x, y_1, y_2) + D_x P(t, F(t, x))\psi_2(t, x, y_1, y_2) \\ &\quad + D_x^2 P(t, F(t, x))(D_x F(t, x)y_1, D_x F(t, x)y_2) \\ \psi_2(0, x, y_1, y_2) &= 0. \end{aligned} \tag{A.8}$$

Notice that the inhomogeneous part, $D_x^2 P(t, F(t, x))(D_x F(t, x)y_1, D_x F(t, x)y_2)$, is continuous in $t \in [0, T]$ by induction. Hence $\psi_2(\cdot, x, y_1, y_2) \in C([0, T], X)$ is the unique mild solution of (A.8) by Theorem A.1. Now let $t \in [0,T]$. One shows first that $\psi_2(t,\cdot,y_1,y_2) : U \to X$ is continuous, uniformly in $y_1, y_2 \in X$ with $\|y_1\| \le 1$, $\|y_2\| \le 1$ (see (A.6)). Then the identity $D_x^2 F(t,x)(y_1,y_2) = \psi_2(t,x,y_1,y_2)$ is proved (see (A.7)), whence $F(t,\cdot) \in C^2(U,X)$. □

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