EXISTENCE OF INVARIANT MANIFOLDS FOR STOCHASTIC EQUATIONS IN INFINITE DIMENSION

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ABSTRACT. We provide a Frobenius type existence result for finite-dimensional invariant submanifolds for stochastic equations in infinite dimension, in the spirit of Da Prato and Zabczyk [5]. We recapture and make use of the convenient calculus on Fréchet spaces, as developed by Kriegl and Michor [16]. Our main result is a weak version of the Frobenius theorem on Fréchet spaces.

As an application we characterize all finite-dimensional realizations for a stochastic equation which describes the evolution of the term structure of interest rates.

1. INTRODUCTION

In this article we investigate the existence of finite-dimensional invariant manifolds for a stochastic equation of the type

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))\,dt + \sum_{j=1}^{d} \sigma_j(r_t)\,dW_t^j \\ r_0 = h_0 \end{cases} \tag{1.1}$$

on a separable Hilbert space H, in the spirit of Da Prato and Zabczyk [5]. The operator $A : D(A) \subset H \to H$ generates a strongly continuous semigroup on H. Here $d \in \mathbb{N}$, and $W = (W^1, \dots, W^d)$ denotes a standard d-dimensional Brownian motion defined on a fixed reference probability space (see [5]). The mappings $\alpha : H \to H$ and $\sigma = (\sigma_1, \dots, \sigma_d) : H \to H^d$ satisfy a smoothness condition, to be defined precisely in what follows (Section 4). We distinguish, in decreasing order of generality, between (local) mild, weak and strong solutions of equation (1.1). The reader is referred to [5] or [8] for the precise definitions.

Our motivation is coming from the theory of interest rates. The basic interest rate contracts are the zero coupon bonds. The price at time t of a zero coupon bond with maturity $T \geq t$ is given by

$$P(t,T) = \exp\left(-\int_0^{T-t} r_t(x)\,dx\right),$$

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where $r_t(x)$ denotes the instantaneous forward rate at time t for date $t+x$ (this notion has been introduced by Musiela [19]). Within the framework of Heath, Jarrow and Morton (henceforth HJM) [14], for every $T \geq 0$, the real-valued process $(r_t(T-t))_{0\leq t\leq T}$ is an Itô processes satisfying the so called HJM drift condition, which assures the absence of arbitrage. It is shown in [8] that the stochastic evolution of the entire *forward curve*, $x \mapsto r_t(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}$, can be described by a stochastic equation of the above type (1.1), where H consists of real-valued continuous functions on $\mathbb{R}_{\geq 0}$, the operator $A = d/dx$ is the generator of the shift-semigroup $S_t h = h(t+\cdot)$, and $\alpha = \alpha_{HJM}$ is completely determined by σ according to the HJM drift condition. We will be more precise about the HJM setup in Section 4 below.

There are several reasons why in practice one is interested in such HJM models which admit a *finite-dimensional realization (FDR)* at every initial curve $r_0 \in H$, see [1, 7, 8, 12]. The formal definition of an FDR is as follows.

Definition 1.1. *Let $m \in \mathbb{N}$ and $h_0 \in H$. An* m-dimensional realization *for (1.1) at h_0 is a pair (V, ϕ), where $V \subset \mathbb{R}^m$ is open, $\phi : V \to H$ is a smooth immersion, such that $h_0 \in \phi(V)$ and, for every $h \in \phi(V)$, there exists a V-valued Itô process Z such that $\phi(Z)$ is a local weak solution to (1.1) with $r_0 = h$.*

The notion of a smooth immersion is recaptured in Section 3 (see Lemma 3.1). By convention, "smooth" is a synonym for C^∞ (see Section 2 for a thorough discussion on differential calculus).

Definition 1.2. *A subset U of H is called* locally invariant *for (1.1) if, for every initial point $h_0 \in U$, there exists a continuous local weak solution r to (1.1) with lifetime τ such that $r_{t\wedge\tau} \in U$, for all $t \geq 0$.*

For the notion of a finite-dimensional submanifold $\mathcal{M}$ of a Hilbert space and its tangent spaces $T_h\mathcal{M}$, $h \in \mathcal{M}$, we refer to Section 3. Finite-dimensional locally invariant submanifolds for (1.1) have been characterized in [10], see also [8]. Here we restate [10, Theorem 3].

Theorem 1.3. *Suppose that α is locally Lipschitz continuous and locally bounded, and σ is C^1. Let $\mathcal{M}$ be an m-dimensional submanifold of H. Then the following conditions are equivalent:*

i) *$\mathcal{M}$ is locally invariant for (1.1)*
ii) *$\mathcal{M} \subset D(A)$ and*

$$\mu(h) := Ah + \alpha(h) - \frac{1}{2}\sum_{j=1}^{d} D\sigma_j(h)\sigma_j(h) \in T_h\mathcal{M} \tag{1.2}$$

$$\sigma_j(h) \in T_h\mathcal{M}, \quad j = 1, \dots, d, \tag{1.3}$$

for all $h \in \mathcal{M}$.

Hence the stochastic invariance problem to (1.1) is equivalent to the deterministic invariance problems related to the vector fields $\mu, \sigma_1, \dots, \sigma_d$.

An FDR is essentially equivalent to a finite-dimensional invariant submanifold in the following sense. If (V, ϕ) is an m-dimensional realization for (1.1) at some $h_0 \in H$, then there exists an open neighborhood V_0 of $\phi^{-1}(h_0)$ in $\mathbb{R}^m$ such that $\phi(V_0)$ is an m-dimensional submanifold, which is locally invariant for (1.1). The converse is given by the following result, which is a restatement of [8, Theorem 6.4.1].

Theorem 1.4. *Let α, σ and $\mathcal{M}$ be as in Theorem 1.3. Suppose $\mathcal{M}$ is locally invariant for (1.1). Then, for any $h_0 \in \mathcal{M}$, there exists an m-dimensional realization (V, ϕ) for (1.1) at h_0 such that $\phi(V) = U \cap \mathcal{M}$, where U is an open set in H.*

Theorem 1.3 provides conditions for the invariance of a given submanifold $\mathcal{M}$. However, it does not say anything about the *existence* of an FDR for (1.1). This issue will be exploited in the present article.

The *FDR-problem* consists of finding sufficient conditions on $\mu, \sigma_1, \dots, \sigma_d$ for the existence of FDRs. Björk et al [1], [3] translated this into an appropriate geometric language. In [3] they completely solved the FDR-problem for equations (1.1) of HJM type on a very particular Hilbert space. Their key argument is the classical Frobenius theorem (see for example [17]), since they are looking for foliations (which is the appropriate notion for the FDR-problem on Hilbert spaces). Therefore they define a Hilbert space, $\mathcal{H}$, on which $A = d/dx$ is a bounded linear operator. As a consequence $\mathcal{H}$ consists solely of entire analytic functions (see [3, Proposition 4.2]). It is well known however that the forward curves implied by a Cox–Ingersoll–Ross (CIR) [4] short rate model are of the form $r_t = g_0 + r_t(0) g_1$ where

$$g_0(x) = d\,\frac{e^{ax} - 1}{e^{ax} + c} \quad \text{and} \quad g_1(x) = \frac{b e^{ax}}{(e^{ax} + c)^2},$$

for some $a, b, c > 0$ and $d \geq 0$ (see e.g. [8, Section 7.4.1]). Since both g_0 and g_1, when extended to $\mathbb{C}$, have a singularity at $x = (\log(c) + i\pi)/a$, they cannot be entire analytic. Hence the CIR forward curves do not belong to $\mathcal{H}$. Since the CIR model is one of the basic HJM models, the Björk-Svensson [3] setting is too narrow for the HJM framework, even though all geometric ideas are already formulated there.

To overcome this difficulty we have to choose a larger forward curve space. But we cannot do without the Frobenius theorem. The problem is that A is typically an unbounded operator on H, so μ is not continuous and not even defined on the whole space H (the choice of $H = \mathcal{H}$ in [3] is exactly made to overcome this problem). The appropriate framework for an extended version of the Frobenius theorem is thus given by the Fréchet space

$$D(A^\infty) := \bigcap_{n \in \mathbb{N}} D(A^n),$$

equipped with the family of seminorms

$$p_n(h) = \sum_{i=0}^{n} \|A^i h\|_H, \quad n \in \mathbb{N}_0.$$

We prove the existence of FDRs on this space under additional technical assumptions on α and $\sigma_1, \dots, \sigma_d$. They have to map $D(A^\infty)$ into itself and generate local flows on $D(A^\infty)$. However, as typical for Frécht spaces, smoothness of α and σ on $D(A^\infty)$ is not enough to guarantee the existence of local flows. Thus we shall provide sufficient conditions on the coefficients, which can be found in Hamilton [13] (α and σ have to be so called Banach maps). Then the existence of FDRs on an open subset U in $D(A^\infty)$ is essentially equivalent to the boundedness of the dimension of the Lie algebra generated by $\mu, \sigma_1, \dots, \sigma_d$ on U. We do not obtain a true foliation of U as in the finite-dimensional case, which is due to the fact that μ merely admits a local semiflow on U and not a local flow. So we are led to the notion of a "weak foliation".

We then exemplify the use of these results with the HJM framework. Here we eventually obtain a striking global result. HJM models that admit an FDR at any initial curve r_0 are necessarily affine term structure models, in a sense to be explained in Section 4 (see Remark 4.14).

The remainder of the paper is organized as follows. In Section 2 we provide a convenient differential calculus on Fréchet spaces (and more general locally convex spaces), as developped in [16]. We discuss the existence of local (semi)flows related to smooth vector fields on a Fréchet space, based on the Banach map principle (Theorems 2.10 and 2.13). In Section 3 we recapture the notion of a finite-dimensional submanifold, and the Lie bracket of two smooth vector fields in a Fréchet space. We point out the crucial fact that the Lie bracket of a Banach map with a bounded linear operator is a Banach map (Lemma 3.4). After the definition of a finite-dimensional *weak foliation* (Definition 3.7) we prove a Frobenius theorem on Fréchet spaces (Theorem 3.9). In Section 4 we provide the rigorous setup for HJM models. Under the appropriate assumptions we solve the FDR-problem and give a global characterization of all finite-dimensional weak foliations.

2. Analysis on Fréchet Spaces

For the purposes of analysis on open subsets of Fréchet spaces we shall follow two equivalent approaches. The classical Gateaux-approach as outlined in [13] and so called "convenient analysis" as in [16]. On Fréchet spaces these two notions of smoothness coincide and convenient calculus is an appropriate extension of analysis to more general locally convex spaces. Furthermore these methods allow simple and elegant calculations. The main advantage of convenient calculus is however, that one can give a precise analytic meaning (in simple terms) to geometric objects on Fréchet spaces as for example vector fields, differential forms (see [16]).

Definition 2.1. *Let E, F be Fréchet spaces and $U \subset E$ an open subset. A map $P : U \to F$ is called* Gateaux-C^1 *if*

$$DP(f)h := \lim_{t\to 0} \frac{P(f+th) - P(f)}{t}$$

exists for all $f \in U$ and $h \in E$ and $DP : U \times E \to F$ is a continuous map.

For the definition of Gateaux-C^2-maps the ambiguities of calculus on Fréchet spaces already appear. Since there is no Fréchet space topology on the vector space of continuous linear mappings $L(E, F)$ one has to work by point evaluations:

Definition 2.2. *Let E, F be Fréchet spaces and $U \subset E$ an open subset. A map $P : U \to F$ is called* Gateaux-C^2 *if*

$$D^2P(f)(h_1, h_2) := \lim_{t\to 0} \frac{DP(f+th_2)h_1 - DP(f)h_1}{t}$$

exists for all $f \in U$ and $h_1, h_2 \in E$ and $D^2P : U \times E \times E \to F$ is a continuous map. Higher derivatives are defined in a similar way. A map is called Gateaux-smooth *or* Gateaux-C^∞ *if it is Gateaux-C^n for all $n \geq 0$.*

The next Theorem collects all essential results of Gateaux-Calculus for our purposes (see [13], pp. 73–84, pp. 99–100):

Theorem 2.3. *Let E, F, G be Fréchet spaces and $U \subset E$ be open in E. Let $P : U \subset E \to F$ and $Q : V \subset F \to G$ be continuous maps:*

i) *If P and Q are Gateaux-C^n, then $Q \circ P$ is Gateaux-C^n and the usual chain rule holds.*
ii) *Let U be convex: P is Gateaux-C^1 if and only if there exists a continuous map $L : U \times E \times E \to F$, linear in the last variable, such that for all $f_1, f_2 \in U$*
$$P(f_1) - P(f_2) = L(f_1, f_2)(f_1 - f_2).$$
iii) *If P is Gateaux-C^1, then for $f_0 \in U$ and a continuous seminorm q on F, there is a continuous seminorm p on E and $\varepsilon > 0$ such that*
$$q(P(f_1) - P(f_2)) \leq p(f_1 - f_2)$$
for $p(f_i - f_0) < \varepsilon$, $i = 1, 2$.

For the construction of differential calculus on locally convex spaces we need the concept of smooth curves into locally convex spaces and the concept of smooth maps on open subsets of locally convex spaces. We remark that already on Fréchet spaces the situation concerning analysis was complicated and unclear until convenient calculus was invented (see [16], pp. 73–77, for extensive historical remarks). The reason for inconsistencies can be found in the fundamental difference between bounded and open subsets.

We denote the set of continuous linear functionals on a locally convex space E by E'_c. A subset $B \subset E$ is called *bounded* if $l(B)$ is a bounded subset of $\mathbb{R}$ for all $l \in E'_c$. A multilinear map $m : E_1 \times ... \times E_n \to F$ is called *bounded* if bounded sets $B_1 \times ... \times B_n$ are mapped onto bounded subsets of F. Continuous linear functionals are clearly bounded. The locally convex vector space of bounded linear operators with uniform convergence on bounded sets is denoted by $L(E, F)$, the dual space formed by bounded linear functionals by E'. These spaces are locally convex vector spaces we shall need for analysis (see [16], 3.17).

Definition 2.4. *Let E be a locally convex space, then $c : \mathbb{R} \to E$ is called* smooth *if all derivatives exist as limits of difference quotients. The set of smooth curves is denoted by $C^\infty(\mathbb{R}, E)$.*

*A subset $U \subset E$ is called c^∞-*open *if $c^{-1}(U)$ is open in $\mathbb{R}$ for all $c \in C^\infty(\mathbb{R}, E)$. The generated topology on E is called c^∞-*topology *and E equiped with this topology is denoted by $c^\infty E$.*

If U is c^∞-open, a map $f : U \subset E \to \mathbb{R}$ is called smooth *if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in C^\infty(\mathbb{R}, E)$.*

These definitions work for any locally convex vector space, but for the following theorem we need a weak completeness assumption. A locally convex vector space E is called *convenient* if the following property holds: a curve $c : \mathbb{R} \to E$ is smooth if and only if it is weakly smooth, i.e. $l \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in E'$. This is equivalent to the assertion that any smooth curve $c : \mathbb{R} \to E$ can be (Riemann-)integrated in E on compact intervals (see [16], 2.14). The spaces $L(E, F)$ and E' are convenient vector spaces (see [16], 3.17), if E and F are convenient.

Theorem 2.5. *Let E, G, H be convenient vector spaces, $U \subset E$, $V \subset G$ c^∞-open subsets:*

i) *Smooth maps are continuous with respect to the c^∞-topology.*
ii) *Multilinear maps are smooth if and only if they are bounded.*

iii) *If $P : U \to G$ is smooth, then $DP : U \to L(E, G)$ is smooth and bounded linear in the second component, where*

$$DP(f)h := \frac{d}{dt}|_{t=0} P(f + th).$$

iv) *The chain rule holds.*

v) *Let $[f, f + h] := \{f + sh \text{ for } s \in [0, 1]\} \subset U$, then Taylor's formula is true at $f \in U$, where higher derivatives are defined as usual (see iii.),*

$$P(f + h) = \sum_{i=0}^{n} \frac{1}{i!} D^i P(f) h^{(i)} + \int_0^1 \frac{(1 - t)^n}{n!} D^{n+1} P(f + th)\,(h^{(n+1)}) dt$$

for all $n \in \mathbb{N}$.

vi) *There are natural convenient locally convex structures on $C^\infty(U, F)$ and we have* cartesian closedness

$$C^\infty(U \times V, H) \simeq C^\infty(U, C^\infty(V, H)).$$

via the natural map $f \mapsto \check{f} : U \to C^\infty(V, H)$ for $f \in C^\infty(U \times V, H)$. This natural map is well defined and a smooth linear isomorphism.

vii) *The evaluation and the composition*

$$\mathrm{ev} : C^\infty(U, F) \times U \to F, \quad (P, f) \mapsto P(f)$$
$$. \circ . : C^\infty(F, G) \times C^\infty(U, F) \to C^\infty(U, G), \quad (Q, R) \mapsto Q \circ R$$

are smooth maps.

viii) *A map $P : U \subset E \to L(G, H)$ is smooth if and only if* $(\mathrm{ev}_g \circ P)$ *is smooth for all $g \in G$.*

Proof. For the proofs see [16] in Subsections 3.12, 3.13, 3.18, 5.11, 5.12, 5.18. □

Convenient Calculus is an extension of the Gateaux-Calculus to locally convex spaces, where all necessary tools for analysis are preserved. Since typically vector spaces like $C^\infty(U, F)$ or $L(E, F)$ are not Fréchet spaces, this extension is very useful for the analysis of the geometric objects in Section 3.

Theorem 2.6. *Let E, F be Fréchet spaces and $U \subset E$ a c^∞-open subset, then U is open and $P : U \subset E \to F$ is Gateaux-smooth if and only if P is smooth (in the convenient sense).*

Proof. By Theorem 4.11 of [16] we get that U is open since $c^\infty E = E$. Assume that P is Gateaux-smooth, then by the chain rule for Gateaux-C^n maps (see Theorem 2.3, i.) the composition $P \circ c$ is Gateaux-C^n for all $n \geq 0$ and all smooth curves $c \in C^\infty(\mathbb{R}, E)$, so P is smooth in the convenient sense. If P is smooth in the convenient sense, then the first derivative DP as defined in Theorem 2.5 exists and is continuous as map $DP : U \times E \to F$ by cartesian closedness and the fact that $c^\infty E = E$ (see Theorem 2.5, i.). The same reasoning holds for higher derivatives, so we obtain that P is Gateaux-C^n for all $n \geq 0$. □

Since we shall calculate with semiflows and semigroups of bounded linear operators, we shall need convenient calculus on domains of the form $[0, \varepsilon[\times U$, where U is open in a Fréchet space E. The definition of smooth maps is straightforward due to the simple strucure of convenient calculus.

Let K be a convex set with non-void interior K° in a Fréchet space E and F a convenient vector space, then $f : K \to F$ is called smooth if $f \circ c : \mathbb{R} \to F$ is smooth for all smooth curves $c \in C^\infty(\mathbb{R}, E)$ with $c(\mathbb{R}) \subset K$. We have the following properties (for a proof see [16], pp. 247–254):

Theorem 2.7. *Let K be a convex subset with non-void interior K° in a Fréchet space E, F a convenient space and $P : K \to F$ a map.*

i) *P is smooth if and only if P is smooth on K° and all derivatives $D^n(P|_{K^\circ})$ extend continuously (with respect to the c^∞-topology to K) to K.*
ii) *If P is smooth and $DP : K \to L(E, F)$ a continuous extension of $D(P|_{K^\circ})$, then the chain rule holds, i.e. for $c \in C^\infty(\mathbb{R}, E)$ with $c(\mathbb{R}) \subset K$ we have $(P \circ c)'(t) = DP(c(t)) \cdot c'(t)$.*
iii) *There exists a bounded linear extension operator*

$$C^\infty([0, \varepsilon[, F) \to C^\infty(\mathbb{R}, F).$$

Consequently we can reformulate all assertions of Theorem 2.5 for maps on $[0, \varepsilon[\times U$ with U open in a Fréchet space, since $([0, \varepsilon[\times U)^\circ =]0, \varepsilon[\times U$, in particular, the chain rule and cartesian closedness hold. The time derivatives at 0 can be calculated as right derivatives by the bounded linear extension operator. This convenient approach is through its generality and simplicity much more practical than the equivalent Gateaux approach.

In the sequel we shall apply concepts from both approaches: Gateaux-smoothness for existence theorems and convenient analysis for the sake of generality, simplicity and elegance. Notice that convenient calculus provides a very powerful tool for analysis in concrete calculations, too (see [16] for many examples and [23] for a particularly simple proof of a general Frobenius Theorem).

Concerning differential equations, there are possible counterexamples on non-normable Fréchet spaces in all directions, which causes some problems in the foundations of differential geometry (see [16] and the excellent review article [18]). Nevertheless a useful generalization of the existence theorem for differential equations on Banach spaces is given by the following Banach map principle (see [13] for details, compare also [16], 32.14 for weaker results in a more general situation).

If not otherwise stated, E and F denote Fréchet spaces and B a Banach space in what follows. Given $P : U \subset E \to E$ a smooth map. We are looking for solutions of the ordinary differential equation with initial value $g \in U$

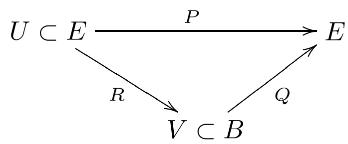
$$f :] - \varepsilon, \varepsilon[\to U \text{ smooth}$$
$$\frac{d}{dt} f(t) = P(f(t))$$
$$f(0) = g \in U.$$

If for any initial value g in a small neighborhood V of $f_0 \in U$ there is a unique smooth solution $t \mapsto f_g(t)$ for $t \in] - \varepsilon, \varepsilon[$ depending smoothly on the initial value g, then $Fl(t, g) := f_g(t)$ defines a *local flow*, i.e. a smooth map

$$Fl :] - \varepsilon, \varepsilon[\times V \to E$$
$$Fl(0, g) = g$$
$$Fl(t, Fl(s, g)) = Fl(s + t, g)$$

if $s, t, s+t \in]-\varepsilon, \varepsilon[$ and $Fl(s, g) \in V$. If there is a local flow around $f_0 \in U$ (this shall mean once and for all: "in an open, convex neighborhood of f_0"), the differential equation is uniquely solvable around $f_0 \in U$ and the dependence on initial values is smooth (see Lemma 2.11 for the proof). Notice at this point that it is irrelevant if we define "smooth dependence" on initial values via maps $V \to C^\infty(]-\varepsilon,\varepsilon[,E)$ or $V\times]-\varepsilon,\varepsilon[\to E$ by cartesian closedness. We shall denote $f_g(t) = Fl_t(g) = Fl(t,g)$.

Definition 2.8. *Given a Fréchet space E, a smooth map $P : U \subset E \to E$ is called a* Banach map *if there are smooth (not necessarily linear) maps $R : U \subset E \to B$ and $Q : V \subset B \to E$ such that $P = Q \circ R$*



where B is a Banach space and $V \subset B$ is an open set.

A *vector field* P on an open subset $U \subset E$ is a smooth map $P : U \to E$. We denote by $\mathcal{B}(U)$ the set of Banach map vector fields and by $\mathfrak{X}(U)$ the convenient space of all vector fields on an open subset of a Fréchet space E.

Theorem 2.9. *$\mathcal{B}(U)$ is a $C^\infty(U,\mathbb{R})$-submodule of $\mathfrak{X}(U)$.*

Proof. We have to show that for $\psi,\eta \in C^\infty(U,\mathbb{R})$ and $P_1,P_2 \in \mathcal{B}(U)$ the linear combination $\psi P_1 + \eta P_2 \in \mathcal{B}(U)$. Given $P_i = Q_i \circ R_i$ for $i = 1,2$ with intermediate Banach spaces B_i, then $\psi P_1+\eta P_2 = Q\circ R$ with $Q : \mathbb{R}^2\times V_1\times V_2 \subset \mathbb{R}^2\times B_1\times B_2 \to E$ and $R : U \to \mathbb{R}^2 \times B_1 \times B_2$ such that

$$Q(r,s,v_1,v_2) = rQ_1(v_1) + sQ_2(v_2)$$
$$R(f) = (\psi(f),\eta(f),R_1(f),R_2(f))$$

So the sum $\psi P_1 + \eta P_2$ is a Banach map and therefore the set of all Banach map vector fields carries the asserted submodule structure. □

Theorem 2.10 (Banach map principle)**.** *Let $P : U \subset E \to E$ be a Banach map, then P admits a local flow around any point $g \in U$.*

Proof. For the proof see [13], Theorem 5.6.3. □

Parameters and time-dependence are treated in the following way. Given an open subset of parameters $Z \subset V$ of a Banach space V and $P : I \times Z \times U \to E$, where I is a open set in $\mathbb{R}$ and U is open in E, such that $P_{t,p} = Q_{t,p} \circ R_{t,p}$, where Q and R depend smoothly on time and parameters, P admits a unique smooth solution for any initial value $f_0 \in U$ at any time point $t_0 \in I$ depending smoothly on parameters, time and initial values.

For the proof of this assertion we look at the extended space $G := \mathbb{R} \times V \times E$ with $\widetilde{P}(t,p,f) = (1,0,P_{t,p}(f))$ and

$$\widetilde{Q}(t,p,z) = (1,0,Q_{t,p}(f))$$
$$\widetilde{R}(t,p,f) = (t,p,R_{t,p}(f))$$

with Banach space $\widetilde{B} := \mathbb{R} \times V \times B$.

We can replace in the above definition of a local flow the interval $]-\varepsilon,\varepsilon[$ by $[0,\varepsilon[$ to obtain *local semiflows*, see Theorem 2.7 for details in calculus. The initial value problem

$$\begin{aligned} f : [0, \varepsilon[\to U \text{ smooth} \\ \frac{d}{dt} f(t) = P(f(t)) \\ f(0) = g \in U. \end{aligned}$$

admits unique solutions around an initial value depending smoothly on the initial values if and only if a local semiflow exists. The notion of a local semiflow is redundant on Banach spaces.

Lemma 2.11. *Let Fl be a local semiflow on $[0, \varepsilon[\times U \to E$, then the map $P(f) := \frac{d}{dt}|_{t=0} Fl(t, f)$ is a well defined smooth vector field. We obtain*

$$DFl_t(f)P(f) = P(Fl_t(f))$$

and the initial value problem has unique solutions for small times which coincide with the given semiflow.

Proof. The equation follows by the flow property and the definition of P immediately:

$$DFl_t(f)P(f) = \frac{d}{ds} Fl_t(Fl_s(f))|_{s=0} = \frac{d}{ds} Fl_{t+s}(f)|_{s=0} = P(Fl_t(f)).$$

Given a solution $f : [0, \delta[\to E$ of the initial value problem associated to P with $f(0) = f_0 \in U$, then

$$\begin{aligned} \frac{d}{ds} Fl_{t-s}(f(s)) = -P(Fl_{t-s}(f(s))) + P(Fl_{t-s}(f(s))) = 0 \\ Fl_{t-s}(f(s)) = f(t) \end{aligned}$$

for all $0 \leq s \leq t$, whence uniqueness for the solutions of the initial value problem. □

We are in particular interested in special types of differential equations on Fréchet spaces E, namely Banach map perturbed bounded linear equations. Given a bounded linear operator $A : E \to E$, the abstract Cauchy problem associated to A is given by the initial value problem associated to A. We assume that there is a smooth semigroup of bounded linear operators $S : \mathbb{R}_{\geq 0} \to L(E, E)$ such that

$$\lim_{t \downarrow 0} \frac{S_t - id}{t} = A$$

which is a global semiflow for the linear vector field $f \mapsto Af$. Notice that the theory of bounded linear operators on Fréchet spaces contains as a special case Hille-Yosida-Theory of unbounded operators on Banach spaces (see for example [22]).

Given a strongly continuous semigroup S_t for $t \geq 0$ of bounded linear operators on a Banach space B, then $D(A^n)$ with the respective operator norms $p_n(f) := \sum_{i=0}^{n} ||A^i f||$ for $n \geq 0$ and $f \in D(A^n)$ is a Banach space, where the semigroup restricts to a strongly continuous semigroup $S^{(n)}$. Consequently the semigroup restricts to the Fréchet space $D(A^\infty)$. This semigroup is now smooth, since it is sufficient – by Theorem 2.5, viii. – to show smoothness of $t \mapsto S_t f$ for all $f \in D(A^\infty)$. This is true since $A^n S_t f = S_t A^n f$ for $t \geq 0$ and $f \in D(A^\infty)$.

For the purposes of classification in Section 4 we shall need the following result.

Lemma 2.12. *Let A be the generator of a strongly continuous semigroup S on a Banach space B, then the operator $A : D(A^\infty) \to D(A^\infty)$ is a Banach map if and only if $A : B \to B$ is bounded.*

Proof. For the properties of $D(A^\infty)$ see [20], in particular it is a Fréchet space with seminorms $p_n(f) = \sum_{i=0}^n ||A^i f||$. If $A : D(A^\infty) \to D(A^\infty)$ is a Banach map in a neighborhood U of a point f_0, then there are smooth maps $R : U \subset D(A^\infty) \to X$ and $Q : V \subset X \to D(A^\infty)$ such that $A = Q \circ R$ and X is a Banach space. By differentiation at f_0 we obtain

$$A = DQ(f_0) \cdot DR(f_0)$$

which means in particular by continuity that there exists $n \geq 0$ such that $DR(f_0)$ can be extended continuously to a linear mapping $\overline{DR(f_0)} : D(A^n) \to X$ (see Theorem 2.3, iii.). So $A : D(A^n) \to D(A^n)$ is a continuous mapping.

We recall the Sobolev Hierarchy for strongly continuous semigroups (see [20]) defined by the following commutative diagram

$$\begin{array}{ccc}
B & \xrightarrow{S_t^{(0)}} & B \\
\downarrow R(\lambda) & & \downarrow R(\lambda) \\
D(A) & \xrightarrow{S_t^{(1)}} & D(A) \\
\downarrow R(\lambda) & & \downarrow R(\lambda) \\
D(A^2) & \xrightarrow{S_t^{(2)}} & D(A^2) \\
\downarrow & & \downarrow \\
\dots & & \dots
\end{array}$$

Here $R(\lambda) := (\lambda - A)^{-1}$ denotes the resolvent at a point of the resolvent set, which defines an isomorphism from $D(A^n)$ to $D(A^{n+1})$. The semigroups $S^{(n)}$ are defined by restriction and are strongly continuous in the respective topologies. The generator of $S^{(n)}$ is given through A restricted to $D(A^{n+1})$. If A is continuous on $D(A^n)$ then $S^{(n)}$ is a smooth group, so by climbing up through the isomorphisms $S^{(0)}$ is a smooth group and therefore the infinitesimal generator is continuous, since it is everywhere defined, by the closed graph theorem. □

Given a Banach map $P : U \subset E \to E$, we want to investigate the solutions of the initial value problem

$$\frac{d}{dt} f(t) = Af(t) + P(f(t)), \quad f(0) = f_0.$$

Theorem 2.13. *Let E be a Fréchet space and A be the generator of a smooth semigroup $S : \mathbb{R} \to L(E)$ of bounded linear operators on E. Let $P : U \subset E \to E$ be a Banach map. For any $f_0 \in U$ there is $\varepsilon > 0$ and an open set V containing f_0 and a local semiflow $Fl : [0, \varepsilon[\times V \to U$ satisfying*

$$\frac{d}{dt} Fl(t, f) = AFl(t, f) + P(Fl(t, f))$$
$$Fl(0, f) = f$$

for all $(t, f) \in [0, \varepsilon[\times V$.

Proof. The arguments follow a proof for the case $A = 0$ in [13]. We prove the theorem by constructing a solution to the integral equation arising from variation of constants:

$$Fl(t,f) = S_t f + \int_0^t S_{t-s} P(Fl(s,f)) ds$$

for small positive time intervals and an open neighborhood of a given initial value f_0. Given $f_0 \in U$ there exists a seminorm p on E and $\delta > 0$ such that

$$||R(f_1) - R(f_2)|| \leq p(f_1 - f_2)$$

for $p(f_i - f_0) < \delta$ and $i = 1, 2$, where $||.||$ denotes the norm on B. Furthermore given $g_0 \in B$, then for any seminorm q on F there are constants C_q and δ_q such that

$$q(Q(g_1) - Q(g_2)) \leq C_q ||g_1 - g_2||$$

for $||g_i - g_0|| < \delta_q$ and $i = 1, 2$. Both assertions follow from Theorem 2.3,iii. By the uniform boundedness principle the set of continuous linear operators $\{S_t\}_{0 \leq t \leq T}$ is uniformly bounded for any fixed $T \geq 0$, i.e. for any seminorm p on E there is a seminorm q_p such that

$$p(S_t f) \leq q_p(f)$$

for $t \leq T$ and for all $f \in E$. We denote by $C([0,\varepsilon], B)$ continuous curves on the interval $[0,\varepsilon]$ to B, $g_0 := R(f_0)$. Without any restriction we can assume that $f_0 = 0$ and $g_0 = 0$ by translations. We can then define a mapping

$$M : U' \times V' \subset E \times C([0,\varepsilon], B) \to V'$$

such that $M(f,h)(t) = R(S_t f + \int_0^t S_{t-s} Q(h(s)) ds)$ for $t \in [0,\varepsilon]$. Given $h \in C([0,\varepsilon], B)$ such that $||h(t)|| \leq \theta$ for $0 \leq t \leq \varepsilon$ with $\{h | \sup_t ||h(t)|| \leq \theta\} \subset V'$, we have

$$p(S_t f + \int_0^t S_{t-s} Q(h(s)) ds) \leq q_p(f) + \varepsilon(q_p(Q(g_0)) + C_{q_p} \theta)$$

provided $\theta \leq \delta_{q_p}$. This can be made smaller than θ if ε is appropriately small and $U' := \{f \in E \text{ with } q_p(f) < \eta\}$ with η appropriately small. In particular $C_{q_p} \varepsilon < 1$. If we assume these conditions, then M is well defined, continuous and furthermore

$$\begin{aligned} \sup_t ||M(f,h_1)(t) - M(f,h_2)(t)|| &\leq \varepsilon \sup_t q_p(Q(h_1(t)) - Q(h_2(t))) \\ &\leq C_{q_p} \varepsilon \sup_t ||h_1(t) - h_2(t)|| \end{aligned}$$

Consequently $M(f,.)$ is a contraction in V' with contraction constant bounded uniformly in $f \in U'$ by a constant strictly smaller than 1. It follows that there is a unique $h(t,f)$ for any $f \in U'$ depending continuously on f, such that

$$M(f,h) = h$$

by the contraction mapping theorem. We define

$$Fl(t,f) := S_t f + \int_0^t S_{t-s} Q(h(s,f)) ds$$

and obtain

$$Fl(t,f) = S_t f + \int_0^t S_{t-s} P(Fl(s,f)) ds$$

since $R(Fl(t,f)) = h(t,f)$ by construction. Any solution of the initial value problem is therefore unique by the Banach contraction principle. By induction with Theorem

2.3, i.), smoothness with respect to time is easily established. For the first derivative one can calculate the limits directly.

Concerning smoothness with respect to the initial value, we proceed in the following way. We show that there exist directional derivatives and calculate them. By Taylor's formula we obtain

$$P(f_1) - P(f_2) = L(f_1, f_0) \cdot (f_1 - f_2)$$

where $L(x_1, x_2) \cdot h := \int_0^1 DP(x_0 + s(x_1 - x_0)) \cdot hds$ is a Banach map in all three variables (see Theorem 2.3, ii.). So we can solve the system given by

$$(f_0, f_1, h) \mapsto (Af_0 + P(f_0), Af_1 + P(f_1), Ah + L(f_1, f_2) \cdot h)$$

with the "flow"-construction from above

$$Fl(t, f_0, f_1, h) = (Fl(t, f_0), Fl(t, f_1), M(t, f_0, f_1, h)),$$

smooth in time and continuous in initial values, where the dependence on h is homogenous, so the "flow" can be defined everywhere in h. By uniqueness of the "flow" the identity

$$\frac{d}{dt}(Fl(t, f_0) - Fl(t, f_1))$$
$$= A(Fl(t, f_0) - Fl(t, f_1)) + L(Fl(t, f_0), Fl(t, f_1)) \cdot (Fl(t, f_0) - Fl(t, f_1))$$

leads to

$$M(t, f_0, f_1, f_0 - f_1) = Fl(t, f_0) - Fl(t, f_1).$$

By homogenity in h we obtain the existence of the directional derivatives and its continuity in point and direction at the domain of definition, so the solution is Gateaux-C^1, by induction we can proceed since we can write down an initial value problem for the derivative DFl_t which has the same form as the treated equation on an extended phase space. □

3. Submanifolds and Weak Foliations in Fréchet Spaces

We are interested in the geometry generated by a finite number of vector fields given on an open subset of a Fréchet space E. Therefore we need the notions of finite-dimensional submanifolds (with boundary) of a Fréchet space (see [16] for all details and more). Here and subsequent E denotes a Fréchet space.

A *chart* on a set M is a bijective mapping $u : U \to u(U) \subset E_U$, where E_U is a Fréchet space and $U \subset M$, $u(U) \subset E_U$ is open. We shall denote a chart by (U, u) or $(u, u(U))$. For two charts (U_α, u_α), (U_β, u_β) the chart changing are given by $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. An atlas is a collection of charts such that the U_α form a cover of M and the chart changings are defined on open subsets of the respective Fréchet spaces. A C^∞-atlas is an atlas with smooth chart changings. Two C^∞-atlases are equivalent is their union is an C^∞-atlas. A maximal C^∞-atlas is called a C^∞-structure on M (maximal is understood with respect to some carefully chosen universe of sets). A (smooth) *manifold* is a set together with a C^∞-structure.

A smooth mapping $F : M \to N$ between smooth manifolds is defined in the canonical way, i.e. for any $m \in M$ there is a chart (V, v) with $F(m) \in V$, a chart (U, u) of M with $m \in U$ and $F(U) \subset V$, such that $v \circ F \circ u^{-1}$ is smooth. This is the case if and only if $F \circ c$ is smooth for all smooth curves $c : \mathbb{R} \to M$, where the concept of a smooth curve is easily set upon.

The final topology with respect to smooth curves or equivalently the final topology with respect to all inverses of chart mappings is the canonical topology of the smooth manifold. We assume manifolds to be smoothly Hausdorff (see the discussion in [16], p. 265), i.e. the real valued smooth functions on M separate points.

A *submanifold* N of a Fréchet manifold M is given by a subset $N \subset M$, such that for each $n \in N$ there is a chart $(u, u(U))$, a splitting $E = E' \times E''$ and $u(U) = V \times W$ with $u(N) = V \times \{u(n)''\}$. By a splitting we shall always understand E' and E'' as closed subspaces of E.

An n-dimensional manifold with *boundary* is defined as ordinary manifold except that we take open subsets in a halfspace $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \text{ with } x_n \geq 0\}$. For the notion (without surprises) of smooth mappings on such open sets see any textbook on differential geometry, for example [17]. The boundary $\{x \in \mathbb{R}^n \text{ with } x_n = 0\}$ of the subspace models the boundary ∂N of the manifold N, which is canonically a manifold without boundary of dimension $n-1$. We denote the interior by $N^\circ := N \setminus \partial N$. A submanifold with boundary is given by the analogue submanifold structure.

We restrict ourselves to finite-dimensional submanifolds with boundary M of Fréchet spaces: A *parametrization* of M is an injective, smooth mapping $\phi : U \subset \mathbb{R}^n_+ \to E$ such that $\phi(U) \subset M$ is open in M and $D\phi(u)$ is injective for all $u \in U$. In this case $\phi(U)$ naturally is a submanifold with boundary again. Given a finite dimensional submanifold with boundary M, then the map $u^{-1}|_{V \times u(n)''} : V \to E$ is a parametrization of M. The *tangent space* $T_m M$ of a finite dimensional submanifold with boundary is defined by parametrizations: given a parametrization ϕ of M with $m \in \phi(U)$, then $T_{\phi(u)} M := D\phi(u)(\mathbb{R}^n)$ for $u \in U$. The tangent space $T_r M$ is certainly independent of the chosen parametrization, since it is equally given at interior points by the space of all vectors $c'(0)$ with $c : \mathbb{R} \to E$ smooth, $c(\mathbb{R}) \subset M$ and $c(0) = r$ by the submanifold property.

Therefore a smooth map $F : M \to N$, where M and N are submanifolds defines a linear map $T_r F : T_r M \to T_{F(r)} N$ via $c'(0) \mapsto (F \circ c)'(0)$, which is given through $DF(r) \cdot c'(0)$.

Lemma 3.1 (Submanifolds by Parametrization)**.** *Let E be a Fréchet space and $\phi : U \subset \mathbb{R}^n_+ \to E$ a smooth immersion, i.e. for $u \in U$ the map $D\phi(u)$ is injective, then for any $u_0 \in U$ there is a small open neighborhood V of u_0 such that $\phi(V)$ is a submanifold with boundary of E and $\phi|_V$ is a parametrization.*

Proof. We assume – by translation – $\phi(u_0) = 0$, since it is a local result. Given a linear basis $e_1, ..., e_n$ of $\mathbb{R}^n$, we get linearly independent vectors $D\phi(u_0)(e_i) =: f_i \in E$. We choose $l_1, ..., l_m$ linearly independent linear functionals, such that $l_i(f_j) = \delta_{ij}$ and get a splitting $E = E' \times E''$ with $\dim E' = n$ via $E'' := \cap_{i=1}^m \ker l_i$. The projection on the first variable p_1 induces a local diffeomorphism $p_1 \circ \phi$ on a small open neighborhood V of $u_0 \in U$ by the classical inverse function theorem and the extension result in Theorem 2.7. The inverse is denoted by $\psi : V' \subset E' \to V$. Now we construct a new diffeomorphism

$$\eta(u, f'') = (p_1 \circ \phi(u), f'' + p_2 \circ \phi(u))$$

on $V \times W''$, which is invertible by the above considerations:

$$\eta^{-1}(g', g'') = (\psi(g'), g'' - p_2 \circ \phi(\psi(g'))),$$

η^{-1}defines a submanifold chart for $\phi(V)$ since

$$\eta^{-1}(\phi(u)) = (u, 0)$$

for $u \in V$ by definition. □

Definition 3.2. *A vector field X on a open subset $U \subset E$ of a Fréchet space is a smooth map $X : U \to E$. The set of all vector fields on U is denoted by $\mathfrak{X}(U)$. Given a diffeomorphism $F : U \to V$, i.e. F and F^{-1} are smooth, the map*

$$(F^*Y)(f) := DF(f)^{-1}(Y(F(f)))$$

is well defined for $Y \in \mathfrak{X}(V)$ and defines a bounded linear isomorphism $F^* : \mathfrak{X}(V) \to \mathfrak{X}(U)$ by cartesian closedness (see Theorem 2.5). It is called the pull-back *of vector fields, furthermore $F_* := (F^*)^{-1}$ is called the* push forward. *The* Lie bracket *of two vector fields $X, Y \in \mathfrak{X}(U)$ is defined by the following formula:*

$$[X, Y](f) = DX(f) \cdot Y(f) - DY(f) \cdot X(f)$$

and is a bounded, skew-symmetric bilinear map from $\mathfrak{X}(U) \times \mathfrak{X}(U)$ into $\mathfrak{X}(U)$.

We can treat the pull back as in finite dimensional analysis due to convenient calculus. In the Gateaux approach we are forced to formulate each of these results by point evaluations. Nevertheless it is natural to talk of analytic properties of the objects themselves.

Proposition 3.3. *Let $U \subset E$ be an open subset. Given two vector fields $X, Y \in \mathfrak{X}(U)$, where X admits a local flow $Fl^X : I \times U \to E$, then*

$$[X, Y] = \frac{d}{dt}(Fl^X_{-t})^* Y|_{t=0}.$$

Furthermore for any diffeomorphisms $F : U \to V$, $G : V \to W$

$$F^*[X, Y] = [F^*X, F^*Y]$$

and

$$(G \circ F)^* = F^* \circ G^*, \quad (G \circ F)_* = G_* \circ F_*.$$

Consequently the pull back is a bounded Lie algebra isomorphism, since vector fields constitute a Lie algebra with the Lie bracket. Finally we obtain the useful formula for a smooth map $H : S \to U$, where $S \subset E$ is open:

$$\frac{d}{dt} F \circ Fl^X_t \circ H = (F_*X)(F \circ Fl^X_t \circ H),$$

where we only assume that X generates a semiflow $Fl^X : I \times U \to E$.

Proof. (see [16], 32.15) We can calculate directly with the flow Fl^X for the vector field X

$$\begin{aligned}
\frac{d}{dt}(Fl^X_{-t})^* Y(f)|_{t=0} &= \frac{d}{dt} DFl^X_t(Fl^X_{-t}(f)) \cdot Y(Fl^X_{-t}(f))|_{t=0} \\
&= DX(f) \cdot Y(f) - D^2 Fl^X_0(f)(X(f), Y(f)) - DY(f) \cdot X(f) \\
&= [X, Y](f)
\end{aligned}$$

for $f \in U$. We applied the flow property $(Fl_{-t})^{-1} = Fl_t$ for small t and the chain rule of convenient analysis. The interchange of $\frac{d}{dt}$ and D is possible due to the symmetry of second derivatives. The two following properties are clear by calculating both sides directly. The last equation can is proved by

$$\frac{d}{dt} F \circ Fl_t^X(H(f)) = DF\left(Fl_t^X(H(f))\right) \cdot X(Fl_t^X(H(f))) = (F_* X)\left(F(Fl_t^X(H(f)))\right)$$

for $f \in U$ by the definition of the push forward. □

The following crucial lemma collects algebraic properties of Banach map vector fiels.

Lemma 3.4. *Let U be an open set in a Fréchet space E, then $\mathcal{B}(U)$ is a subalgebra with respect to the Lie bracket. Let A be a bounded linear operator on E, then $[A, \mathcal{B}(U)] \subset \mathcal{B}(U)$. Consequently the Lie algebra $L(E)$ acts on $\mathcal{B}(U)$ by the Lie bracket.*

Proof. Given two Banach maps P_1 and P_2, $DP_1(f) \cdot P_2(f) = DQ_1(R_1(f)) \cdot DR_1(f) \cdot P_2(f)$ holds, which can be written as composition of $DQ_1(v) \cdot w$ for $v, w \in B$ and $(R_1(f), DR_1(f) \cdot P_2(f))$ for $f \in U$. So the Lie bracket lies in $\mathcal{B}(U)$. Given $A \in L(E)$, we see that $AP_1(f) - DP_1(f) \cdot Af$ is a Banach map by an obvious decomposition. □

We denote by $\langle \ldots \rangle$ the generated vector space over the reals $\mathbb{R}$. which means that D_f is vector space generated by the set S of local vector fields at $f \in U$:

Definition 3.5. *Let E be a Fréchet space, U an open subset. A* distribution *on U is a collection of vector subspaces $D = \{D_f\}_{f \in U}$ of E. A vector field $X \in \mathfrak{X}(U)$ is said to take values in D if $X(f) \in D(f)$ for $f \in U$. A distribution D on U is said to be* involutive *if for any two locally given vector fields X, Y with values in D the Lie bracket $[X, Y]$ has values in D.*

A distribution is said to have constant rank *if $\dim_{\mathbb{R}} D_f$ is locally constant $f \in U$. A distribution is called* smooth *if there is a set S of local vector fields on U such that*

$$D_f = \langle \{X(f) | (X : U_X \to E) \in S \text{ and } f \in U_X\} \rangle .$$

We say that the distribution admits local frames *on U if for any $f \in U$ there is an open neighborhood $f \in V \subset U$ and n smooth, pointwise linearly independent vector fields $X_1, ..., X_n$ on V with*

$$\langle X_1(g), ..., X_n(g) \rangle = D_g$$

for $g \in V$.

Remark 3.6. *Given a distribution D on U generated by a set of local vector fields S, such that the dimensions of D_f are bounded by a fixed constant N. Let $f \in U$ be a point with maximal dimension $n_f = \dim_{\mathbb{R}} D_f$, then there are n_f smooth local vector fields $X_1, ..., X_{n_f} \in S$ with common domain of definition U' such that*

$$\langle X_1(f), ..., X_n(f) \rangle = D_f .$$

Choosing n_f continuous linear functionals $l_1, ..., l_{n_f} \in E'$ with $l_i(X_j(f)) = \delta_{ij}$, then the continuous mapping $M : U' \to L(\mathbb{R}^{n_f})$, $\quad g \mapsto (l_i(X_j(g)))$ has range in the invertible matrices in a small neighborhood of f. Consequently in this neighborhood the dimension of D_g is at least n_f. It follows by maximality of n_f that it is exactly n_f. In particular the distribution admits a local frame at f.

The concept of weak foliations will be perfectly adapted to the FDR-problem:

Definition 3.7. *A* weak foliation $\mathcal{F}$ *of dimension* n *on an open subset* U *of a Fréchet space* E *is a collection of submanifolds with boundary* $\{M_r\}_{r\in U}$ *such that*

i) *For all* $r \in U$ *we have* $r \in M_r$ *and the dimension of* M_r *is* n.

ii) *The distribution*

$$D(\mathcal{F})(f) := \langle T_f M_r \text{ for all } r \in U \text{ with } f \in M_r \rangle$$

has dimension n *for all* $f \in U$, *i.e. given* $f \in U$ *the tangent spaces* $T_f M_r$ *agree for all* $M_r \ni f$. *This distribution is called the* tangent distribution *of* $\mathcal{F}$.

Given any distribution D *we say that* D *is* tangent *to* $\mathcal{F}$ *if* $D(f) \subset D(\mathcal{F})(f)$ *for all* $f \in U$.

Classically one is interested in the existence of tangent weak foliations for a given distribution of minimal dimension m. Therefore we shall need the following essential lemma.

Proposition 3.8. *Let* D *be an involutive, smooth distribution of constant rank* n *on an open subset* U *of a Fréchet space* E. *Let* X *and* Y *be vector fields with values in* D *and let* X *admit a local flow, then*

$$(Fl_t^X)^*(Y)(f) \in D_f$$

for $f \in U$, *where it is defined.*

Proof. Given a local frame $X_1, ..., X_n$ on an open neighborhood V of f_0, we have by involutivity that $[X, X_i] = \sum_{k=1}^n P_i^k X_k$. Notice that P_i^k are smooth functions locally on V. Given $g \in V$ and n linear independent functionals l_m such that $l_m(X_j(g)) = \delta_{mj}$, then

$$l_m([X, X_i](f)) = \sum_{k=1}^n P_i^k(f) l_m(X_k(f))$$

for all $f \in V$. Since the matrix $M(f) := (l_m(X_k(f)))$ is invertible at g and has smooth entries, it is invertible on an open neighborhood of g, and the inverse has smooth entries. The smooth inverse matrix applied to the left hand vector proves smoothness of P_i^k. With the above formula and Lemma 3.3 we get

$$\begin{aligned}
\frac{d}{dt}(Fl_t^X)^*(X_i) &= -\frac{d}{ds}(Fl_{t-s}^X)^*(X_i)|_{s=0} \\
&= -\frac{d}{ds}(Fl_{-s}^X)^*(Fl_t^X)^*(X_i)|_{s=0} \\
&= -[X, (Fl_t^X)^*(X_i)] \\
&= -(Fl_t^X)^*[X, X_i] \\
&= -\sum_{k=1}^n P_i^k \circ Fl_t^X (Fl_t^X)^*(X_k)
\end{aligned}$$

which is a linear equation with time-dependent real valued coefficients $g_i^k(t) := -P_i^k(Fl_t^X(f))$ on E^n for $h_i(t) := (Fl_t^X)^*(X_i)(f)$ at any point f in an open neighborhood of f_0, namely

$$\frac{d}{dt}h_i(t) = \sum_{k=1}^n g_i^k(t) h_k(t)$$

with $h_i(0) \in E$. The solution of this differential equation is given by the classical time dependent flow associated to the smooth matrix $t \mapsto (g_i^k(t))$ applied to a vector in E^n. If we are given a flow for a vector field the solutions are unique due to Lemma 2.11. Consequently provided the initial values lie in D_f^n the solution lies in D_f^n for small times by the subspace property, but $(Fl_0^X)^*(X_i)(f) = X_i(f) \in D_f$ for $f \in U$. □

Theorem 3.9. *Let D be an smooth distribution of constant rank n on an open subset U of a Fréchet space E. Assume that for any point f_0 the distribution admits a local frame of vector fields $X_1, ..., X_n$, where $X_1, ..., X_{n-1}$ admit local flows $Fl_t^{X_i}$ and X_n admits a local semiflow $Fl_t^{X_n}$. Then D is involutive if and only if it is tangent to an n-dimensional weak foliation.*

Proof. We suppose that D is involutive. Let $f_0 \in U$ be fixed, then there are on some open set V, with $f_0 \in V$, n linearly independent vector fields $X_1, ..., X_n$ generating each D_f for $f \in V$. Furthermore local flows $Fl^{X_i} :]-\varepsilon, \varepsilon[\times V_0 \to V$ for $i = 1, ..., n-1$ and a local semiflow $Fl^{X_n} : [0, \varepsilon[\times V_0 \to V$ exist for $V_0 \subset V$ open with $f_0 \in V_0$ and some $\varepsilon > 0$. We define the candidate parametrization $\alpha(u, r) = Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}(r)$ on $W_1 \times V_1$, where V_1 open with $f_0 \in V_1$ and $W \subset \mathbb{R}_+^n$ open, convex around 0. This is possible due to continuity of the (semi-)flows.

We can calculate the tangent spaces on the canonical basis of $\mathbb{R}^n$: by cartesian closedness we obtain the derivative of $Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}$ with respect to u_i immediately by Proposition 3.3

$$\frac{\partial}{\partial u_i} Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n} = \left((Fl_{u_1}^{X_1})_* ... (Fl_{u_{i-1}}^{X_{i-1}})_* X_i \right) \circ Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_n}^{X_n}$$

with $F = Fl_{u_1}^{X_1} \circ ... \circ Fl_{u_{i-1}}^{X_{i-1}}$ and $H = Fl_{u_{i+1}}^{X_{i+1}} \circ ... \circ Fl_{u_n}^{X_n}$. So we arrive at

$$D_1\alpha(u, r)(e_i) = \left((Fl_{u_1}^{X_1})_* ... (Fl_{u_{i-1}}^{X_{i-1}})_* X_i \right) (\alpha(u, r))$$

for $1 \leq i \leq n$. By Proposition 3.8 these vectors lie in $D_{\alpha(u,r)}$ and are linearly independent for $u \in W_2$ and $r \in V_2$ with $W_2 \subset W_1$ open, convex around 0 and $V_2 \subset V_1$ open around f_0. They generate the distribution in a small neighborhood by a dimension argument. It is essential that the first $n-1$ vector fields admit a local flow. So we obtain a family of tangential manifolds for D. Each parametrization for fixed r defines locally a smooth submanifold with boundary $\alpha(u_1, .., u_{n-1}, 0, r)$ by redoing the proof of Lemma 3.1 with continuously parametrized immersions. Consequently we can find an open set $V_2 \subset E$ and $W_2 \subset \mathbb{R}_+^n$ such that $\alpha|_{W_2 \times V_2}$ defines a weak foliation with tangent distribution D.

Suppose now that there is a weak foliation $\mathcal{F} = \{M_r\}_{r \in U}$ of dimension n. We apply the above notation on a subset V, where we have a local frame $X_1, ..., X_n$ with the stated properties. Given $r \in V$, there exists a finite dimensional submanifold with boundary M_r and $X_i(f) \in T_f M_r$ for $1 \leq i \leq n$ at any interior point $f \in {M_r}^\circ$. By Lemma 2.11 the local flows $Fl_t^{X_i}$ restrict locally to ${M_r}^\circ$ for $1 \leq i \leq n-1$ since the vector fields X_i are tangent to M_r and admit local flows around any interior point of M_r.

So for small t and $Y \in \mathfrak{X}(U)$ with values in D the pull back $(Fl_t^{X_i})^* Y(f)$ takes values in D_f for f in the interior of M_r, since it can be calculated as pull back of the restriction $Fl_t^{X_i}|_{M_r}$. The smooth map $t \mapsto (Fl_t^{X_i})^* Y(f)$ takes values in the finite dimensional space D_f, so the derivative lies there by closedness, but the derivative equals $[X_i, Y](f)$ by Proposition 3.3.

We do not know whether $r \in M_r$ lies on the boundary of M_r or not, but we can approximate r by interior points $f_m \to r$ as $m \to \infty$. At r the vector space D_r has a basis $X_1(r), ..., X_n(r)$: given n linearly independent linear functionals $l_1^r, ..., l_n^r$ with $l_i^r(X_j(r)) = \delta_{ij}$. We can choose V sufficiently small such that the smooth matrix $f \mapsto M(f) := (l_i(X_j(f))$ is invertible for $f \in V$, hence the inverse matrix $N(f) := M(f)^{-1}$ defines linear functionals $l_i^f := \sum_{k=1}^n N(f)_{ik} l_k^r$ for $1 \leq i \leq n$, which depend smoothly on f and satisfy $l_i^f(X_j(f)) = \delta_{ij}$ for $f \in V$. The associated projections $p^f := id - \sum_{k=1}^n l_k^f X_k(f)$ detect whether a vector field on V takes values in D_f or not: for $Z \in \mathfrak{X}(V)$ we obviously have $p^f(Z(f)) = 0$ if and only if $Z(f) \in D_f$, for $f \in V$. However, $p^{f_m}([X_i, Y](f_m)) = 0$ for $m \geq 1$ as calculated above, so by continuity $p^r([X_i, Y](r)) = 0$ for $1 \leq i \leq n-1$.

Hence $[X_i, X_j]$ takes values in D locally for $1 \leq i, j \leq n$ and D is therefore involutive since we can due the procedure everywhere on U. □

Remark 3.10. *For details on Frobenius theorems in the classical setting see* [15]. *The phenomenon that there is no Frobenius chart is due to the fact that there is one vector field among the vector fields X_1,...,X_n (generating the distribution D) admitting only a local semiflow. If all of them admitted flows, there would exist a Frobenius chart, which can be given by a construction outlined in* [23]. *The non-existence of a Frobenius-chart means that the leafs cannot be parallelized, since they follow semiflows, which means that "gaps" between two leafs can occur and leafs can touch. This is an infinite dimensional phenomenon, which does not appear in finite dimensions.*

4. Finite-dimensional Realizations for HJM Models

In this section we apply the preceding results to characterize those HJM models that satisfy the appropriate Frobenius condition (see condition (F) below), which is essentially equivalent to the existence of FDRs at any inital curve. We will demonstrate that this condition yields a very particular geometry of the invariant submanifolds – loosely speaking, each of them is a band of copies of an affine submanifold.

Remark 4.1. *Although we subsequently focus on HJM models, many arguments can be carried over to more general stochastic equations (1.1) in the spirit of Da Prato and Zabczyk* [5].

First we provide the rigorous setup for HJM models, summarizing [8]. The Hilbert space H of forward curves is characterized by the properties

- **(H1):** $H \subset C(\mathbb{R}_{\geq 0}; \mathbb{R})$ with continuous embedding (that is, for every $x \in \mathbb{R}_{\geq 0}$, the pointwise evaluation $\mathrm{ev}_x : h \mapsto h(x)$ is a continuous linear functional on H), and $1 \in H$ (the constant function 1).
- **(H2):** The family of right-shifts, $S_t f = f(t + \cdot)$, for $t \in \mathbb{R}_{\geq 0}$, forms a strongly continuous semigroup S on H.
- **(H3):** There exists a closed subspace H_0 of H such that
$$\mathcal{S}(f, g)(x) := f(x) \int_0^x g(\eta)\, d\eta,$$
defines a continuous bilinear mapping $\mathcal{S} : H_0 \times H_0 \to H$.

We write shortly $\mathcal{S}(f)$ for $\mathcal{S}(f,f)$. We assume that the volatility coefficients σ_j map H into H_0. Then the HJM drift coefficient

$$\alpha = \alpha_{HJM} := \sum_{j=1}^{d} \mathcal{S}(\sigma_j) : H \to H \tag{4.1}$$

is a well-defined map. Hence an HJM model is uniquely determined by the specification of its volatility structure $\sigma = (\sigma_1, \dots, \sigma_d)$.

As an illustration we shall always have the following example in mind (see [8, Section 5]).

Example 4.2. *Let $w : \mathbb{R}_{\geq 0} \to [1, \infty)$ be a non-decreasing C^1-function such that $w^{-1/3} \in L^1(\mathbb{R}_{\geq 0})$. We may think of $w(x) = e^{\alpha x}$ or $w(x) = (1 + x)^\alpha$, for $\alpha > 0$ or $\alpha > 3$, respectively. The space H_w consisting of absolutely continuous functions h on $\mathbb{R}_{\geq 0}$ and equipped with the norm*

$$\|h\|_w^2 := |h(0)|^2 + \int_{\mathbb{R}_{\geq 0}} \left| \frac{d}{dx} h(x) \right|^2 w(x)\, dx$$

is a Hilbert space satisfying (H1)–(H2). Property (H3) is satisfied for $H_0 = H_{w,0} := \{h \in H_w \mid \lim_{x\to\infty} h(x) = 0\}$.

The operator A is the generator of the shift semigroup S. It is easy to see that $D(A) \subset \{h \in H \cap C^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \mid (d/dx)h \in H\}$ and $Ah = (d/dx)h$. Without much loss of generality we shall in fact assume

(H4): $D(A) = \{h \in H \cap C^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \mid (d/dx)h \in H\}$.

Also (H4) is satisfied for the spaces H_w from Example 4.2.

Denote by $A_0 : D(A_0) \subset H_0 \to H_0$ the restriction of A to H_0. That is, $D(A_0) = \{h \in D(A) \cap H_0 \mid Ah \in H_0\}$. The definition of the Fréchet space $D(A_0^\infty) := \cap_{n\in\mathbb{N}} D(A_0^n)$ is obvious. The next result follows immediately from (H1), (H3) and (H4).

Lemma 4.3. *For any $f, g \in D(A_0)$ we have $\mathcal{S}(f,g) \in D(A)$ and*

$$A\mathcal{S}(f,g) = \mathcal{S}(Af,g) + \mathcal{S}(f,Ag) + f \,\mathrm{ev}_0(g).$$

Hence $\mathcal{S} : D(A_0^\infty) \times D(A_0^\infty) \to D(A^\infty)$ is a continuous bilinear mapping.

The preceding specifications for σ are still too general for concrete implementations. We actually have the idea of σ being sensitive with respect to functionals of the forward curve. That is, $\sigma_j(h) = \phi_j(\ell_1(h), \dots, \ell_p(h))$, for some $p \geq 1$, where $\phi_j : \mathbb{R}^p \to D(A_0^\infty)$ is a smooth map and $\ell_1, \dots, \ell_p$ denote continuous linear functionals on H (or even on $C(\mathbb{R}_{\geq 0}; \mathbb{R})$). We may think of $\ell_i(h) = (1/x_i) \int_0^{x_i} h(\eta)\, d\eta$ (benchmark yields) or $\ell_i(h) = \mathrm{ev}_{x_i}(h)$ (benchmark forward rates). This idea is (generalized and) expressed in terms of the following regularity and non-degeneracy assumptions:

(A1): $\sigma_j = \phi_j \circ \ell$ where $\ell \in L(H, \mathbb{R}^p)$, for some $p \in \mathbb{N}$, and $\phi_j : \mathbb{R}^p \to D(A_0^\infty)$ are smooth and pointwise linearly independent maps, $1 \leq j \leq d$. Hence $\sigma : H \to D(A_0^\infty)^d$ is a Banach map (see Definition 2.8).

(A2): For every $q \geq 0$, the linear map $(\ell, \ell \circ A, \dots, \ell \circ A^q) : D(A^\infty) \to \mathbb{R}^{p(q+1)}$ is open.

(A3): A is unbounded; that is, $D(A)$ is a strict subset of H. Equivalently, $A : D(A^\infty) \to D(A^\infty)$ is not a Banach map (see Lemma 2.12).

We believe that this setup is flexible enough to capture any reasonable HJM model. Assumption (A2) is essential for the strong characterization result in Theorem 4.10 below. Intuitively, (A2) says that the following interpolation problem is well-posed on $D(A^\infty)$: given a smooth curve $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$, for any finite number of data of the form $\mathcal{I} = (\ell(g), \ell((d/dx)g), \dots, \ell((d/dx)^q g)) \in \mathbb{R}^{p(q+1)}$ we can find an interpolating function $h \in D(A^\infty)$ with $(\ell(h), \dots, \ell \circ A^q(h)) = \mathcal{I}$. Notice, however, that degenerate examples such as the following are excluded: let $p = 3$ and $\ell(h) = (\mathrm{ev}_0(h), \mathrm{ev}_1(h), \int_0^1 h(x)\,dx)$. Then $\ell \circ A(h) = (\mathrm{ev}_0(Ah), \mathrm{ev}_1(Ah), \mathrm{ev}_1(h) - \mathrm{ev}_0(h))$. Thus the rank of $(\ell, \ell \circ A)$ is at most 5, and $(\ell, \ell \circ A) : D(A^\infty) \to \mathbb{R}^6$ cannot be an open map.

Combining Lemma 4.3 and (A1) yields

Lemma 4.4. *$\mathcal{S}(\sigma_j) : H \to D(A^\infty)$ is a Banach map, for every $1 \leq j \leq d$, hence also α_{HJM}.*

By the discussion after the proof of Lemma 2.11, the semigroup S leaves $D(A^\infty)$ invariant and is smooth on $D(A^\infty)$. Hence by (A1) and Lemma 4.4 the assumptions of Theorem 2.13 are satisfied, and the vector field $h \mapsto \mu(h) = Ah + \alpha_{HJM}(h) - (1/2)\sum_{j=1}^d D\sigma_j(h)\sigma_j(h)$, see (1.2), admits a local semiflow on $D(A^\infty)$.

Lemma 4.5. *Let $X_1, \dots, X_k$ be linearly independent Banach maps on an open set U in $D(A^\infty)$, for some $k \in \mathbb{N}$. Then the set*

$$\mathcal{N} = \{h \in U \mid \mu(h) \in \langle X_1(h), \dots, X_k(h)\rangle\}$$

is closed and nowhere dense in U.

Proof. Clearly, $\mathcal{N}$ is closed by continuity of μ and $X_1, \dots, X_k$. Now suppose there exists a set $V \subset \mathcal{N}$ which is open in $D(A^\infty)$. For every $h \in V$ there exist unique numbers $c_1(h), \dots, c_k(h)$ such that

$$\mu(h) = \sum_{j=1}^k c_j(h) X_j(h). \tag{4.2}$$

We can choose linear functionals $\xi_1, \dots, \xi_k$ on $D(A^\infty)$ such that the $k \times k$-matrix $M_{ij}(h) := \xi_i(X_j(h))$ is smooth and invertible on V (otherwise we choose a smaller open subset V). Hence

$$c_i(h) = \sum_{j=1}^k M_{ij}^{-1}(h)\xi_j(\mu(h))$$

are smooth functions on V. Then (4.2) implies that A is a Banach map on V. But this contradicts (A3), whence the claim. □

The vector fields $\mu, \sigma_1, \dots, \sigma_d$ induce two distributions on $D(A^\infty)$: their linear span $D = \langle \mu, \sigma_1, \dots, \sigma_d\rangle$, and the Lie algebra D_{LA} generated by all multiple Lie brackets of these vector fields. As a consequence of (A1) and Lemma 4.5 there exists a closed and nowhere dense set $\mathcal{N}$ in $D(A^\infty)$ such that

$$\dim D_{LA} \geq \dim D = d + 1 \quad \text{on } D(A^\infty) \setminus \mathcal{N}. \tag{4.3}$$

Remark 4.6. *The preceding observation proves a conjecture in* [3]*, namely that every nontrivial generic short rate model is of dimension 2 (see* [3, Remark 7.1]*).*

The existence of an FDR at some initial point is a singular event, in general. The concept of a finite-dimensional weak folitation is thus appropriate for the FDR-problem. By Definition 3.7 an n-dimensional weak foliation $\mathcal{F}$ on some open subset U in $D(A^\infty)$ is a collection $\{\mathcal{M}_h\}_{h\in U}$ of n-dimensional submanifolds with boundary in $D(A^\infty)$. Notice that by the canonical embedding $D(A^\infty) \hookrightarrow H$ every $\mathcal{M}_h$ is also a submanifold with boundary in H.

Remark 4.7. *We are thus looking for FDRs in $D(A^\infty)$. This seems to be a restriction since the original HJM model (1.1) is defined on H. However, as it was stated in Theorem 1.3, any finite-dimensional invariant submanifold $\mathcal{M}$ in H lies* necessarily *in $D(A)$. Under the preceding assumptions on σ, we show in* [11] *that* necessarily *$\mathcal{M} \subset D(A^\infty)$ (as a set), and if* $\dim \mathcal{M} = d+1$ *then $\mathcal{M}$ is even a submanifold in $D(A^\infty)$. From this point of view the following results are essentially optimal.*

The following is a modification of the necessary condition in Theorem 3.9.

Proposition 4.8. *Let U be an open set in $D(A^\infty)$, and $\mathcal{F}$ an n-dimensional weak foliation on U, for some $n \in \mathbb{N}$. If D is tangent to $\mathcal{F}$ then $D_{LA} \subset D(\mathcal{F})$ on U.*

Proof. Let X and Y be vector fields on U with values in $D(\mathcal{F})$, and such that X admits a local flow on U. Then it follows as in the second part of the proof of Theorem 3.9 that $[X, Y]$ takes values in $D(\mathcal{F})$ on U. Hence by the very defintion of D_{LA}, Lemma 3.4 and Theorem 2.10 we obtain, by induction, that $D_{LA} \subset D(\mathcal{F})$ on U, and the proposition is proved. □

Let U denote an open connected set in $D(A^\infty)$ in what follows. Proposition 4.8 tells us that boundedness of $\dim D_{LA}$ on U is a necessary condition for the existence of a finite-dimensional weak foliation on U. To avoid difficult to analyse degenerate situations where $\dim D_{LA}$ is not constant on U, we shall only consider the non-degenerate case. This is our appropriate Frobenius condition

(F): D_{LA} has constant finite dimension N_{LA} on U.

Here and subsequently, we let (F) be in force. In view of (4.3) we have $N_{LA} \geq d+1$.

Proposition 4.9. *We have*

$$\mu(h) \notin \langle \sigma_1(h), \dots, \sigma_d(h) \rangle, \quad \forall h \in U. \tag{4.4}$$

Moreover, for any $h_0 \in U$ there exists an open neighborhood V and Banach maps $X_{d+1}, \dots, X_{N_{LA}-1}$ on V such that

$$D_{LA} = \langle \mu, \sigma_1, \dots, \sigma_d, X_{d+1}, \dots, X_{N_{LA}-1} \rangle \quad \text{on } V.$$

In particular, D_{LA} is tangent to an N_{LA}-dimensional weak foliation $\mathcal{F}$ on U.

Proof. Suppose $\mu(h_0) \in \langle \sigma_1(h_0), \dots, \sigma_d(h_0) \rangle$, for some $h_0 \in U$. By the definition of D_{LA} and Lemma 3.4 there exist $N_{LA} - d$ Banach maps $X_{d+1}, \dots, X_{N_{LA}}$ on U such that

$$D_{LA}(h) = \langle \sigma_1(h), \dots, \sigma_d(h), X_{d+1}(h), \dots, X_{N_{LA}}(h) \rangle,$$

for $h = h_0$, and hence for all h in a neighborhood of h_0, by continuity. But this implies that $\mu(h)$ lies in the span of Banach maps, for all h in an open set. This contradicts Lemma 4.5, whence (4.4). The rest of the proposition follows by Remark 3.6 and Theorem 3.9. □

In the following theorem we provide the full classification of $\mathcal{F}$.

Theorem 4.10. *Under the above assumptions there exist linearly independent constant vectors* $\lambda_1, \dots, \lambda_{N_{LA}-1} \in D(A^\infty)$ *such that* $D_{LA} = \langle \mu, \lambda_1, \dots, \lambda_{N_{LA}-1} \rangle$ *and*

$$\sigma_j \in \langle \lambda_1, \dots, \lambda_{N_{LA}-1} \rangle, \quad 1 \le j \le d, \tag{4.5}$$

on U.

Proof. Define the smooth map $\Gamma := \sum_{j=1}^d \Gamma_j : \mathbb{R}^p \to D(A^\infty)$ by

$$\Gamma_j(y) := \mathcal{S}(\phi_j(y)) - \frac{1}{2} D\phi_j(y) \left(\ell(\phi_j(y)) \right).$$

So that we can write $\mu(h) = Ah + \Gamma(\ell(h))$. Let $1 \le i, j \le d$. We already know from Lemma 3.4 that $[\sigma_i, \sigma_j]$ and $[\mu, \sigma_j]$ are Banach maps. In fact, a straightforward calculation yields the decompositions

$$[\sigma_i, \sigma_j] = \phi_{ij} \circ \ell : D(A^\infty) \to \mathbb{R}^p \to D(A^\infty),$$
$$[\mu, \sigma_j] = \delta_j \circ (\ell, \ell \circ A) : D(A^\infty) \to \mathbb{R}^{2p} \to D(A^\infty),$$

for smooth maps $\phi_{ij} : \mathbb{R}^p \to D(A^\infty)$ and $\delta_j : \mathbb{R}^{2p} \to D(A^\infty)$. Here the linearity of ℓ is essential, see (A1). Now fix $h_0 \in U$. By induction of the preceding argument and Proposition 4.9 there exists an open neighborhood V of h_0, an integer $q \ge -1$, and linearly independent Banach maps $X_1, \dots, X_{N_{LA}-1}$ with decomposition

$$X_i = \Psi_i \circ (\ell, \dots, \ell \circ A^q) : D(A^\infty) \to \mathbb{R}^{p(q+1)} \to D(A^\infty), \tag{4.6}$$

for smooth maps $\Psi_i : \mathbb{R}^{p(q+1)} \to D(A^\infty)$ such that

$$D_{LA} = \langle \mu, X_1, \dots, X_{N_{LA}-1} \rangle \quad \text{on } V. \tag{4.7}$$

Notice that the case $q = -1$ is included in a consistent way: it simply means that X_i in (4.6) is constant.

There exists a minimal integer, still denoted by q, with the above properties. We shall show that $q = -1$.

We argue by contradiction and suppose that $q \ge 0$. We claim that then there exists smooth maps $\tilde{\Psi}_i : \mathbb{R}^{pq} \to D(A^\infty)$ such that we can replace X_i in (4.7) with $\tilde{X}_i = \tilde{\Psi}_i \circ (\ell, \dots, \ell \circ A^{q-1})$. Indeed, since $[\mu, X_i]$ is a Banach map on V (see Lemma 3.4), for every $h \in V$ there exists numbers $c_{ij}(h)$ such that

$$[\mu, X_i](h) = \sum_{j=1}^{N_{LA}-1} c_{ij}(h) X_j(h), \quad 1 \le i \le N_{LA} - 1. \tag{4.8}$$

As in the proof of Lemma 4.5 we find linear functionals $\xi_1, \dots, \xi_{N_{LA}-1}$ on $D(A^\infty)$ such that the $(N_{LA} - 1) \times (N_{LA} - 1)$-matrix $M_{ij}(y) := \xi_i(\Psi_j(y))$ is smooth and invertible on $W := (\ell, \dots, \ell \circ A^q)(V)$, which is an open set in $\mathbb{R}^{p(q+1)}$ by (A2). By explicit calculation we obtain

$$[\mu, X_i] = \Delta_i \circ (\ell, \dots, \ell \circ A^{q+1}) \tag{4.9}$$

where

$$\begin{aligned} \Delta_i(y, z) &= (A + D\Gamma(y_0) \circ \ell) \cdot \Psi_i(y) \\ &\quad - D\Psi_i(y) \cdot \left(\begin{pmatrix} y_1 \\ \vdots \\ y_q \\ z \end{pmatrix} + \begin{pmatrix} \ell(\Gamma(y_0)) \\ \vdots \\ \vdots \\ (\ell \circ A^q)(\Gamma(y_0)) \end{pmatrix} \right), \end{aligned} \tag{4.10}$$

for $(y,z) = (y_0,\dots,y_q,z) \in \mathbb{R}^{p(q+1)} \times \mathbb{R}^p$. Equating (4.8) and (4.9), applying the functionals ξ_k and inverting gives that

$$(y,z) \mapsto \gamma_{ij}(y,z) := \sum_{k=1}^{N_{LA}-1} M^{-1}_{kj}(y)\xi_k(\Delta_i(y,z))$$

are smooth functions from $W' := (\ell,\dots,\ell\circ A^{q+1})(V)$ into $\mathbb{R}$, and they satisfy $c_{ij}(h) = \gamma_{ij}\circ(\ell,\dots,\ell\circ A^{q+1})(h)$ on V, hence

$$\Delta_i(y,z) = \sum_{j=1}^{N_{LA}-1} \gamma_{ij}(y,z)\Psi_j(y), \quad \forall (y,z)\in W'. \tag{4.11}$$

Differentiating (4.11) with respect to z (which makes sense since W' is open by (A2)) yields, see (4.10),

$$D_{y_q}\Psi_i(y) = \sum_{j=1}^{N_{LA}-1} D_z\gamma_{ij}(y,z)\Psi_j(y), \quad \forall (y,z)\in W'.$$

Arguing again by linear independence of $\Psi_1,\dots,\Psi_{N_{LA}-1}$ we see that the $\mathbb{R}^p$-valued maps

$$D_z\gamma_{ij}(y,z) \equiv: \beta_{ij}(y)$$

depend only on y. We may assume that $W = W_0\times W_1$ where $W_0\subset\mathbb{R}^{pq}$ and $W_1\subset\mathbb{R}^p$ are open such that $(y_0,z_0) := ((\ell,\dots,\ell\circ A^{q-1})(h_0), \ell\circ A^q(h_0)) \in W_0\times W_1$, and W_1 is star-shaped with respect to z_0 (otherwise replace V accordingly). Now let $(y,z)\in W_0\times W_1$ and define $\psi(t) := \Psi((y, z_0 + t(z-z_0)))$. Then there exists an open interval I containing $[0,1]$ such that

$$\begin{aligned} \frac{d}{dt}\psi_i(t) &= \sum_{j=1}^{N_{LA}-1} \left(\beta_{ij}((y,z_0+t(z-z_0)))\cdot(z-z_0)\right)\psi_j(t) \\ \psi_i(0) &= \Psi_i(y,z_0), \quad i=1,\dots,N_{LA}-1, \end{aligned}$$

for $t\in I$. This system of differential equations has a unique solution, which is of the form

$$\psi_i(t) = \sum_{j=1}^{N_{LA}-1} \alpha_{ij}(t)\psi_j(0),$$

for some smooth curves $\alpha_{ij} : I\to\mathbb{R}$. In particular, for $t=1$,

$$\Psi_i(y,z) = \sum_{j=1}^{N_{LA}-1} \alpha_{ij}(1)\Psi_j(y,z_0).$$

This way we find a smooth matrix-valued map, again denoted by (α_{ij}), on $W_0\times W_1$ such that

$$\Psi_i(y,z) = \sum_{j=1}^{N_{LA}-1} \alpha_{ij}(y,z)\Psi_j(y,z_0), \quad \forall (y,z)\in W_0\times W_1.$$

But this implies that μ and the Banach maps $\Psi_j(\cdot,z_0)\circ(\ell,\dots,\ell\circ A^{q-1})$ span the Lie algebra D_{LA} on V. Whence the claim.

But q was supposed to be minimal – a contradiction. Hence $q=-1$; that is, $X_1,\dots,X_{N_{LA}-1}$ in (4.7) can be chosen constant on some neighborhood of h_0. Since

$h_0 \in U$ was arbitrary and U is connected, the theorem now follows by a continuity argument. □

Theorem 4.10 is a global result in so far as it holds for every open connected set U in $D(A^\infty)$ where (F) is satisfied. We now are interested in the question whether U can be chosen to be the entire space $D(A^\infty)$. In other words, whether there exist a priori structural restrictions on the choice of U. In view of (F) and Theorem 4.10 it is clear that U must not intersect with the singular set

$$\Sigma := \{h \in D(A^\infty) \mid \mu(h) \in \langle \lambda_1, \dots, \lambda_{N_{LA}-1} \rangle\}. \tag{4.12}$$

By Lemma 4.5, Σ is closed and nowhere dense in $D(A^\infty)$.

Lemma 4.11. *If (4.5) holds on $D(A^\infty)$, then Σ lies in a finite-dimensional linear subspace $\mathcal{O}$ in $D(A^\infty)$ with $N_{LA} \le \dim \mathcal{O} \le N_{LA} + (N_{LA}-1)^2$.*

Proof. Since σ is continuous, (4.5) holds on H. Assumption (A1) yields

$$\langle \sigma_1(h), \dots, \sigma_d(h) \rangle \subset D(A_0^\infty), \quad \forall h \in H.$$

Hence there exists $d \le d^* \le N_{LA} - 1$ such that (after a change of coordinates if necessary) $\lambda_1, \dots, \lambda_{d^*} \in D(A_0^\infty)$, and

$$\sigma_i(h) = \sum_{j=1}^{d^*} \beta_{ij}(h)\lambda_j, \quad 1 \le i \le N_{LA} - 1, \quad \forall h \in H, \tag{4.13}$$

for smooth functions $\beta_{ij} : H \to \mathbb{R}$. Moreover, $D\sigma_i(h)\sigma_i(h) \in \langle \lambda_1, \dots, \lambda_{N_{LA}-1} \rangle$, for all $h \in H$. By (1.2) hence

$$\Sigma = \{h \in D(A^\infty) \mid \nu(h) \in \langle \lambda_1, \dots, \lambda_{N_{LA}-1} \rangle\},$$

where $\nu := A + \alpha_{HJM}$. Since $\Lambda_{ij} := \mathcal{S}(\lambda_i, \lambda_j)$ is a well-defined element in $D(A^\infty)$, for all $1 \le i, j \le d^*$, we obtain

$$\nu(h) = Ah + \sum_{i,j=1}^{d^*} a_{ij}(h)\Lambda_{ij}, \quad \forall h \in D(A^\infty), \tag{4.14}$$

where $a_{ij}(h) := \sum_{k=1}^{d} \beta_{ki}(h)\beta_{kj}(h)$, see (4.1). Hence $h \in \Sigma$ if and only if there exist real numbers $c_1(h), \dots, c_{N_{LA}-1}(h)$ such that

$$Ah + \sum_{i,j=1}^{d^*} a_{ij}(h)\Lambda_{ij} = \sum_{i=1}^{N_{LA}-1} c_i(h)\lambda_i. \tag{4.15}$$

Let $\mathcal{R}$ be the subspace spanned by $\lambda_1, \dots, \lambda_{N_{LA}-1}$ and $\Lambda_{11}, \dots, \Lambda_{d^*d^*}$, and let I be a set of indices (i,j) such that $\{\lambda_1, \dots, \lambda_{N_{LA}-1}, \Lambda_{ij} \mid (i,j) \in I\}$ is linear independent and spans $\mathcal{R}$. In view of (4.15) it is clear that Σ lies in $\mathcal{O} := A^{-1}(\mathcal{R})$. Since the kernel of A is spanned by 1 (see (H1)), the dimension of $\mathcal{O}$ is $1 + \dim \mathcal{R} = N_{LA} + |I|$. □

Hence the maximal possible choice of U is $D(A^\infty) \setminus \Sigma$. In this case we can say more about Σ.

Lemma 4.12. *Suppose that $U = D(A^\infty) \setminus \Sigma$. Then $h \in \Sigma$ implies*

$$h + \langle \lambda_1, \dots, \lambda_{N_{LA}-1} \rangle \subset \Sigma.$$

Proof. By Theorem 4.10 and since $[\mu, \lambda_i]$ is a Banach map on U (see Lemma 3.4), we have

$$[\mu, \lambda_i](h) = D\mu(h)\lambda_i \in \langle \lambda_1, \ldots, \lambda_{N_{LA}-1} \rangle, \tag{4.16}$$

for all $h \in D(A^\infty) \setminus \Sigma$, and hence for all $h \in D(A^\infty)$, by smoothness of μ. Now let $h \in \Sigma$ and $u \in \mathbb{R}^{N_{LA}-1}$. Using Taylor's formula (Theorem 2.5) we calculate

$$\mu\left(h + \sum_{i=1}^{N_{LA}-1} u_i \lambda_i\right) = \mu(h) + \sum_{i=1}^{N_{LA}-1} u_i \int_0^1 D\mu\left(h + t\sum_{i=1}^{N_{LA}-1} u_i \lambda_i\right)\lambda_i \, dt, \tag{4.17}$$

which lies in $\langle \lambda_1, \ldots, \lambda_{N_{LA}-1} \rangle$ by (4.16), and the lemma follows. □

We now can give the classification of the corresponding HJM models as well.

Theorem 4.13. *Suppose (F) holds on $U = D(A^\infty) \setminus \Sigma$, where Σ is given by (4.12). Then, for every $h_0 \in D(A^\infty)$, there exists an $\mathbb{R}^{N_{LA}-1}$-valued diffusion process Y with $Y_0 = 0$ such that*

$$r_t = Fl_t^\mu(h_0) + \sum_{i=1}^{N_{LA}-1} Y_t^i \lambda_i \tag{4.18}$$

is the unique continuous local solution to (1.1) with $r_0 = h_0$. If $h_0 \in \Sigma$ we can even choose Y such that

$$r_t = h_0 + \sum_{i=1}^{N_{LA}-1} Y_t^i \lambda_i. \tag{4.19}$$

In particular, Σ is locally invariant for (1.1).

The coordinate process Y will be explicitely constructed in the proof below (see (4.24)).

Remark 4.14. *HJM models that satisfy (4.18), or (4.19), are known in the finance literature as* affine term structure models. *Hence Theorem 4.13 can be roughly reformulated in the following way:* HJM models that admit an FDR at every inital point $h_0 \in D(A^\infty)$ are necessarily affine term structure models.

Affine term structure models have been extensively studied in [7], [8], [6] *(see also references therein).*

Proof. By smoothness of σ and μ, (4.5) and (4.16) hold on H and $D(A^\infty)$, respectively. Let $h_0 \in D(A^\infty) \setminus \Sigma$ and $\mathcal{M}_{h_0}$ a leaf of the weak foliation $\mathcal{F}$ through h_0 (see Proposition 4.9). As in the proof of Theorem 3.9 we obtain a parametrization of $\mathcal{M}_{h_0}$ at h_0 by

$$\alpha(u, h_0) = Fl_{u_0}^\mu(h_0) + \sum_{i=1}^{N_{LA}-1} u_i \lambda_i, \quad u = (u_0, \ldots, u_{N_{LA}-1}) \in [0, \varepsilon) \times V, \tag{4.20}$$

for some $\varepsilon > 0$ and some open neighborhood V of 0 in $\mathbb{R}^{N_{LA}-1}$, where Fl^μ is the local semiflow induced by μ. (Strictly speaking, $\alpha(\cdot, h_0)$ is a parametrization of a submanifold with boundary of $\mathcal{M}_{h_0}$.) Now we proceed as in [8, Section 6.4] to find the appropriate coordinate process Y. Using Taylor's formula we obtain as in (4.17)

$$\begin{aligned} \mu(\alpha(u, h_0)) &= \mu\left(Fl_{u_0}^\mu(h_0)\right) + \sum_{i=1}^{N_{LA}-1} \tilde{b}_i(u, h_0)\lambda_i \\ &= D\alpha(u, h_0) \cdot (1, \tilde{b}_1(u, h_0), \ldots, \tilde{b}_{N_{LA}-1}(u, h_0)), \end{aligned} \tag{4.21}$$

where $\tilde{b}_i(\cdot, h_0) : [0, \varepsilon) \times V \to \mathbb{R}$ are smooth maps well specified by

$$\sum_{i=1}^{N_{LA}-1} \tilde{b}_i(u, h_0)\lambda_i := \sum_{i=1}^{N_{LA}-1} u_i \int_0^1 D\mu \left(Fl^{\mu}_{u_0}(h_0) + t \sum_{i=1}^{N_{LA}-1} u_i \lambda_i \right) \lambda_i \, dt.$$

On the other hand, we have

$$\sigma_i(\alpha(u, h_0)) = D\alpha(u, h_0) \cdot (0, \rho_i(u, h_0), 0, \ldots, 0), \quad 1 \leq i \leq d, \tag{4.22}$$

where $\rho_i(\cdot, h_0) = (\rho_{i1}(\cdot, h_0), \ldots, \rho_{id^*}(\cdot, h_0)) : [0, \varepsilon) \times V \to \mathbb{R}^{d^*}$ are smooth maps given by

$$\rho_{ij}(u, h_0) := \beta_{ij}(\alpha(u, h_0)),$$

see (4.13). Define the smooth map $b_i(\cdot, h_0) : [0, \varepsilon) \times V \to \mathbb{R}$ by

$$b_i(u, h_0) := \begin{cases} \tilde{b}_i(u, h_0) + \frac{1}{2} \sum_{j=1}^{d} D\rho_{ji}(u, h_0) \cdot \rho_j(u, h_0), & 1 \leq i \leq d^*, \\ \tilde{b}_i(u, h_0), & d^* < i \leq N_{LA} - 1. \end{cases} \tag{4.23}$$

Then the stochastic differential equation

$$\begin{cases} dY_t^i = b_i((t, Y_t), h_0) \, dt + \sum_{j=1}^{d} \rho_{ji}((t, Y_t), h_0) \, dW_t^j, & 1 \leq i \leq d^*, \\ dY_t^i = b_i((t, Y_t), h_0) \, dt, & d^* < i \leq N_{LA} - 1, \\ \ \ Y_0 = 0, \end{cases} \tag{4.24}$$

has a unique V-valued continuous local solution. By Itô's formula it follows that $r_t = \alpha((t, Y_t), h_0)$ is the unique continuous local solution to (1.1), see [8, Section 6.4], whence the theorem is proved for $h_0 \in D(A^\infty) \setminus \Sigma$.

Now let $h_0 \in \Sigma$. By Lemma 4.12, the $(N_{LA} - 1)$-dimensional affine submanifold $\mathcal{N}_{h_0} := h_0 + \langle \lambda_1, \ldots, \lambda_{N_{LA}} \rangle$ lies in Σ. Since (1.2) and (1.3) are clearly satisfied for all $h \in \mathcal{M} = \mathcal{N}_{h_0}$, Theorem 1.3 gives that $\mathcal{N}_{h_0}$ is locally invariant for (1.1). Replace α in (4.20) by

$$\tilde{\alpha}(u, h_0) := h_0 + \sum_{i=1}^{N_{LA}-1} u_i \lambda_i, \quad u = (u_1, \ldots, u_{N_{LA}-1}) \in \mathbb{R}^{N_{LA}-1},$$

which is a parametrization of $\mathcal{N}_{h_0}$. A similar procedure as above yields an $\mathbb{R}^{N_{LA}-1}$-valued diffusion process Y such that $r_t = \tilde{\alpha}(Y_t, h_0)$ is the unique continuous local solution to (1.1), whence (4.19). (Notice that, by construction, Y is time-homogeneous.) Since $Fl^{\mu}_t(h_0) \in \mathcal{N}_{h_0}$, for all $t \geq 0$ where it is defined, it is easy to modify Y such that (4.18) is satisfied too. □

We remark that the form of the FDRs, (4.18) and (4.24), has already been derived in [2] and [3] under the *assumption* of (4.5) and $D_{LA} = \langle \mu, \lambda_1, \ldots, \lambda_{N_{LA}-1} \rangle$. In this article we provided the sufficiency and *necessity* of these conditions and its consequences in a more general (and appropriate) functional-analytic setup.

We finally show that $\lambda_1, \ldots, \lambda_{N_{LA}-1}$ have to satisfy a functional relation which depends on β_{ij} (see (4.13)). Let the assumptions of Theorem 4.13 be in force. As shown in the proof of Lemma 4.11 we obtain $D_{LA} = \langle \nu, \lambda_1, \ldots, \lambda_{N_{LA}-1} \rangle$ on $D(A^\infty)$.

Hence, as in (4.16), there exist smooth functions c_{ij} on $D(A^\infty)$ such that

$$D\nu(h)\lambda_i = A\lambda_i + \sum_{k,l=1}^{d^*} (Da_{kl}(h)\lambda_i)\,\Lambda_{kl} = \sum_{j=1}^{N_{LA}-1} c_{ij}(h)\lambda_j, \quad \forall h \in D(A^\infty). \tag{4.25}$$

Here we have used the notation from the proof of Lemma 4.11, see (4.14). Now fix $h \in D(A^\infty)$. Expressed as a point-wise equality for functions, (4.25) reads

$$\frac{d}{dx}\left(\lambda_i(x) + \frac{1}{2}\sum_{k,l=1}^{d^*} (Da_{kl}(h)\lambda_i)\,\Lambda_k(x)\Lambda_l(x)\right) = \sum_{j=1}^{N_{LA}-1} c_{ij}(h)\lambda_j(x), \quad \forall x \in \mathbb{R}_{\geq 0},$$

where $\Lambda_i(x) := \int_0^x \lambda_i(\eta)\,d\eta$. Integration with respect to x yields

$$\frac{d}{dx}\Lambda_i(x) = -\frac{1}{2}\sum_{k,l=1}^{d^*} (Da_{kl}(h)\lambda_i)\,\Lambda_k(x)\Lambda_l(x) + \sum_{j=1}^{N_{LA}-1} c_{ij}(h)\Lambda_j(x) + \lambda_i(0),$$

for all $x \in \mathbb{R}_{\geq 0}$. Thus every $h \in D(A^\infty)$ implies a system of ODEs (Riccati equations) for the functions $\Lambda_1, \dots, \Lambda_{N_{LA}-1}$, which have to hold simultaneously for all $h \in D(A^\infty)$.

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