Stable Diffusions Interacting through Their Ranks, as Models of Large Equity Markets

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Log-Log Capital Distribution Curves I

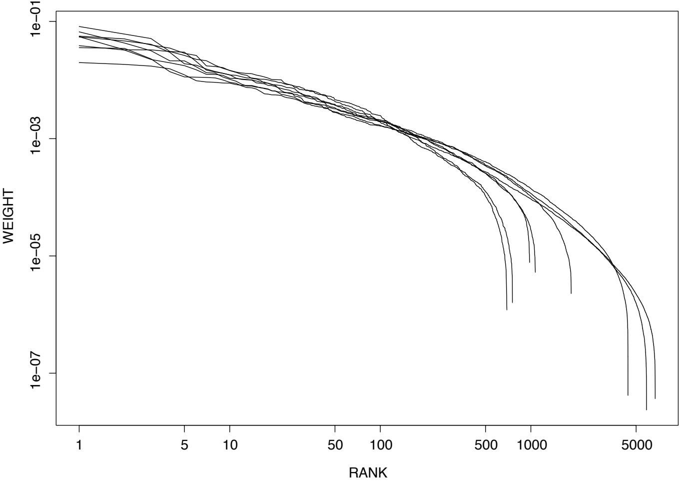


Figure: U.S. equity market, 1929-1999 (E.R. Fernholz (2002), p. 95)

Log-Log Capital Distribution Curves II

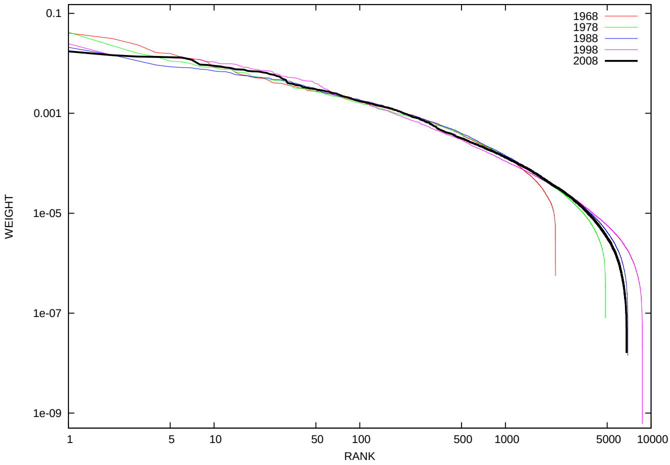


Figure: Capital distribution curves, U.S. equity market, 1968-2008

What kinds of models can describe this long-term stability?

Definition of Hybrid Atlas Model

- Capitalizations $\mathfrak{X} := \left\{ \left(X_1(t), \ldots, X_n(t) \right) , \;\; 0 \le t < \infty \right\}$.
- Descending Order Statistics (lexicographic tie-breaks):

$$\max_{1 \le i \le n} X_i(t) =: X_{(1)}(t) \ge X_{(2)}(t) \ge \cdots \ge X_{(n)}(t) := \min_{1 \le i \le n} X_i(t) \,.$$

The curves of the previous slides are (smoothed) maps

$$\log k \;\longmapsto\; \frac{1}{T} \int_0^T \log \left(\frac{X_{(k)}(t)}{X_1(t) + \cdots + X_n(t)} \right) dt \,,$$

for $k = 1, 2, \cdots , n$ over different decades $[0, T]$ (for instance, Jan 1969 – Dec 1978; of course, each decade has its own, associated market “size” n).

Log-Capitalizations

Log-capitalizations $Y_i(t) := \log X_i(t)$.

Reverse-Order Statistics: $Y_{(1)}(t) \geq \cdots \geq Y_{(n)}(t)\,, \quad 0 \leq t < \infty\,.$

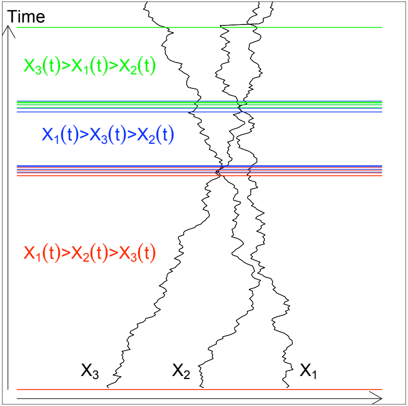
Postulated Dynamics for Log-Capitalizations:

$$\mathrm{d}Y_i(t) = (\gamma + \gamma_i + g_k)\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_i(t) \qquad \text{if} \quad Y_{(k)}(t) = Y_i(t)\,;$$

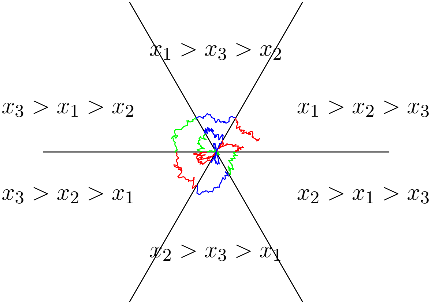
for $1 \leq i\,, k \leq n$, $0 \leq t < \infty$, where $W(\cdot)$ is $n-$dim. B.M.
System of Brownian particles interacting through their ranks.
Unique weak solution (Bass & Pardoux, PTRF '87).

	company name i	k^{th} ranked company
Drift ("mean") Diffusion ("variance")	γ_i	g_k $\sigma_k > 0$

Illustration ($n = 3$) of Interactions through Rank: Linear and Kaleidoscopic Views



Paths in $\mathbb{R}_+ \times$ time



A path in different wedges of $\mathbb{R}^n$

Permutations and Polyhedral Chambers

For $\mathbf{p} \in \Sigma_n$ (symmetric group on n elements), define wedge

$$\mathcal{R}_{\mathbf{p}} := \left\{ \xi \in \mathbb{R}^n : \xi_{\mathbf{p}(1)} \geq \xi_{\mathbf{p}(2)} \geq \cdots \geq \xi_{\mathbf{p}(n)} \right\},$$

a polyhedral chamber consisting of all points $\xi \in \mathbb{R}^n$ such that $\xi_{\mathbf{p}(k)}$ is ranked k^{th} among $\xi_1, \cdots, \xi_n$. We resolve ties "lexicographically", always in favor of the lowest index ("name") i. This results in a partition of $\mathbb{R}^n$ into pairwise-disjoint chambers.

(To wit: $\mathbf{p}(k)$ is the "name" (index) of the "particle" (coördinate) that occupies the k^{th} rank in the permutation $\mathbf{p} \in \Sigma_n$.)

Define also the "coarser" chambers

$$\begin{aligned} Q_k^{(i)} &:= \left\{ \xi \in \mathbb{R}^n : \xi_i \text{ is ranked } k^{th} \text{ among } \xi_1, \ldots, \xi_n \right\} \\ &= \bigcup_{\{\mathbf{p} \in \Sigma_n : \mathbf{p}(k) = i\}} \mathcal{R}_{\mathbf{p}} ; \quad 1 \leq i, k \leq n. \end{aligned}$$

Vector Representation as a System of Diffusions

$$\mathrm{d}Y(t) \;=\; \mathbf{C}\big(Y(t)\big)\,\mathrm{d}t + \mathbf{S}\big(Y(t)\big)\,\mathrm{d}W(t)\,; \quad 0 \le t < \infty$$

with Mean-Field-Type, but "rough", Interactions:

$$\mathbf{C}(y) = \sum_{\mathbf{p}\in\Sigma_n} \left(g_{\mathbf{p}^{-1}(1)} + \gamma_1 + \gamma\,, \ldots, \, g_{\mathbf{p}^{-1}(n)} + \gamma_n + \gamma\right)' \cdot \mathbf{1}_{\mathcal{R}_\mathbf{p}}(y)$$

$$= \sum_{k=1}^{n} \Big(\big(g_k + \gamma_1 + \gamma\big) \cdot \mathbf{1}_{Q_k^{(1)}}(y)\,, \ldots\,, \, \big(g_k + \gamma_n + \gamma\big) \cdot \mathbf{1}_{Q_k^{(n)}}(y) \Big)',$$

$$\mathbf{S}(y) = \sum_{\mathbf{p}\in\Sigma_n} \underbrace{\mathrm{diag}\left(\sigma_{\mathbf{p}^{-1}(1)}\,, \ldots, \, \sigma_{\mathbf{p}^{-1}(n)}\right)}_{\mathfrak{s}_\mathbf{p}} \cdot \mathbf{1}_{\mathcal{R}_\mathbf{p}}(y)\,; \qquad y \in \mathbb{R}^n$$

$$= \mathrm{diag}\left(\sum_{k=1}^{n} \sigma_k \cdot \mathbf{1}_{Q_k^{(1)}}(y)\,, \ldots\,, \sum_{k=1}^{n} \sigma_k \cdot \mathbf{1}_{Q_k^{(n)}}(y) \right).$$

Semimartingale Representation of Ranked Processes

Recall $Y_{(1)}(t) \geq \cdots \geq Y_{(n)}(t)$, and denote by

$$\Lambda^{k,\ell}(t) \;:=\; L^{Y_{(k)}-Y_{(\ell)}}(t)$$

the local time accumulated at the origin by the nonnegative semimartingale $Y_{(k)}(\cdot) - Y_{(\ell)}(\cdot)$ up to time t, for $1 \leq k < \ell \leq n$.

Lemma: For $k = 1, \ldots, n$, $0 \leq t \leq T$, we have

$$\mathrm{d}Y_{(k)}(t) = \Big(\gamma + g_k + \sum_{i=1}^{n} \gamma_i \, \mathbf{1}_{Q_k^{(i)}}\big(Y(t)\big)\Big)\, \mathrm{d}t + \sigma_k \, \mathrm{d}B_k(t) + \frac{1}{2}\Big[\mathrm{d}\Lambda^{k,k+1}(t) - \mathrm{d}\Lambda^{k-1,k}(t)\Big],$$

with the independent Brownian Motions (P. Lévy's theorem)

$$B_k(\cdot) \;:=\; \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{Q_k^{(i)}}\big(Y(t)\big)\, \mathrm{d}W_i(t), \qquad k = 1, \cdots, n.$$

Local Time

Reminder: The "right" Local Time at the origin, accumulated on $[0, t]$ by a continuous semim'gale $Y(\cdot) = Y(0) + M(\cdot) + V(\cdot)$, is

$$L^Y(t) := Y^+(t) - Y^+(0) - \int_0^t \mathbf{1}_{\{Y(s)>0\}} \, \mathrm{d}Y(s)$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{4\,\varepsilon} \int_0^t \mathbf{1}_{\{-\varepsilon<Y(s)<\varepsilon\}} \, \mathrm{d}\langle M \rangle(s) \, .$$

The resulting process $L^Y(\cdot)$ is increasing, continuous, flat off the set $\{t \geq 0 : Y(t) = 0\}$. If $Y(\cdot) \geq 0$, this becomes

$$L^Y(t) = \int_0^t \mathbf{1}_{\{Y(s)=0\}} \, \mathrm{d}Y(s) = \int_0^t \mathbf{1}_{\{Y(s)=0\}} \, \mathrm{d}V(s) \, .$$

- Banner & Ghomrasni (2008) provide semimartingale representations for the ranked processes in terms of these *Collision Local Times:*

$$\mathrm{d}Y_{(k)}(t) = \sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}(Y(t))\,\mathrm{d}Y_i(t)$$

$$+ \sum_{\ell=k+1}^{n} \frac{1}{\mathcal{N}_k(t)}\,\mathrm{d}\Lambda^{k,\ell}(t) - \sum_{\ell=1}^{k-1} \frac{1}{\mathcal{N}_k(t)}\,\mathrm{d}\Lambda^{k,\ell}(t)\,.$$

Please note the tug-of-war going on: an "upward pressure" coming from the lower ranks ($\ell = k+1, \cdots, n$), and a "downward pressure" from the upper ranks ($\ell = 1, \cdots, k-1$).

- Here we keep track of the "size of the crowd" in rank k via

$$\mathcal{N}_k(t) := \#\{\, i : Y_i(t) = Y_{(k)}(t)\,\}\,;$$

we also assume that all the semimatingales' bounded variation parts are absolutely continuous w.r.t. Lebesgue measure, and that for all (i,j) we have $\mathrm{Leb}(\{t \geq 0 : Y_i(t) = Y_j(t)\}) = 0\,.$

Lemma: For $k = 1, \ldots, n$, $0 \le t \le T$, we have

$$\mathrm{d}Y_{(k)}(t) = \Big(\gamma + g_k + \sum_{i=1}^{n} \gamma_i \, \mathbf{1}_{Q_k^{(i)}}\big(Y(t)\big)\Big)\,\mathrm{d}t + \sigma_k\,\mathrm{d}B_k(t)$$
$$+ \frac{1}{2}\Big[\mathrm{d}\Lambda^{k,k+1}(t) - \mathrm{d}\Lambda^{k-1,k}(t)\Big],$$

Idea of Proof of Lemma: Why only the "nearest neighbor" (or "simple collision") local times $\Lambda^{k,k+1}(\cdot)$ and $\Lambda^{k-1,k}(\cdot)$?

Does this mean that triple (and higher-order) collisions do not occur ? Far from it (example of BASS & PARDOUX (1987)).

Reason: For any $1 \le i, j, m \le n$, the “rank-gap" process

$$\max_{\nu=i,j,m} Y_\nu(\cdot) - \min_{\nu=i,j,m} Y_\nu(\cdot)$$

turns out (some hard work here...) to dominate a Bessel process in dimension $\delta > 1$, and analysis of its local time shows

$$L^{Y_{(k)} - Y_{(\ell)}}(\cdot) \equiv \Lambda^{k,\ell}(\cdot) \equiv 0\,, \qquad |\,k - \ell\,| \ge 2\,.$$

Serendipity (and relief): *even if* triple (or higher-order) collisions occur, they just *do not matter* for the respective local times.

Recent work with T. Ichiba & M. Shkolnikov on the absence of triple collisions: A *necessary* condition is the concavity of the variances $k \mapsto \sigma_k^2$, $k = 1, \cdots, n$.

A *sufficient* condition is the concavity of the graph of $k \mapsto \sigma_k^2$, $k = 0, 1 \cdots, n, n+1$, where $\sigma_0^2 = \sigma_{n+1}^2 = 0$.

These Local Times can be estimated...

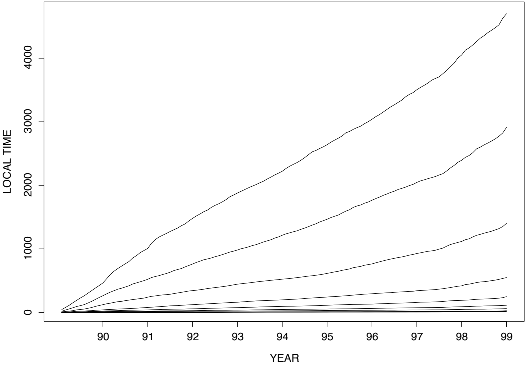


Figure: The estimated local time or “turnover" processes $\Lambda^{k,k+1}(\cdot)$ for $k = 10, 20, 40, \ldots, 5120$; U.S. CRSP data, Jan 1990 – Dec 1999. (From E.R. FERNHOLZ (2002) *Stochastic Portfolio Theory*, page 107.)

Local Times as Cumulative Turnover across Ranks

Discussion: Such estimation comes from the construction of rank-based portfolios that invest in an index-like fashion (according to relative capitalization) in, say, the top k stocks.

The performance of such a portfolio relative to the entire market involves a *leakage* term proportional to the local time $\Lambda^{k,k+1}(\cdot)$ which measures the “turnover" between ranks k and $k+1$; this can then be estimated based on observables.

Please note that this kind of turnover tends to increase, the lower one goes down the ranks (that is, with increasing k) .

- The apparent linearity of the growth of local times is yet another indication of an underlying stability or ergodic behavior.

(Recall that for, say, Brownian motion, local time grows like $\sqrt{T}$; whereas for processes with an invariant distribution and stochastic stability, local time grows like T.)

What kinds of conditions can ensure such stochastic stability?

STABILITY CONDITIONS

Very roughly speaking: *Assign big growth rates (and big variances) to the smallest stocks; then a stable capital distribution does indeed emerge.*

In particular, we shall assume, for every $k = 1, \dots, n-1$ and $\mathbf{p} \in \Sigma_n$:

$$\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0\,, \qquad \sum_{\ell=1}^{k} \left(g_\ell + \gamma_{\mathbf{p}(\ell)}\right) < 0\,.$$

Interpretation: “the cloud of particles" will stick together: no sub-collection of particles can “form its own galaxy", as it were, and drift apart without ever again making contact with the rest.

Example 1 – Atlas model:

$$g_1 = \cdots = g_{n-1} = -\mathfrak{g} < 0;$$
$$g_n = (n-1)\mathfrak{g} > 0;$$
$$\gamma_1 = \cdots = \gamma_n = 0.$$

The company with the lowest capitalization provides all the growth – or support, as with the Titan of mythical lore – for the entire structure. *(Here, companies are totally “anonymous" as far as their growth rates are concerned.)*

Example 2 – Atlas model with stock-specific drifts:

$$g_1, \ldots, g_n \quad \text{as above;} \qquad \sum_{i=1}^{n} \gamma_i = 0\,, \quad \max_{1 \le i \le n} \gamma_i < \mathfrak{g}\,.$$

For instance:

$$\gamma_i = \mathfrak{g} \left(1 - \frac{2i}{n+1} \right), \quad 1 \le i \le n\,.$$

Stochastic Stability

The average (center of gravity)

$$\overline{Y}(\cdot) := \frac{1}{n} \sum_{i=1}^{n} Y_i(\cdot)$$

of the log-capitalizations

$$\overline{Y}(t) = \overline{Y}(0) + \gamma\, t + \frac{1}{n} \sum_{k=1}^{n} \sigma_k\, B_k(t)$$

is Brownian motion with variance $\sum_{k=1}^{n} (\sigma_k / n)^2$, drift γ.

Recall here the independent Brownian Motions

$$B_k(\cdot) = \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{Q_k^{(i)}}(Y(t))\, \mathrm{d}W_i(t)\,, \qquad k = 1, \cdots, n\,.$$

The stability conditions

$$\boxed{\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0\,, \qquad \sum_{\ell=1}^{k} \left(g_\ell + \gamma_{\mathbf{p}(\ell)}\right) < 0\,.}$$

imply that the process of deviations from the center of gravity

$$\widetilde{Y}(\cdot) := \left(Y_1(\cdot) - \overline{Y}(\cdot), \ldots, Y_n(\cdot) - \overline{Y}(\cdot)\right)$$

is positive recurrent, uniformly over compact sets.

From the theory of R.Z. Khas'minskii (1960, 1980) we have then the following stochastic stability result:

Proposition: The process $\widetilde{Y}(\cdot)$ is stable in distribution; to wit, there is a unique invariant probability measure $\mu(\cdot)$ such that for every bounded, measurable $f : \Pi \to \mathbb{R}$ we have, with $\Pi := \{y \in \mathbb{R}^n : y_1 + \cdots + y_n = 0\}$, the Strong Law of Large Numbers

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\widetilde{Y}(t)\right) \mathrm{d}t = \int_\Pi f(y)\, \mu(\mathrm{d}y), \quad a.s.$$

Can we describe this invariant measure ?

Average Occupation Times

Setting $f(\cdot) = \mathbf{1}_{\mathcal{R}_\mathbf{p}}(\cdot)$ (respectively, $\mathbf{1}_{Q_k^{(i)}}(\cdot)$), we define the average occupation times of $X(\cdot)$ in the polyhedral chambers $\mathcal{R}_\mathbf{p}$ (respectively, $Q_k^{(i)}$):

$$\theta_\mathbf{p} := \mu(\mathcal{R}_\mathbf{p}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{R}_\mathbf{p}}(X(t))\,\mathrm{d}t\,, \qquad \mathbf{p} \in \Sigma_n\,,$$

$$\vartheta_{k,i} := \mu(Q_k^{(i)}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{Q_k^{(i)}}(X(t))\,\mathrm{d}t\,, \quad 1 \le k, i \le n\,.$$

Equilibrium Identity:

$$\boxed{\gamma_i + \sum_{k=1}^{n} g_k\, \vartheta_{k,i} \;=\; 0\,; \quad i = 1, \ldots, n\,.}$$

Strong Laws of Large Numbers

Stability implies an *SLLN* for Local Times, $\forall\ k = 1, \cdots, n-1$:

$$\lim_{T\to\infty} \frac{1}{T}\,\Lambda^{k,k+1}(T) \;=\; -2\sum_{\ell=1}^{k}\Big(g_\ell + \sum_{i=1}^{n} \vartheta_{\ell,i}\,\gamma_i\Big)$$

$$= \;-\,2\sum_{\mathbf{p}\in\Sigma_n} \theta_{\mathbf{p}} \sum_{\ell=1}^{k} \Big(g_\ell + \gamma_{\mathbf{p}(\ell)}\Big) \;>\; 0\,, \qquad \text{a.s.}$$

A quantity that increases with rank k, for instance under the condition (satisfied in Examples 1, 2):

$$g_k + \gamma_i < 0\,; \qquad \forall\ \ 1 \le k \le n-1\,,\ \ 1 \le i \le n\,.$$

What can be said about $\vartheta_{k,i}$ and the invariant measure μ ?

Linearly Growing Variances

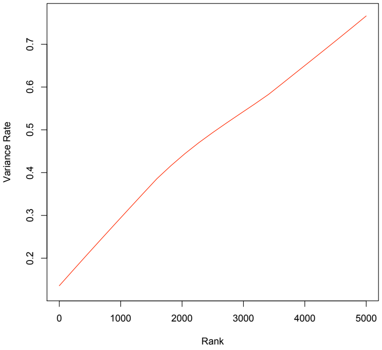


Figure: Smoothed variance by rank, U.S. Equity market, 1990-1999.

We shall assume that variances grow linearly by rank:

$$\sigma_2^2 - \sigma_1^2 = \sigma_3^2 - \sigma_2^2 = \cdots = \sigma_n^2 - \sigma_{n-1}^2 \geq 0\,.$$

Semimartingale Reflected Brownian Motions

Recall the ranked semimartingale decomposition

$$\mathrm{d}Y_{(k)}(t) = \sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}(Y(t))\,\mathrm{d}Y_i(t) + \frac{1}{2}\left[\mathrm{d}\Lambda^{k,k+1}(t) - \mathrm{d}\Lambda^{k-1,k}(t)\right].$$

The vector $\Xi(\cdot)$ of "**Gaps**" $\Xi_k(\cdot) := Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot) \geq 0$ with

$$\Xi_k(\cdot) = \Xi_k(0) + \sum_{i=1}^{n} \int_0^{\cdot} \left(\mathbf{1}_{Q_k^{(i)}} - \mathbf{1}_{Q_{k+1}^{(i)}}\right)\left(Y(t)\right) \cdot \mathrm{d}Y_i(t)$$

$$-\frac{1}{2}\left[\Lambda^{k-1,k}(\cdot) + \Lambda^{k+1,k+2}(\cdot)\right] + \Lambda^{k,k+1}(\cdot), \quad 1 \leq k \leq n-1$$

can be seen as a **semimartingale reflected Brownian motion** in the nonnegative orthant (Harrison, Reiman, Williams).

- Finally, we define the *indicator map* $\boxed{\mathbb{R}^n \ni \xi \mapsto \mathfrak{p}^{\xi} \in \Sigma_n}$

$$\xi_{\mathfrak{p}^{\xi}(1)} \geq \xi_{\mathfrak{p}^{\xi}(2)} \geq \cdots \geq \xi_{\mathfrak{p}^{\xi}(n)}, \quad \text{so that} \quad \mathfrak{p}^{\xi} = \mathbf{p} \iff \xi \in \mathcal{R}_{\mathbf{p}},$$

where $\mathfrak{p}^{\xi}(k)$ is the name (index) of the coördinate that occupies the k^{th} rank among $\xi_1, \cdots, \xi_n$.

We introduce also the **Index Process**

$$\boxed{\mathfrak{P}_t := \mathfrak{p}^{Y(t)}} \qquad 0 \leq t < \infty,$$

with values in the symmetric group Σ_n. The definition implies

$$Y_{\mathfrak{P}_t(1)} = Y_{(1)}(t) \geq \cdots \geq Y_{(n)}(t) = Y_{\mathfrak{P}_t(n)}, \qquad 0 \leq t < \infty.$$

Keeps track of “who occupies a particular rank k at any given time".

Invariant Distribution for Adjacent Gaps and Indices

Proposition: Under the stability and linearly-growing-variance conditions, the invariant distribution $\nu(\cdot)$ of $(\Xi(\cdot), \mathfrak{P}.)$ is

$$\nu(A \times B) = \Big(\sum_{\mathbf{p} \in \Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{p},k}^{-1} \Big)^{-1} \cdot \sum_{\mathbf{p} \in B} \int_A \exp\big(- \langle \lambda_{\mathbf{p}}, z \rangle \big)\, dz$$

for every measurable set $A \times B \in (\mathbb{R}_+)^{n-1} \times \Sigma_n$. Here we have set $\lambda_{\mathbf{p}} := (\lambda_{\mathbf{p},1}, \ldots, \lambda_{\mathbf{p},n-1})'$ to be the vector of components:

$$\lambda_{\mathbf{p},k} := \frac{-2 \sum_{\ell=1}^{k} \big(g_\ell + \gamma_{\mathbf{p}(\ell)} \big)}{(\sigma_k^2 + \sigma_{k+1}^2)/2} > 0\,; \quad \mathbf{p} \in \Sigma_n\,,\ 1 \le k \le n-1\,.$$

Discussion: The particular form of $\nu(\cdot\,,\cdot)$ leads to the density

$$\mathbb{P}\big(\Xi(t) \in A\big) = \Big(\sum_{\mathbf{p} \in \Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{p},k}^{-1} \Big)^{-1} \cdot \sum_{\mathbf{p} \in \Sigma_n} \int_A \exp(-\langle \lambda_{\mathbf{p}}, z \rangle)\, dz$$

of sums-of-products-of-exponentials type, for the distribution of the semimartingale reflected Brownian motion process

$$\Xi(\cdot) := (\Xi_1(\cdot), \ldots, \Xi_{n-1}(\cdot))'$$

of adjacent gaps

$$\Xi_k(\cdot) := Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot) \geq 0\,, \qquad k = 1, \ldots, n-1$$

under the invariant measure $\nu(\cdot\,,\cdot)$.

Discussion (cont'd): The assumption of *linearly growing variances* is crucial in the Proposition.

Average Occupation Times

Corollary: The long-term-average occupation times are

$$\boxed{\theta_{\mathbf{p}} = \mu(\mathcal{R}_{\mathbf{p}}) = \nu(\mathfrak{S}, \{\mathbf{p}\}) = \Bigl(\sum_{\mathbf{q}\in\Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{q},k}^{-1} \Bigr)^{-1} \cdot \prod_{k=1}^{n-1} \lambda_{\mathbf{p},k}^{-1}}$$

for each chamber $\mathcal{R}_{\mathbf{p}}$ ($\mathbf{p} \in \Sigma_n$), and

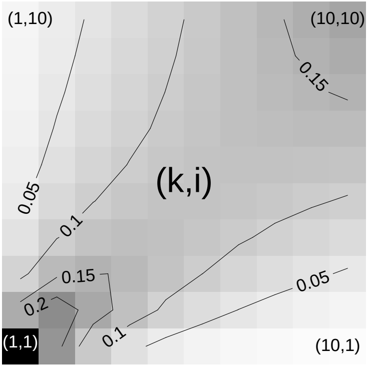
$$\underbrace{\vartheta_{k,i} = \sum_{\{\mathbf{p}\in\Sigma_n : \mathbf{p}(k)=i\}} \theta_{\mathbf{p}}} , \quad i = 1, \ldots, n .$$

These DO satisfy (sanity check) the equilibrium identities

$$\gamma_i + \sum_{k=1}^{n} g_k \, \vartheta_{k,i} = 0 ; \quad i = 1, \ldots, n .$$

- If all $\gamma_i = 0$, then $\vartheta_{k,i} = \frac{1}{n}$ for $1 \leq k, i \leq n$ (first-order model of BFK (2005), includes the simple Atlas model as a special case).

- Heat map of $\vartheta_{k,i}$ when $n = 10$, $\sigma_k^2 = 1 + k$, $g_k = -1$ for $1 \leq k \leq 9$, $g_{10} = 9$, and $\gamma_i = 1 - (2i)/(n+1)$ for $i = 1, \ldots, 10$.



Distribution of Ranked Market Weights

Corollary: The invariant distribution of ranked market weights

$$M_{(k)}(\cdot) := \frac{X_{(k)}(\cdot)}{X_1(\cdot) + \cdots + X_n(\cdot)} \;; \quad k = 1, \ldots, n$$

has probability density function $\wp(m_1, \ldots, m_{n-1})$ given by

$$\boxed{\sum_{\mathbf{p} \in \Sigma_n} \theta_{\mathbf{p}} \frac{\lambda_{\mathbf{p},1} \cdots \lambda_{\mathbf{p},n-1}}{m_1^{\lambda_{\mathbf{p},1}+1} \cdot m_2^{\lambda_{\mathbf{p},2}-\lambda_{\mathbf{p},1}+1} \cdots m_{n-1}^{\lambda_{\mathbf{p},n-1}-\lambda_{\mathbf{p},n-2}+1} m_n^{-\lambda_{\mathbf{p},n-1}+1}}} \,,$$

$$0 < m_n \leq m_{n-1} \leq \ldots \leq m_1 < 1 \,, \;\; m_n := 1 - (m_1 + \cdots + m_{n-1}) \,.$$

- This is a distribution of ratios of Pareto type.

Capital Distribution Curves

$$\wp(m_1, \ldots, m_{n-1}) =$$

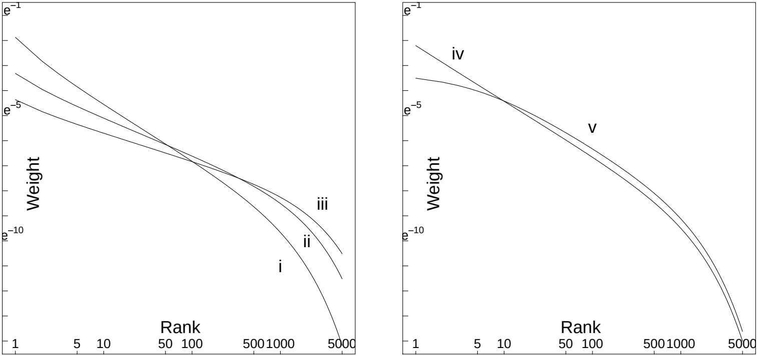
$$= \sum_{\mathbf{p}\in\Sigma_n} \theta_{\mathbf{p}} \frac{\lambda_{\mathbf{p},1} \cdots \lambda_{\mathbf{p},n-1}}{m_1^{\lambda_{\mathbf{p},1}+1} \cdot m_2^{\lambda_{\mathbf{p},2}-\lambda_{\mathbf{p},1}+1} \cdots m_{n-1}^{\lambda_{\mathbf{p},n-1}-\lambda_{\mathbf{p},n-2}+1} m_n^{-\lambda_{\mathbf{p},n-1}+1}}$$

The invariant probability density for the ranked market weights from the previous slide, allows us to describe the long term average (and "expected") **slope** of the capital distribution curve at the various ranks k, thus also its shape:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\log M_{(k+1)}(t) - \log M_{(k)}(t)}{\log(k+1) - \log k} \, dt =$$

$$\mathbb{E}^{\nu}\left(\frac{\log M_{(k+1)} - \log M_{(k)}}{\log(k+1) - \log k}\right) = \frac{-\mathbb{E}^{\nu}(\Xi_k)}{\log(1+k^{-1})} = -\frac{\sum_{\mathbf{p}\in\Sigma_n} \theta_{\mathbf{p}} \, \lambda_{\mathbf{p},k}^{-1}}{\log(1+k^{-1})} \, .$$

Illustrations



- $n = 5000$, $g_n = c_*(2n-1)$, $g_k = 0$, $1 \le k \le n-1$, $\gamma_1 = -c_*$, $\gamma_i = -2c_*$, $2 \le i \le n$, $\sigma_k^2 = 0.075 + 6k \times 10^{-5}$, $1 \le k \le n$. (i) $c_* = 0.02$, (ii) $c_* = 0.03$, (iii) $c_* = 0.04$.
- (iv) $c_* = 0.02$, $g_1 = -0.016$, $g_k = 0$, $2 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.016$,
- (v) $g_1 = \cdots = g_{50} = -0.016$, $g_k = 0$, $51 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.8$.

PORTFOLIO AND WEALTH PROCESSES

Portfolio rule $\Pi(\cdot) := (\Pi_1(\cdot), \ldots, \Pi_n(\cdot))$, $0 \le t < \infty$ is an adapted, locally square-integrable process with

$$\Pi_1(\cdot) + \cdots + \Pi_n(\cdot) = 1 .$$

We call the portfolio “long-only", if all its components are nonnegative. The associated wealth $V^{\Pi}(\cdot)$ is defined by

$$\frac{\mathrm{d}\, V^{\Pi}(t)}{V^{\Pi}(t)} = \sum_{i=1}^{n} \Pi_i(t) \frac{\mathrm{d}\, X_i(t)}{X_i(t)} ; \quad 0 \le t < \infty, \quad V^{\Pi}(0) = 1 .$$

- Examples of long-only portfolios are
  - **Market** $M(\cdot) = (M_1(\cdot), \cdots, M_n(\cdot))$, $M_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}$.
  - **Constant proportion** portfolio:

    $$\Pi(\cdot) \equiv \pi = (\pi_1, \ldots, \pi_n) \in \Gamma_+^n ,$$

    - especially, equally weighted portfolio $\pi_1 = \cdots = \pi_n = 1/n$,

where $\Gamma_+^n := \{ x \in [0,1]^n \,:\, x_1 + \cdots + x_n = 1 \}$.

Constant-Proportion Portfolios

The stability conditions make this theory ideal for the application of the Empirical Bayes “Universal Portfolios" (Cover (1991), Jamshidian (1992)), in the framework of Stochastic Portfolio Theory (SPT).

- Under these conditions, the wealth process $V^{\pi}(\cdot)$ of the constant proportion portfolio $\Pi(\cdot) \equiv \pi \in \Gamma_{+}^{n}$ satisfies

$$\lim_{T \to \infty} \frac{1}{T} \log V^{\pi}(T) = \gamma + \gamma_{\pi}^{\infty} \,, \ \text{a.s.},$$

where

$$\gamma_{\pi}^{\infty} := \frac{1}{2} \sum_{i=1}^{n} \pi_i \left(1 - \pi_i\right) a_{ii}^{\infty}$$

is the asymptotic “excess growth rate" of SPT, and

$$a_{ii}^{\infty} := \lim_{T \to \infty} a_{ii}(T) = \lim_{T \to \infty} \frac{1}{T} \langle Y_i \rangle (T) = \sum_{k=1}^{n} \sigma_k^2 \, \vartheta_{k,i} \,, \quad i = 1, \ldots, n \,.$$

are the asymptotic variance rates.

Asymptotic Target Portfolio

The Asymptotic Target Portfolio (ATP) maximizes long-term growth rate among all constant-proportion portfolios. It is given as $\overline{\pi} := \arg\max_{\pi \in \Gamma_+^n} \gamma_\pi^\infty$, with

$$\overline{\pi}_i = \frac{1}{2}\left[1 - \frac{n-2}{a_{ii}^\infty}\Big(\sum_{j=1}^n \frac{1}{a_{jj}^\infty}\Big)^{-1}\right]; \quad i = 1, \ldots, n\,.$$

Needs knowledge of the variance structure. Over long time horizons, this portfolio outperforms the overall market rather significantly:

$$\lim_{T\to\infty} \frac{1}{T} \log \frac{V^{\overline{\pi}}(T)}{V^M(T)} = \frac{1}{2}\sum_{i=1}^n \overline{\pi}_i \left(1 - \overline{\pi}_i\right) a_{ii}^\infty \geq \frac{n-1}{2}\Big(\sum_{i=1}^n \frac{1}{a_{ii}^\infty}\Big),$$

where $V^M(\cdot)$ is the wealth process of market portfolio $M(\cdot)$.

Asymptotically Active Market

If all the weights in the asymptotic target portfolio $\overline{\pi} = (\overline{\pi}_1, \ldots, \overline{\pi}_n)$ are positive, that is,

$$\overline{\pi}_i = \frac{1}{2}\Big[1 - \frac{n-2}{a_{ii}^{\infty}}\Big(\sum_{j=1}^{n}\frac{1}{a_{jj}^{\infty}}\Big)^{-1}\Big] > 0\,; \quad 1 \le i \le n\,,$$

we call the market is asymptotically active.

Sufficient conditions for the asymptotic active market are

- Constant variance $\sigma_1^2 = \cdots = \sigma_n^2$ or
- (Pure) Atlas model $\gamma_1 = \cdots = \gamma_n = 0\,.$

In general, the drift and volatility coefficients have non-trivial effects on this condition.

Universal Portfolio (Cover('91) & Jamshidian('92))

Universal portfolio $\widehat{\Pi}(\cdot)$ and its wealth $V^{\widehat{\Pi}}(\cdot)$ are given as performance-weighted averages of constant-proportion portfolios

$$\widehat{\Pi}_i(\cdot) := \frac{\int_{\Gamma_+^n} \pi_i V^{\pi}(\cdot)\,\mathrm{d}\pi}{\int_{\Gamma_+^n} V^{\pi}(\cdot)\,\mathrm{d}\pi}\,, \; 1 \le i \le n\,, \qquad V^{\widehat{\Pi}}(\cdot) = \frac{\int_{\Gamma_+^n} V^{\pi}(\cdot)\,\mathrm{d}\pi}{\int_{\Gamma_+^n} \mathrm{d}\pi}\,.$$

Great advantage: completely model-free.

Proposition. Under the Hybrid Atlas model with the stability condition and the asymptotically active market condition, the Asymptotic Target Portfolio (ATP) and Universal Portfolio (UP) are asymptotically equivalent in the first order, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\widehat{\Pi}}(T)}{V^{\bar{\pi}}(T)}\right) = 0\,, \quad a.s.$$

Growth-Optimal Portfolio

We shall call growth optimal a portfolio $\varpi(\cdot)$ that satisfies

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\Pi}(T)}{V^{\varpi}(T)}\right) \le 0\,, \quad a.s.,$$

for any portfolio $\Pi(\cdot)$. Maximizing growth rates, we obtain

$$\varpi_i(\cdot) = \frac{1}{2} + \frac{1}{a_{ii}(\cdot)}\Big[\widetilde{\gamma}_i(\cdot) + \Big(\sum_{j=1}^{n} \frac{1}{a_{jj}(\cdot)}\Big)^{-1}\Big(1 - \frac{n}{2} - \sum_{j=1}^{n} \widetilde{\gamma}_j(\cdot)\Big)\Big]$$

where $\widetilde{\gamma}_i(\cdot) = g_k + \gamma_i + \gamma$, if $X_i(\cdot) = X_{(k)}(\cdot)\,; \quad 1 \le i, k \le n$.
Extremely model-dependent.

- If $\sigma_1^2 = \cdots = \sigma_n^2 = \sigma^2$, this reduces to

$$\varpi_i(\cdot) = \frac{1}{n}\Big[-\frac{n\,\widetilde{\gamma}_i(\cdot)}{\sigma^2} + 1 + \frac{1}{\sigma^2}\sum_{j=1}^{n} \widetilde{\gamma}_j(\cdot)\Big]\,; \quad 1 \le i \le n\,.$$

- Let us specialize to the equal-variance case, i.e.,

$$\sigma_1^2 = \cdots = \sigma_n^2 = \sigma^2 .$$

The performance of the growth-optimal portfolio

$$\lim_{T\to\infty} \frac{1}{T} \log V^{\varpi}(T) = \gamma + \frac{\sigma^2}{2}\Big(1 - \frac{1}{n}\Big) + \frac{1}{2\sigma^2}\Big(\sum_{k=1}^{n} g_k^2 - \sum_{i=1}^{n} \gamma_i^2\Big)$$

dominates the performance of the universal portfolio and the asymptotic target portfolio

$$\lim_{T\to\infty} \frac{1}{T} \log V^{\overline{\pi}}(T) = \lim_{T\to\infty} \frac{1}{T} \log V^{\widehat{\Pi}}(T) = \gamma + \frac{\sigma^2}{2}\Big(1 - \frac{1}{n}\Big);$$

indeed, $\sum_{k=1}^{n} g_k^2 > \sum_{i=1}^{n} \gamma_i^2$ holds under the stability conditions.

- Big advantage of universal portfolio:

No need to know any of the model parameter.

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