Optimal Numeraires for Risk Measures*

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Abstract

Can the usage of a risky numeraire with a greater than risk free expected return reduce the capital requirements in a solvency test? I will show that this is not the case. In fact, under a reasonable technical condition, there exists no optimal numeraire which yields smaller capital requirements than any other numeraire.

1 Statement and Proof of the Result

Can the usage of a risky numeraire with a greater than risk free expected return reduce the capital requirements in a solvency test? I will show that this is not the case. In fact, under a reasonable technical condition, there exists no optimal numeraire which yields smaller capital requirements than any other numeraire.

We consider a one period setup. Terminal nominal values are modelled as essentially bounded random variables $X \in L^\infty$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Random variables that coincide almost surely are identified. The riskiness of a portfolio is quantified by a convex risk measure ρ on L^∞ satisfying the following "coherence" axioms (introduced by Artzner et al. [1] and further extended to the convex case by Föllmer and Schied [5, 6]):

$$\text{convexity:} \quad \rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \quad \text{for } \lambda \in [0,1], \tag{1}$$

$$\text{monotonicity:} \quad \rho(X) \geq \rho(Y) \quad \text{if } X \leq Y, \tag{2}$$

$$\text{cash-invariance:} \quad \rho(X+m) = \rho(X) - m \quad \text{for } m \in \mathbb{R}, \tag{3}$$

$$\text{normality:} \quad \rho(0) = 0. \tag{4}$$

It is legitimate practice to discount terminal values by a numeraire — one euro tomorrow is less than one euro today. We denote by $r \geq 0$ the prevailing risk free rate. The regulatory required capital (the "solvency capital

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requirement") an insurance company must have available at the beginning of the accounting period is

$$\rho(X/\mathrm{e}^r - x) = x + \rho(X/\mathrm{e}^r), \tag{5}$$

where $x \in \mathbb{R}$ and $X \in L^\infty$ denote initial and terminal nominal value of the company's portfolio, respectively. That is, $\rho(X/\mathrm{e}^r)$ equals the amount of risk free bonds the company needs in addition (can withdraw, if negative) at inception to become (remain) acceptable.

Can we replace the risk free bond by a risky numeraire and achieve a reduction of capital requirements? Indeed, let $U > 0$ denote the terminal nominal value of a traded financial instrument. Since used as a numeraire, we can normalize it and assume that its initial value is one. The required capital becomes $x + \rho(X/U)$. Obviously, one would chose a numeraire with a greater than risk free expected return, i.e. $\mathbb{E}[U] > \mathrm{e}^r$. However, it turns out that there is no optimal numeraire, as the following theorem indicates:

Theorem 1.1. *Assume that ρ is sensitive, that is,*

$$\rho(-1_A) > 0 \quad \textit{for all } A \in \mathcal{F} \textit{ with } \mathbb{P}[A] > 0. \tag{6}$$

Let $U, V > 0$ be two random variables and denote

$$\mathcal{M} := \{Z \mid Z/U \in L^\infty \textit{ and } Z/V \in L^\infty\}.$$

Then $\rho(Z/U) \leq \rho(Z/V)$ for all $Z \in \mathcal{M}$ if and only if $U = V$.

Proof. Sufficiency of the statement is clear.

To prove necessity, we first recall the well known representation result for convex risk measures on L^∞ (see e.g. [6] or [3]). Let $(L^\infty)^*$ denote the dual space of L^∞, that is, the space of bounded finitely additive measures ν which are absolutely continuous with respect to $\mathbb{P}$. We define the convex set

$$\mathcal{C} := \{\nu \in (L^\infty)^* \mid \langle \nu, 1 \rangle = -1 \text{ and } \langle \nu, Y \rangle \leq 0 \text{ for all } Y \geq 0\}. \tag{7}$$

Then, for all $Y \in L^\infty$,

$$\rho(Y) = \max_{\nu \in \mathcal{C}} \left(\langle \nu, Y \rangle - \rho^*(\nu) \right) \tag{8}$$

where ρ^* denotes the convex conjugate of ρ, which, in view of (4), is positive:

$$\rho^*(\nu) = \sup_{Z \in L^\infty} \langle \nu, Z \rangle - \rho(Z) \geq \langle \nu, 0 \rangle - \rho(0) = 0. \tag{9}$$

Now let $n \in \mathbb{N}$ and denote $A_n := \{\frac{1}{n} \leq U\frac{n+1}{n} \leq V \leq n\}$. We argue by contradiction and suppose $\mathbb{P}[A_n] > 0$. Clearly, $Z := -V 1_{A_n} \in \mathcal{M}$. The above results (8), (9) and (6) therefore imply

$$0 < \rho(Z/V) = \langle -\mu, 1_{A_n} \rangle - \rho^*(\mu) \leq \langle -\mu, 1_{A_n} \rangle \tag{10}$$

for some $\mu \in \mathcal{C}$. Since, moreover, $1 + \frac{1}{n} \leq V/U$ on A_n we infer that $\langle -\mu, 1_{A_n} \rangle < \langle -\mu, 1_{A_n} V/U \rangle$ and therefore

$$\rho(Z/V) < \langle -\mu, 1_{A_n} V/U \rangle - \rho^*(\mu) = \langle \mu, Z/U \rangle - \rho^*(\mu) \leq \rho(Z/U).$$

But this contradicts the assumption of the theorem, whence $\mathbb{P}[A_n] = 0$. By letting $n \to \infty$, we conclude $U \geq V$.

This also implies $V \in \mathcal{M}$ and hence $\rho(V/U) \leq \rho(V/V) = -1$. Define $B := \{U > V\}$. If $\mathbb{P}[B] > 0$ then, by (6),

$$0 < \rho(-1 + V/U) = 1 + \rho(V/U) \leq 1 - 1 = 0,$$

a contradiction. Hence $\mathbb{P}[B] = 0$ and thus $U = V$. □

Remark 1.2. Condition (6) is satisfied by many known convex risk measures, such as expected shortfall (see e.g. [6]). Expected shortfall is the underlying risk measure in the Swiss Solvency Test [7], the new regulatory framework for Swiss insurance companies. Moreover, it is internally used by some major insurance companies (see [4]).

Remark 1.3. The conclusion of the theorem becomes stronger the smaller the set $\mathcal{M}$ of "test positions" is. An inspection of the proof shows that it would suffice to consider elements $Z \in \mathcal{M}$ with $Z/V \leq \epsilon$, for some $\epsilon > 0$.

Remark 1.4. The risk measure considered the theorem, $\rho_U(Z) := \rho(Z/U)$, satisfies convexity (1), monotonicity (2) and normality (4). However, cash-invariance (3) has to be replaced by U-invariance:

$$\rho_U(Z + mU) = \rho_U(Z) - m, \quad \text{for } m \in \mathbb{R}.$$

For a more detailed study of such risk measures see [3].

Remark 1.5. Artzner et al. [2] (henceforth ADK) also examine the effect of a change of numeraire on risk measures, albeit in a different context. Indeed, after a slight adaptation of notation, they fix a set $\mathcal{A}$ of acceptable terminal nominal portfolio values and a pair of numeraires $U, V > 0$ with initial value one. Let $\mathcal{M}_0 = \{x(U - V) \mid x \in \mathbb{R}\}$ denote the space of portfolios in U and V with zero initial value. In fact, ADK consider more than two tradeable assets, but the minimum set consists of U and V. For any terminal nominal value X, ADK define the minimum additional capital invested in U and V at inception for X to become $\mathcal{A}$-acceptable

$$\begin{aligned}\rho_{ADK}(X) &= \inf\{m \mid X + xU + yV \in \mathcal{A}, \text{ for some } x + y = m\} \\ &= \inf\{m \mid X + mU \in \mathcal{A} + \mathcal{M}_0\} = \inf\{m \mid X + mV \in \mathcal{A} + \mathcal{M}_0\}.\end{aligned}$$

Obviously, the risk measure ρ_{ADK} is both U- and V-invariant (see Remark 1.4). In this sense, the augmented acceptance set $\mathcal{A} + \mathcal{M}_0$ is "numeraire invariant" with respect to U and V.

Our approach is different as we started with a fixed convex risk measure ρ, satisfying axioms (1)–(4). Any choice of a numeraire U induced a corresponding set of acceptable nominal portfolio values $\mathcal{A}^U = \{X \mid \rho(X/U) \le 0\} = U\mathcal{A}^1$. Our objective was then to find an optimal numeraire, which in particular would maximize the acceptance set $\mathcal{A}^U$. This approach is closer to practice, where it is more common to explicitly specify a risk measure (a "simple" object) first, which then implies an acceptance set (a "complex" object), than the other way round.

Finally, let us consider a somewhat related problem: for two convex risk measures ρ and σ on L^∞, does $\sigma \le \rho$ imply $\sigma = \rho$? The answer is no. Actually, any subgradient $\sigma \in \partial\rho(0) := \{\nu \in (L^\infty)^* \mid \langle \nu, Z\rangle \le \rho(Z)\ \forall Z \in L^\infty\}$ defines a convex risk measure with this property. Indeed, it is well known (see e.g. [3]) that $\emptyset \ne \partial\rho(0) \subset \mathcal{C}$, see (7).

2 Conclusion

I have shown that, under a reasonable technical condition, there is no optimal numeraire that yields lower solvency capital requirements than any other numeraire. In particular, the greater than risk free expected return of a risky numeraire cannot compensate for the additional risk that is introduced when discounting by its terminal value.

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