Discussion of "Stable Diffusions Interacting through Their Ranks, as Models of Large Equity Markets" by Ioannis Karatzas

Discussant: Semyon Malamud, EPF Lausanne and SFI

Standard Model

$$
\begin{aligned}
d \log B(t) &= r(t)\, dt \\
d \log S_i(t) &= \gamma_i(t)\, dt \, + \, \sum_{\nu} \sigma_{i\nu}(t)\, dW_{\nu}
\end{aligned} \tag{1}
$$

(a_{ij}) is the instantaneous variance-covariance matrix of the stocks

Log Wealth Process of a Portfolio Strategy π

$$d\log(V^{w,\pi}) \;=\; \left(\gamma^{\pi}(t)\,dt \;+\; \sum_{\nu} \sigma^{\pi}_{\nu}(t)\,dW_{\nu}\right)$$

with volatilities

$$\sigma^{\pi}_{\nu}(t) \;=\; \sum_{i} \pi_i(t)\,\sigma_{i\nu}(t)\ ,$$

the growth rate

$$\gamma^{\pi}(t) \;=\; \sum_{i} \pi_i(t)\gamma_i(t) \;+\; \gamma^{\pi}_{*}$$

and **the excess growth rate**

$$\gamma^{\pi}_{*} \;=\; \frac{1}{2}\left(\sum_{i} a_{ii}(t)\,\pi_i(t) \;-\; \frac{1}{2}\sum_{i,j} \pi_i(t)\pi_j(t)\,a_{ij}(t)\right)$$

Pure diversification gain Theorem: excess growth rate is always positive for a long-only portfolio

The Market Portfolio μ

$$\mu_i(t) \;=\; S_i(t) / \sum_j S_j(t)$$

Order Statistics

$$\mu_{(1)} \;\geq\; \mu_{(2)} \;\geq\; \cdots \;\geq\; \mu_{(n)}$$

Weath process

$$V^{w,\mu} \;=\; \sum_i S_i(t)$$

Questions

- **Can we outperform the market with a simple long only portfolio?**
- **In a model-independent way?**
- **Without any statistical parameter estimation?**

Answers:

- **Yes, we can**
- **Yes, we can**
- **Yes, we can**

But we need ...

Answers:

- **Yes, we can**
- **Yes, we can**
- **Yes, we can**

But we need ...

Diversity over $[0, T]$:

Strong diversity: $\mu_{(1)} \ \leq \ 1 - \delta$ a.s. for some $\delta > 0$

Weak diversity: $\frac{1}{T} \int_0^T \mu_{(1)}(t) dt \ \leq \ 1 - \delta$ a.s.

Definitely True in the Data!

Can we find a simple model with diversity?

- No!
- Because Diversity leads to arbitrage!

  Example: a diversity-weighted portfolio

$$\pi_i^{(p)} = \frac{\mu_i(t)^p}{\sum_j \mu_j(t)^p}, \; p \in (0,1)$$

  outperformes the market with probability 1

- Strict local martingales

Can we find a simple model with diversity?

- No!
- Because Diversity leads to arbitrage!

  Example: a diversity-weighted portfolio

$$\pi_i^{(p)} = \frac{\mu_i(t)^p}{\sum_j \mu_j(t)^p}, \; p \in (0,1)$$

  outperformes the market with probability 1

- Strict local martingales

Pure Diversity Based Portfolios

- Can be constructed
- Are based only on the perfectly observable capital distribution curve $\log k \;\to\; \log \mu_{(k)}$

Local "Rank Occupation" Times

The dynamics of $\mu_{(k)}$ is to a large extent driven by the **directly observable** local times $\Lambda^{k,k+1}(t)$ accumulated at the origin by $\log(\mu_{(k)}/\mu_{(k+1)})$

Empirical Regularities

- Capital distribution curve $\log k \ \rightarrow \ \log \mu_{(k)}$ is extremely stable over time, flat for small k, concave for large k
- Local times $\Lambda^{k,k+1}(t)$ are almost linear in t and decrease very fast with k
- variance is almost linear in the rank

Questions:

- Amazing ?!
- What is the economics behind this?
- Can we build a model reproducing these regularities?

Answers:

- Amazing ?! **Yes!**
- What is the economics behind this? **No idea!**
- Can we build a model reproducing these regularities? This is what Ioannis' talk was about.

Hybrid Atlas Models

- Postulate that

$$d(\log S_i(t)) \;=\; (\gamma + \gamma_i + g_k) \;+\; \sigma_k \, dW_i(t) \;+\; \sum_j \rho_{ij} dW_j(t)$$

  if $\mu_i = \mu_{(k)}$
- stability conditions: **global cancellation:**

$$\sum_k g_k \;+\; \sum_i \gamma_i \;=\; 0$$

- **no** $k \leq n-1$ **stocks dominate**:

$$\sum_{l=1}^{k} (g_l + \gamma_{i_l}) \;<\; 0$$

  for all $k \leq n-1$ and all permutations $\{i_1, \cdots, i_n\}$.
- These conditions imply stochastic stability: there is unique invariant distribution and SLLN holds $\Rightarrow$ local times grow linearly with t
- closed form invariant measure when $\rho_{ij} = 0$

Long-Run Slopes of the Local Times

$$\lim_{T\to\infty} \frac{1}{T}\Lambda^{k,k+1}(T) \;=\; -2\sum_{l=1}^{k}\left(g_l + \sum_{i=1}^{l}\gamma_i\theta_{l,i}\right)$$

where $\theta_{l,i}$ is the long-run occupation time.
Empirically, the slopes decrease fast with k. What does this mean?

Concavity of the Capital Distribution Curve

For any permutation $\mathbf{p}$ define

$$\lambda_{\mathbf{p},k} \;=\; \frac{-4\sum_{l=1}^{k}(g_l + \gamma_{\mathbf{p}(l)})}{\sigma_k^2 + \sigma_{k+1}^2}$$

(the exponents for the invariant distribution).
Then, the curve is concave if the sequence

$$\lambda_{\mathbf{p},k} \log \frac{k+1}{k}$$

is decreasing in k for any $\mathbf{p}$.
What does this condition mean?

Empirical Diversity

Define diversity as the entropy (other definitions also work):

$$\sum_i \mu_i(t)\, \log(\mu_i(t))$$

An additional Risk Factor?

Alpha as diversity beta?
The shape of capital distribution fluctuates over time?

MarketDiversity.jpg 500×194 pixels