

# Credit Derivatives in an Affine Framework (Working Paper Version)

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## Abstract

A general and efficient method for valuing credit derivatives based on multiple entities is developed in an affine framework. This includes interdependence of market and credit risk, joint credit migration and counterparty default risk of multiple firms. As an application we provide closed form expressions for the joint distribution of default times, default correlations, and default swap spreads in the presence of counterparty default risk. An extension of this framework towards modeling default correlations of a large credit portfolio is also discussed.

**Key words:** affine intensity based models, counterparty risk, credit derivatives, default dependence

## 1 Introduction

The rapid growth of the credit derivatives market generates an upsurge for valuation models of various credit derivatives, including credit default swaps (CDSs). This requires analytically tractable models which incorporate an appropriate dependence structure between market and credit risk, credit migration and default risk of multiple firms. These aspects are inevitable for accurately pricing credit derivatives and efficient model calibration.

In this paper we present a general method to value default-sensitive securities based on multiple entities in an affine intensity-based framework. We model risk-free rates and the credit states of multiple firms jointly as an affine state process. Due to a simple mathematical trick, which allows to replace indicator variables by exponential-affine functions of the state process, we obtain closed form expressions for the conditional expectations of a variety of joint

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credit events. This allows us to derive closed form expressions for joint distributions of default times, default correlations, and CDS spreads in the presence of counterparty default risk. Using an affine approximation technique, which goes back to Singleton and Umantsev [22], one can also obtain analytically tractable expressions for swaption prices.

The state of a firm is expressed by a tuple consisting of a credit index and credit indicator. The credit index, as mentioned in [15], is regarded as the firm’s credit score, which can be related to its asset value or its credit rating. For example, it can in principle be obtained by a monotone transformation of the actual credit rating given by Moody’s or S&P; i.e.,  $\mathbb{R}_+$  is decomposed into finitely many non-overlapping intervals  $I_{Aaa}, I_{Aa}, \dots$  with the credit index in  $I_R$  meaning that the firm is  $R$ -rated,  $R \in \{Aaa, Aa, \dots\}$ . Or it can also be determined by the distance to default variable estimated from the firm’s asset information as shown in [7]. In [4], the authors propose a way to determine this credit index variable using the corresponding corporate bond spread. It is further assumed that the higher the credit index value, the worse a firm’s financial situation and zero-value of the corresponding credit index implies the perfect financial health of a firm.

The indicator variable is defined to follow a simple point process starting at zero with a constant jump size one and intensity given by the credit index process. The first jump of this process indicates the default of the corresponding firm. This method is originally proposed in [1] and specified to a doubly stochastic setup in [18]. To model risk-free rates, for simplicity, here we only employ a one-factor affine model and define the factor as the short rate. It is straightforward to include an affine multi-factor interest rate model. Additional, e.g. industry specific, factors can easily be built in, as long as they comply with the affine structure. Although, for simplicity, we consider affine diffusion and simple point processes when it comes to computations, the following carries over to more general affine jump-diffusion state processes. We sketch alternative affine regimes including jumps, which contribute to more weight in the tail distribution of the credit index process.

We do not condition on different filtrations, as this is usually done for the construction of a doubly stochastic intensity based credit risk model, see [9] for an overview. Instead, we model the entire affine state process with respect to one comprehensive filtration in one step. Then we use analytic methods, involving Laplace transforms and ODEs, to obtain the joint distribution of the future default events conditional on the current state of the world. As a result, we provide a general and efficient intensity based valuation method for credit derivatives in an affine framework. We also point out that we allow for simultaneous default of several firms. A very rare, but realistic event, which is often ignored by other models.

In [4], the authors proposed a hybrid of a structural (barrier) and reduced form (intensity based) model. It is worth mentioning that the present setup can in fact be considered as a limit case of the hybrid model in [4]: the credit index process “jumps to infinity” at default, thereby hitting the “barrier at infinity”. However, since in [4] the lifetime of the state process itself is finite

(the process explodes), it is difficult to use that approach for a multiple firm situation. From this limiting point of view, the present intensity based approach keeps a structural flavor, in that one can still identify the firm's default intensity as its credit index variable.

As for the recovery issue of a credit derivative, we adopt the convention of recovery at default and assume that the recovery rate is a random variable depending on both risk-free rates and the credit index of the default firm, which is more reasonable than assuming recovery at maturity as in [17] or that the recovery rate is stochastically independent of default probability and risk-free rates as in [14].

The literature on credit risk modeling is huge and fastly growing. We do not intend to provide a comprehensive reference list. Instead, we refer to the recent books [13, 9, 20, 19, 2] for an overview. Here, we mention in particular Jarrow and Yildirim [16], who introduce correlation between market and credit risk by using an intensity based model where risk-free rates and default intensities depend on some common macroeconomic factors. Meanwhile, motivated by the catenated downfalls of firms during the financial crises in East Asia, Jarrow and Yu [17], and further developed by Yu [23], propose to consider the credit risk induced by the interdependence structure between firms by generalizing the intensity based models to allow a firm exposed to some firm-specific default risk, as well as to common risk factors. However, due to the complexity of the analysis, they confine their discussion to the situation where the default intensity follows a simple point process and only price the "idealized" default swaps with the simplified assumption that the recovery payment is made at the maturity of the CDS. The copula method is used in [21] to introduce default dependency in an intensity based model.

The remainder of the paper is organized as follows. Section 2 introduces the basic three-firm model, based on affine diffusion and simple point processes. We then discuss and illustrate the joint distribution of default times, the density function and default correlation. In Section 3 we derive closed form expressions for the valuation of a CDS in the presence of counterparty risk. In Section 4 we sketch how to price a swaption by affine approximation. Extensions towards modeling large credit portfolios and alternative dynamics including jumps are discussed in Section 5. Section 6 contains brief concluding remarks. For the sake of readability we have postponed some technical parts to the appendix.

## 2 The Basic Three-Firm Model

In this section we describe the basic model incorporating three firms and a one-factor short rate model. The extension to an  $m$ -factor interest rate and  $n$ -firm model along the following lines is straightforward, see also Section 5.2 below.

For background and theory of affine processes we refer the reader to [10]. With  $e_i$  we denote the  $i$ -th standard basis vector in  $\mathbb{R}^7$ ,  $i = 0, 1, \dots, 6$ . Moreover, we shall frequently use the multi-index notation

$$\mathbf{p} = (p_4, p_5, p_6), \mathbf{q} = (q_4, q_5, q_6) \in \mathbf{I} := \{0, 1\}^3.$$

The scalar product of two vectors  $x$  and  $y$  is denoted by  $\langle x, y \rangle$ .

We now consider the affine jump-diffusion process  $X = (X^0, \dots, X^6)$  in  $\mathbb{R}_+^7$  with generator

$$\begin{aligned} \mathcal{A}f(x) &= \sum_{i=0}^3 \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=0}^3 (b_i + \langle \beta_i, x \rangle) \partial_{x_i} f(x) \\ &\quad + \sum_{\mathbf{p} \in \mathbf{I}} (f(x + p_4 e_4 + p_5 e_5 + p_6 e_6) - f(x)) (\ell_{\mathbf{p}} + \langle \lambda_{\mathbf{p}}, x \rangle), \end{aligned} \quad (2.1)$$

where

$$\alpha_i, b_i \geq 0, \quad \beta_i = (\beta_{i0}, \dots, \beta_{i6}) \in \mathbb{R}^7 \text{ with } \beta_{ij} \geq 0, \forall j \neq i, \quad \ell_{\mathbf{p}} \geq 0, \quad \lambda_{\mathbf{p}} \in \mathbb{R}_+^7.$$

We let  $X$  be realized on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual conditions (e.g. the ‘‘canonical’’ space of càdlàg paths in  $\mathbb{R}_+^7$ ). Depending on the context,  $\mathbb{P}$  stands either for the real-world or risk-neutral measure. In the latter case, prices are computed as  $\mathbb{P}$ -(conditional) expectations. Equivalent measure changes which preserve the respective affine structures exist and are feasible. For a discussion we refer to [5] and [6].

$X^0$  denotes the short rate process. The pair  $(X^i, X^{3+i})$  represents the credit state of firm  $i$ , where  $X^i$  denotes the credit index and  $X^{3+i}$  represents the default indicator of firm  $i$ , and we further assume that  $X_0^{3+i} = 0$  for  $i = 1, 2, 3$ . Then the first jump time

$$\tau_i := \inf\{t \mid X_t^{3+i} > 0\}$$

of  $X^{3+i}$  models the default time of firm  $i$ . We see from (2.1) that the firms can default simultaneously in all possible combinations, since we sum up over all jumps in the directions of  $p_4 e_4 + p_5 e_5 + p_6 e_6$ , for  $\mathbf{p} \in \mathbf{I}$  (to exclude one of these combinations, simply set the corresponding intensity coefficients ( $\ell_{\mathbf{p}}$  and  $\lambda_{\mathbf{p}}$ ) to be zero). This is a rare but realistic event.  $X^i$  plays the role of measuring the financial health for firm  $i$ . As mentioned before, the larger  $X^i$  the more likely is a default of firm  $i$  (this effect can be achieved by an appropriate choice of the model parameters  $\lambda_{\mathbf{p},i}$ ).

The generator (2.1) implies a rich interdependence structure between the components  $X^i$ :

- The interest rates,  $X^0$ , influence all credit risk related variables,  $X^1, \dots, X^6$ , by  $\beta_{i0}$  (mean-reversion level of  $X^i$ ) and the respective  $\lambda_{\mathbf{p},0}$  (jump intensity of  $X^{3+i}$ ).
- The credit index of firm  $i$ ,  $X^i$ ,  $i = 1, 2, 3$ , drives the intensities for (joint) defaults of firms 1, 2 and 3 by the respective  $\lambda_{\mathbf{p},i}$ . This type of correlation has already been used by [11].

$X^i$  also influences the mean reversion level for  $X^j$  by  $\beta_{ji}$ ,  $j = 0, \dots, 3$  (however, typically we let the short rates evolve autonomously, that is, we set  $\beta_{0i} = 0$ ).

- The counter process for firm  $i$ ,  $X^{3+i}$ ,  $i = 1, 2, 3$ , influences the intensities for (joint) defaults of firms 1, 2 and 3 by the respective  $\lambda_{\mathbf{p}, 3+i}$ . Note that this introduces “infectious defaults” or a “loop dependent default risk structure” as proposed in [8] and [17], respectively: the default of either firm increases the default intensity of the other firm. See also Example 2 below.

$X^{3+i}$  also influences the mean reversion level for  $X^j$  by  $\beta_{j, 3+i}$ ,  $j = 0, \dots, 3$ , another form of default contagion.

**Remark 2.1.** *Prior to the default of firm  $i \in \{1, 2, 3\}$ ,  $X^i$  and  $X^{i+3}$  represent the credit index and default indicator of the firm. Once firm  $i$  defaults, these two state variables become latent factors that drive the credit index and default dynamics of the remaining firms. One could mitigate the impact of  $X^{i+3}$  on the remaining firms after default of firm  $i$  by adding a zero-mean reverting drift term, say  $-\beta_{i+3}x_{i+3}\partial_{x_{i+3}}$  for some  $\beta_{i+3} > 0$ , to the generator (2.1). For simplicity of exposure we omit this here.*

**Remark 2.2.** *The present framework may be modified to include multiple defaults with constant fractional recovery  $r \in (0, 1)$  of the nominal value. Simply let  $X_t^{i+3}$  be the counter of the number of defaults of firm  $i$  by time  $t$ . At each default the payoff of the firm’s bond is reduced to  $r$  times its previous value. The terminal payoff, at time  $T$  say, is then reduced by the factor  $r^{X_T^{i+3}} = \exp(X_T^{i+3} \log r)$ , which is an affine function of  $X_T$  and hence analytically tractable. See [20, Section 6.1.3] for a more detailed discussion.*

**Remark 2.3.** *With regard to the parameters in (2.1), it is worth mentioning how the model calibration, say under the risk neutral measure, works. First as shown in [4], the parameters  $\{\alpha_i, b_i, \beta_i\}_{0 \leq i \leq 3}$ , the short rate  $X^0$  and individual credit index values  $\{X^i\}_{1 \leq i \leq 3}$  can be estimated using market observations of treasury and corporate bond yields. The remaining parameters that are used to characterize the joint credit migration correlations between different firms can then be calibrated using, e.g., credit default swap (CDS) data in combination with the explicit formula for CDS spreads in Section 3 below.*

*Since market price of risk specifications which preserve the affine structure (2.1) exist ([5, 6]), it is also possible to calibrate the model to real-world default correlations, from which risk premiums can be inferred.*

Fix  $\delta \geq 0$ . The basic affine property of this process reads (see [10])

$$\mathbb{E} \left[ e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_t \rangle} \mid \mathcal{F}_t \right] = e^{\phi(T-t, v; \delta) + \langle \psi(T-t, v; \delta), X_t \rangle} \quad (2.2)$$

for all  $v \in \mathbb{R}_-^7$ , where the  $\mathbb{R}_-$ -valued functions  $\phi = \phi(t, v; \delta)$  and  $\psi_i = \psi_i(t, v; \delta)$

solve the following generalized Riccati equations (GREs)

$$\begin{aligned}
\partial_t \phi &= \sum_{k=0}^3 b_k \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \ell_{\mathbf{p}} (e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1), \\
\phi(0, v; \delta) &= 0, \\
\partial_t \psi_i &= \alpha_i \psi_i^2 + \sum_{k=0}^3 \beta_{ki} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p}, i} (e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1) - \delta \mathbf{1}_{\{i=0\}} \\
\psi_i(0, v; \delta) &= v_i, \\
\partial_t \psi_j &= \sum_{k=0}^3 \beta_{kj} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p}, j} (e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1), \\
\psi_j(0, v; \delta) &= v_j
\end{aligned} \tag{2.3}$$

for  $i = 0, 1, 2, 3$  and  $j = 4, 5, 6$ .

## 2.1 Basic Trick

The following basic trick allows to express a variety of possible joint credit events as (limits) of exponential functions of  $X$ : let  $i, j \in \{1, 2, 3\}$ , then

$$\begin{aligned}
\mathbf{1}_{\{t < \tau_i\}} &= \lim_{k \rightarrow \infty} e^{-k X_t^{3+i}}, \\
\mathbf{1}_{\{s < \tau_i \leq t\}} &= \mathbf{1}_{\{s < \tau_i\}} - \mathbf{1}_{\{t < \tau_i\}} = \lim_{k \rightarrow \infty} (e^{-k X_s^{3+i}} - e^{-k X_t^{3+i}}), \quad s < t, \\
\mathbf{1}_{\{s < \tau_i, t < \tau_j\}} &= \mathbf{1}_{\{s < \tau_i\}} \mathbf{1}_{\{t < \tau_j\}} = \lim_{k \rightarrow \infty} e^{-k(X_s^{3+i} + X_t^{3+j})}, \\
&\text{etc.}
\end{aligned} \tag{2.4}$$

This asks for the following general result, the proof of which is postponed to Section A.

**Proposition 2.4.** *For  $t \leq T$ ,  $v \in \mathbb{R}_-^7$ ,  $\delta \geq 0$  and  $\mathbf{p} \in \mathbf{I}$  we have*

$$\begin{aligned}
&\mathbb{E} \left[ e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \lim_{k \rightarrow \infty} e^{-k(p_4 X_T^4 + p_5 X_T^5 + p_6 X_T^6)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(T-t, v; \delta; \mathbf{p}) + \sum_{i \in \{0, \dots, 3\} \cup J_0(\mathbf{p})} \Psi_i(T-t, v; \delta; \mathbf{p}) X_t^i} \prod_{j \in J_1(\mathbf{p})} \mathbf{1}_{\{X_t^j = 0\}}, \tag{2.5}
\end{aligned}$$

where  $J_0(\mathbf{p}) := \{4 \leq j \leq 6 \mid p_j = 0\}$ ,  $J_1(\mathbf{p}) := \{4 \leq j \leq 6 \mid p_j = 1\}$  and the  $\mathbb{R}_-$ -valued functions

$$\Phi = \Phi(t, v; \delta; \mathbf{p}) \quad \text{and} \quad \Psi_i = \Psi_i(t, v; \delta; \mathbf{p})$$

satisfy

$$\begin{aligned}
\partial_t \Phi &= \sum_{k=0}^3 b_k \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \ell_{\mathbf{q}} (e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1) - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \ell_{\mathbf{q}}, \\
\Phi(0, v; \delta; \mathbf{p}) &= 0, \\
\partial_t \Psi_i &= \alpha_i \Psi_i^2 + \sum_{k=0}^3 \beta_{ki} \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \lambda_{\mathbf{q}, i} (e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1) \\
&\quad - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \lambda_{\mathbf{q}, i} - \delta \mathbf{1}_{\{i=0\}}, \\
\Psi_i(0, v; \delta; \mathbf{p}) &= v_i, \\
\partial_t \Psi_j &= \sum_{k=0}^3 \beta_{kj} \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \lambda_{\mathbf{q}, j} (e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1) - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \lambda_{\mathbf{q}, j}, \\
\Psi_j(0, v; \delta; \mathbf{p}) &= v_j
\end{aligned} \tag{2.6}$$

for  $i = 0, 1, 2, 3$  and  $j \in J_0(\mathbf{p})$ , where  $\mathbf{I}_0(\mathbf{p}) := \{\mathbf{q} \in \mathbf{I} \mid q_j = 0 \ \forall j \in J_1(\mathbf{p})\}$  and  $\mathbf{I}_1(\mathbf{p}) := \mathbf{I} \setminus \mathbf{I}_0(\mathbf{p}) = \{\mathbf{q} \in \mathbf{I} \mid q_j = 1 \text{ for some } j \in J_1(\mathbf{p})\}$ .

**Remark 2.5.** Notice that  $\Phi(t, v; \delta; \mathbf{p})$  and  $\Psi_i(t, v; \delta; \mathbf{p})$  in Proposition 2.4 do in fact not depend on  $v_j$  for  $j \in J_1(\mathbf{p})$ , as becomes clear from the GREs (2.6) and the definition of  $\mathbf{I}_0(\mathbf{p})$ .

**Example 1** Let  $t \leq T$ . The  $\mathcal{F}_t$ -conditional Laplace transform of  $X_T$  with respect to the  $T$ -forward measure conditional on  $\{T < \tau_1 \wedge \tau_2\}$  is

$$\frac{\mathbb{E} \left[ e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \mathbf{1}_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ e^{-\int_t^T X_s^0 ds} \mathbf{1}_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right]}, \quad v \in \mathbb{R}_-^7,$$

where

$$\begin{aligned}
&\mathbb{E} \left[ e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \mathbf{1}_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right] \\
&= \mathbb{E} \left[ e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \lim_{k \rightarrow \infty} e^{-k(X_T^4 + X_T^5)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(T-t, v; 1; 1, 1, 0) + \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, v; 1; 1, 1, 0) X_t^i} \mathbf{1}_{\{X_t^4 = X_t^5 = 0\}}.
\end{aligned}$$

## 2.2 Joint Distribution of Default Times

With the aid of (2.4) and Proposition 2.4 we now discuss the dependence structure of the default times  $\tau_1$  and  $\tau_2$ .

Fix  $s \geq 0$ . For the  $\mathcal{F}_s$ -conditional joint distribution of  $(\tau_1, \tau_2)$  we have

$$\begin{aligned}
F(t, T) &= \mathbb{P}[\tau_1 \leq t, \tau_2 \leq T \mid \mathcal{F}_s] \\
&= 1 - \mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mid \mathcal{F}_s] - \mathbb{E}[\mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s] + \mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s],
\end{aligned} \tag{2.7}$$

for  $t, T \geq s$ . The terms involved are

$$\begin{aligned}\mathbb{E}[1_{\{t < \tau_1\}} | \mathcal{F}_s] &= \mathbb{E} \left[ \lim_{k \rightarrow \infty} e^{-kX_t^4} | \mathcal{F}_s \right] \\ &= e^{\Phi(t-s, 0; 0; 1, 0, 0) + \sum_{i \in \{0, \dots, 3, 5, 6\}} \Psi_i(t-s, 0; 0; 1, 0, 0) X_s^i} 1_{\{X_s^4=0\}}, \\ \mathbb{E}[1_{\{T < \tau_2\}} | \mathcal{F}_s] &= \mathbb{E} \left[ \lim_{k \rightarrow \infty} e^{-kX_T^5} | \mathcal{F}_s \right] \\ &= e^{\Phi(T-s, 0; 0; 0, 1, 0) + \sum_{i \in \{0, \dots, 3, 4, 6\}} \Psi_i(T-s, 0; 0; 0, 1, 0) X_s^i} 1_{\{X_s^5=0\}},\end{aligned}$$

and, for  $t \leq T$ ,

$$\begin{aligned}\mathbb{E}[1_{\{t < \tau_1\}} 1_{\{T < \tau_2\}} | \mathcal{F}_s] &= \mathbb{E} \left[ \lim_{k \rightarrow \infty} e^{-kX_t^4} \mathbb{E} \left[ \lim_{l \rightarrow \infty} e^{-lX_T^5} | \mathcal{F}_t \right] | \mathcal{F}_s \right] \\ &= e^{\Phi(T-t, 0; 0; 0, 1, 0)} \mathbb{E} \left[ \lim_{k \rightarrow \infty} e^{-k(X_t^4 + X_t^5)} e^{\sum_{i \in \{0, \dots, 3, 4, 6\}} \Psi_i(T-t, 0; 0; 0, 1, 0) X_t^i} | \mathcal{F}_s \right] \\ &= e^{\Phi(T-t, 0; 0; 0, 1, 0) + \Phi(t-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, 0; 0; 0, 1, 0) e_i; 0; 1, 1, 0)} \\ &\quad \times e^{\sum_{j \in \{0, \dots, 3, 6\}} \Psi_j(t-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, 0; 0; 0, 1, 0) e_i; 0; 1, 1, 0) X_s^j} 1_{\{X_s^4=X_s^5=0\}},\end{aligned}\tag{2.8}$$

and similarly for  $t \geq T$ ,

$$\begin{aligned}\mathbb{E}[1_{\{t < \tau_1\}} 1_{\{T < \tau_2\}} | \mathcal{F}_s] &= e^{\Phi(t-T, 0; 0; 1, 0, 0) + \Phi(T-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(t-T, 0; 0; 1, 0, 0) e_i; 0; 1, 1, 0)} \\ &\quad \times e^{\sum_{j \in \{0, \dots, 3, 6\}} \Psi_j(T-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(t-T, 0; 0; 1, 0, 0) e_i; 0; 1, 1, 0) X_s^j} 1_{\{X_s^4=X_s^5=0\}}.\end{aligned}\tag{2.9}$$

**Remark 2.6.** For simplicity, we will set  $s = 0$  in what follows and use the convention  $X_0^j = 0$  for  $j = 4, 5, 6$ . All results carry over after a slight modification to the general case  $s \geq 0$ .

### 2.2.1 Joint Density

Notice that the joint distribution function (2.7) is twice continuously differentiable in  $(t, T)$  for  $t \neq T$ , but not on the diagonal  $\Delta := \{(t, t) \mid t \geq 0\}$  in general. Below we illustrate the three cases where i) a jointly continuous density function  $f$  with

$$F(t, T) = \int_0^t \int_0^T f(u, v) dudv, \quad \forall (t, T) \in \mathbb{R}_+^2,\tag{2.10}$$

exists (Example 4), ii)  $f$  exists but is only piecewise continuous (Example 2), iii) the entire mass of the distribution is concentrated on  $\Delta$  and hence a density does not exist (Example 3).



**Example 2** Let  $\ell := \ell_{(1,0,0)} > 0$  and  $\lambda := \lambda_{(0,1,0),4} > 0$  and all the other parameters be zero. Then the generator (2.1) is of the form

$$\mathcal{A}f(x) = (f(x + e_4) - f(x))\ell + (f(x + e_5) - f(x))\lambda x_4.$$

That is, firm 1 defaults with a constant intensity  $\ell$  and the default intensity of firm 2 is zero first, jumps to  $\lambda$  at the default time of firm 1 (“infectious default”) and increases by the amount of  $\lambda$  at any further jump time of  $X^4$ . Accordingly, we have

$$\begin{aligned}\partial_t \Phi(t, v; 0; 1, 0, 0) &= \partial_t \Phi(t, v; 0; 1, 1, 0) = -\ell \\ \partial_t \Phi(t, v; 0; 0, 1, 0) &= \ell (e^{v_4 - \lambda t} - 1) \\ \partial_t \Psi_4(t, v; 0; 1, 0, 0) &= \lambda (e^{v_5} - 1) \\ \partial_t \Psi_4(t, v; 0; 0, 1, 0) &= \partial_t \Psi_4(t, v; 0; 1, 1, 0) = -\lambda\end{aligned}$$

and  $\partial_t \Psi_i(t, v; 0; \mathbf{p}) \equiv 0$  for all  $i \neq 4$ . So that

$$\begin{aligned}\Phi(t, v; 0; 1, 0, 0) &= \Phi(t, v; 0; 1, 1, 0) = -\ell t \\ \Phi(t, v; 0; 0, 1, 0) &= \ell \left( \frac{e^{v_4}}{\lambda} (1 - e^{-\lambda t}) - t \right) \\ \Psi_4(t, v; 0; 1, 0, 0) &= v_4 + \lambda (e^{v_5} - 1) t \\ \Psi_4(t, v; 0; 0, 1, 0) &= \Psi_4(t, v; 0; 1, 1, 0) = v_4 - \lambda t\end{aligned}$$

and  $\Psi_i(t, v; 0; \mathbf{p}) \equiv v_i$  for all  $i \neq 4$ . In view of (2.8) and (2.9) we obtain for  $G(t, T) = \mathbb{E}[1_{\{t < \tau_1\}} 1_{\{T < \tau_2\}}]$ ,

$$G(t, T) = \begin{cases} e^{\frac{\ell}{\lambda}(1 - e^{-\lambda(T-t)}) - \ell T}, & t \leq T, \\ e^{-\ell t}, & t \geq T. \end{cases}$$

It is easy to see that  $\partial_t G(t, T)$  and  $\partial_T G(t, T)$  are jointly continuous in  $(t, T)$  and absolutely continuous in  $T$  and  $t$ , respectively. Hence (2.10) holds. But  $f$  is not continuous at  $\Delta$  since

$$\partial_t \partial_T G(t, T) = \begin{cases} (\ell \lambda e^{-\lambda(T-t)} + \ell e^{-\lambda(T-t)} (1 - e^{-\lambda(T-t)})) G(t, T), & t < T, \\ 0, & t > T, \end{cases}$$

and we see that  $\frac{\partial_t \partial_T G(t, T)}{G(t, T)} = \ell \lambda \neq 0$ .

**Example 3** It is rather obvious that the distribution (2.7) is singular if defaults of different firms can occur simultaneously. For illustration consider the generator

$$\mathcal{A}f(x) = f(x + e_4 + e_5) - f(x),$$

that is, we set  $\ell_{(1,1,0)} = 1$  and all other parameters are zero. A straightforward calculation shows that

$$F(t, T) = 1 - e^{-t \wedge T}.$$

This distribution carries the entire mass on the diagonal  $\Delta$ , and therefore has no density.

**Example 4** As we have seen above, the joint distribution function (2.7) for infectious defaults (Example 2) and simultaneous defaults (Example 3) does not admit a (continuous) density. We now consider an example where  $\tau_1$  and  $\tau_2$  are conditionally independent given the information  $\mathcal{G} = \sigma(X_t^0, \dots, X_t^3 \mid t \geq 0)$  generated by  $X^0, \dots, X^3$ . In other words, the default times  $\tau_1$  and  $\tau_2$  are doubly stochastic driven by the factors  $(X^0, \dots, X^3)$ , see e.g. [9]. We let the generator (2.1) be of the form

$$\begin{aligned} \mathcal{A}f(x) &= \alpha_0 x_0 \partial_{x_0}^2 f(x) + (b_0 + \beta_{00} x_0) \partial_{x_0} f(x) \\ &+ \sum_{i=1}^2 \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=1}^2 (b_i + \beta_{i0} x_0 + \beta_{ii} x_i) \partial_{x_i} f(x) \\ &+ (f(x + e_4) - f(x)) (\lambda_{(1,0,0),0} x_0 + \lambda_{(1,0,0),1} x_1 + \lambda_{(1,0,0),2} x_2) \\ &+ (f(x + e_5) - f(x)) (\lambda_{(0,1,0),0} x_0 + \lambda_{(0,1,0),1} x_1 + \lambda_{(0,1,0),2} x_2), \end{aligned} \quad (2.11)$$

with the symmetric structure

$$\begin{aligned} \alpha_1 &= \alpha_2, \quad b_1 = b_2, \quad \beta_{10} = \beta_{20}, \quad \beta_{11} = \beta_{22}, \\ \lambda_{(1,0,0),0} &= \lambda_{(0,1,0),0}, \quad \lambda_{(1,0,0),1} = \lambda_{(0,1,0),2}, \quad \lambda_{(1,0,0),2} = \lambda_{(0,1,0),1}. \end{aligned} \quad (2.12)$$

Since here we have

$$\mathbb{P}[\tau_1 \leq t, \tau_2 \leq T \mid \mathcal{G}] = \mathbb{P}[\tau_1 \leq t \mid \mathcal{G}] \cdot \mathbb{P}[\tau_2 \leq T \mid \mathcal{G}]$$

and both of the  $\mathcal{G}$ -conditional distribution functions on the right hand side have a  $\mathcal{G}$ -measurable continuous density, it is rather obvious that  $F(t, T) = \mathbb{E}[\mathbb{P}[\tau_1 \leq t, \tau_2 \leq T \mid \mathcal{G}]]$  admits a continuous density.

We write short  $\Phi(v; \mathbf{p}) = \Phi(t, v; 0; \mathbf{p})$  and  $\Psi_i(v, \mathbf{p}) = \Psi_i(t, v; 0; \mathbf{p})$ . The relevant ODEs (2.6) are

$$\begin{aligned} \partial_t \Phi(0; \mathbf{p}) &= b_0 \Psi_0(0; \mathbf{p}) + b_1 (\Psi_1(0; \mathbf{p}) + \Psi_2(0; \mathbf{p})), \\ \partial_t \Psi_0(0; \mathbf{p}) &= \alpha_0 \Psi_0^2(0; \mathbf{p}) + \beta_{00} \Psi_0(0; \mathbf{p}) + \beta_{10} (\Psi_1(0; \mathbf{p}) + \Psi_2(0; \mathbf{p})) - \lambda_{\mathbf{p},0}, \\ \partial_t \Psi_i(0; \mathbf{p}) &= \alpha_1 \Psi_i^2(0; \mathbf{p}) + \beta_{11} \Psi_i(0; \mathbf{p}) - \lambda_{\mathbf{p},i}, \quad i = 1, 2, \end{aligned}$$

for  $\mathbf{p} = (1, 0, 0)$ ,  $(0, 1, 0)$ , and

$$\begin{aligned} \partial_t \Phi(v; 1, 1, 0) &= b_0 \Psi_0(v; 1, 1, 0) + 2b_1 \Psi_1(v; 1, 1, 0), \\ \partial_t \Psi_0(v; 1, 1, 0) &= \alpha_0 \Psi_0^2(v; 1, 1, 0) + \beta_{00} \Psi_0(v; 1, 1, 0) + 2\beta_{10} \Psi_1(v; 1, 1, 0) - 2\lambda_{\mathbf{p},0}, \\ \partial_t \Psi_1(v; 1, 1, 0) &= \alpha_1 \Psi_1^2(v; 1, 1, 0) + \beta_{11} \Psi_1(v; 1, 1, 0) - \lambda_{(1,0,0),1} - \lambda_{(0,1,0),1}, \end{aligned}$$

with  $\Psi_2(v; 1, 1, 0) = \Psi_1(v; 1, 1, 0)$ , by symmetry.

Note that  $\Psi_1$  and  $\Psi_2$  above solve autonomous Riccati equations. The following solution formula is well know:

**Lemma 2.7.** *The function*

$$G = G(t, r_0) = -\frac{2C(e^{\rho t} - 1) - (\rho(e^{\rho t} + 1) + B(e^{\rho t} - 1))r_0}{\rho(e^{\rho t} + 1) - B(e^{\rho t} - 1) - 2A(e^{\rho t} - 1)r_0} \quad (2.13)$$

with  $\rho := \sqrt{B^2 + 4AC}$  is the unique solution of the Riccati equations

$$\partial_t G = AG^2 + BG - C, \quad G(0, r_0) = r_0,$$

for  $A, C \geq 0$ ,  $B \in \mathbb{R}$  and  $r_0 \leq 0$ .

With formula (2.13) at hand it is possible—but cumbersome (we used Mathematica for the formal calculations)—to show that (2.7) is smooth enough to allow for a continuous density. Figures 1–3 show this density function for

$$\begin{aligned} \alpha_0 &= 1.736 \times 10^{-5}, & \alpha_1 &= 3.2648, \\ b_0 &= 0.01167, & b_1 &= 1.6328 \times 10^{-5}, \\ \beta_{00} &= -0.15492, & \beta_{10} &= 0.23006, & \beta_{11} &= -1.472, \\ \lambda_{(1,0,0),0} &= 0.26365, & \lambda_{(1,0,0),1} &= 0.10613, \\ X_0^0 &= 0.0105, & X_0^1 &= X_0^2 = 0.07, & X_0^3 &= 0, \end{aligned} \tag{2.14}$$

and different values for  $\lambda_{(1,0,0),2}$ , the impact of the second firm's credit rating,  $X_t^2$ , on the default intensity of firm 1, and vice versa. The larger  $\lambda_{(1,0,0),2}$ , the stronger the dependence of the default times, which is best seen on the diagonal of the domain in Figures 2 and 3.

**Remark 2.8.** *The above parameters were obtained by the single name model calibration in [4] and the symmetry assumption (2.12).  $X_0^1 = 0.07$  corresponds to Moody's rating class Aaa. For further calibration details we refer to [4].*

## 2.2.2 Default Correlation

The joint distribution function (2.7) contains all the information about the dependence of the default times  $\tau_1$  and  $\tau_2$ . A first (but not sufficient) indicator for this dependence is the correlation of the events  $\{\tau_1 \leq T\}$  and  $\{\tau_2 \leq T\}$ ,

$$\text{corr}(T) = \frac{\text{Cov}_{12}(T)}{\sqrt{\text{Cov}_{11}(T)\text{Cov}_{22}(T)}} \tag{2.15}$$

with

$$\begin{aligned} \text{Cov}_{ij}(T) &:= \mathbb{E}[1_{\{\tau_i \leq T\}} 1_{\{\tau_j \leq T\}}] - \mathbb{E}[1_{\{\tau_i \leq T\}}] \mathbb{E}[1_{\{\tau_j \leq T\}}] \\ &= \begin{cases} \mathbb{E}[1_{\{\tau_i \leq T\}}] - (\mathbb{E}[1_{\{\tau_i \leq T\}}])^2, & i = j, \\ F(T, T) - \mathbb{E}[1_{\{\tau_i \leq T\}}] \mathbb{E}[1_{\{\tau_j \leq T\}}], & i \neq j, \end{cases} \end{aligned}$$

for varying  $T \geq 0$ . The terms involved are

$$\mathbb{E}[1_{\{\tau_i \leq T\}}] = 1 - \mathbb{E} \left[ \lim_{k \rightarrow \infty} e^{-k X_T^{3+i}} \right] = 1 - e^{\Phi(T, 0; 0; \mathbf{p}(i)) + \sum_{j=0}^3 \Psi_j(T, 0; 0; \mathbf{p}(i)) X_0^j},$$

where  $\mathbf{p}(1) := (1, 0, 0)$  and  $\mathbf{p}(2) := (0, 1, 0)$ .

As documented in [21], the default correlations introduced by correlated default rating processes are typically too low to match the empirical correlations observed from markets. However, since in our model the default intensity of one firm depends explicitly, by factor  $\lambda_{(1,0,0),2}$ , on the rating process of the other firm, the resulting default correlation can reach the level of market observations (see [23]). Figure 4 shows the term structure of default correlations (2.15) for the model (2.11), (2.12), and (2.14) with different values for  $\lambda_{(1,0,0),2}$ . This illustrates once more the flexibility of our approach to account for dependence of default.

### 3 Valuing Credit Default Swaps

We now consider the valuation of a plain vanilla credit default swap (CDS) with notional principal \$1. The *seller* (firm 3) of a CDS contract provides the *buyer* (firm 2) insurance against the risk of default of a third party called the *reference entity* (firm 1). In return, the buyer makes periodic payments to the seller. We denote by  $T_0$  the start date of the CDS and the payment dates by  $T_1, \dots, T_n$ . We assume that  $T_k - T_{k-1} \equiv \Delta$  for all  $k = 1, \dots, n$ . We consider a Bermudan setup. That is, cashflows take place at dates  $T_k$  only, given the events that happened in the preceding periods  $(T_{j-1}, T_j]$ ,  $j = 1, \dots, k$ .

At time  $T_k$ :

- if no default has occurred yet ( $T_k < \tau_1 \wedge \tau_2 \wedge \tau_3$ ) then the buyer pays to the seller a fixed rate  $c$ ;
- if the reference entity has defaulted in period  $(T_{k-1}, T_k]$  ( $T_{k-1} < \tau_1 \leq T_k$ ) and the seller has not defaulted yet ( $T_k < \tau_3$ ) and the buyer has not defaulted by  $T_{k-1}$  ( $T_{k-1} < \tau_2$ ) then the seller pays  $1 - G(X_{T_k})$  and the contract terminates, where

$$G(x) = e^{r+(\rho,x)} \leq 1$$

denotes the recovery rate for the bond issued by the reference entity, for some  $r \in \mathbb{R}_-$  and  $\rho \in \mathbb{R}_-^7$ ;

- in all other cases there is no payment and the contract terminates.

The value at time  $t \leq T_0$  of the buyer's payments accordingly is  $cB_t$ , where

$$\begin{aligned} B_t &= \mathbb{E} \left[ \sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} \Delta \mathbf{1}_{\{T_k < \tau_1 \wedge \tau_2 \wedge \tau_3\}} \mid \mathcal{F}_t \right] \\ &= \Delta \sum_{k=1}^n \mathbb{E} \left[ e^{-\int_t^{T_k} X_s^0 ds} \lim_{l \rightarrow \infty} e^{-l(X_{T_k}^4 + X_{T_k}^5 + X_{T_k}^6)} \mid \mathcal{F}_t \right] \\ &= \Delta \sum_{k=1}^n e^{\Phi(T_k - t, 0; 1; 1, 1, 1) + \sum_{i=0}^3 \Psi_i(T_k - t, 0; 1; 1, 1, 1) X_t^i} \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}}. \end{aligned} \tag{3.1}$$

The value at time  $t \leq T_0$  of the seller's payment is

$$\begin{aligned}
S_t &= \mathbb{E} \left[ \sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} (1 - G(X_{T_k})) 1_{\{T_{k-1} < \tau_1 \leq T_k\}} 1_{\{T_{k-1} < \tau_2\}} 1_{\{T_k < \tau_3\}} \mid \mathcal{F}_t \right] \\
&= \sum_{k=1}^n \mathbb{E} \left[ e^{-\int_t^{T_k} X_s^0 ds} (1 - G(X_{T_k})) \right. \\
&\quad \left. \times \lim_{l, m \rightarrow \infty} \left( e^{-lX_{T_{k-1}}^4} - e^{-mX_{T_k}^4} \right) e^{-lX_{T_{k-1}}^5 - mX_{T_k}^6} \mid \mathcal{F}_t \right] \\
&= \sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k}, \tag{3.2}
\end{aligned}$$

for some exponential affine terms  $S_t^{1k}, \dots, S_t^{4k}$ . For sake of readability, the detailed expressions are given in Section B below.

The forward CDS spread  $C(t)$  at time  $t \leq T_0$  is the fixed rate at which we have  $C(t)B_t = S_t$ . From the above, we obtain

**Lemma 3.1.** *The forward CDS spread is given by*

$$C(t) = \frac{S_t}{B_t} = \frac{\sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k}}{\Delta \sum_{k=1}^n e^{\Phi(T_k - t, 0; 1, 1, 1) + \sum_{i=0}^3 \Psi_i(T_k - t, 0; 1, 1, 1) X_t^i}}, \tag{3.3}$$

where the terms  $B_t$  and  $S_t^{ik}$  are given by (3.1) and (B.1)–(B.4), respectively.

Figure 5 shows the CDS spread  $C(T_0)$  for different CDS lengths,  $T_n - T_0$ , for the case of single-party risk (only the reference entity can default, that is,  $X^2 = X^3 = 0$ ) with different rating classes:  $X_{T_0}^1 = 0.07$  (Moody's Aaa), 0.13102 (Aa), 0.465 (A), 0.80907 (Baa) and  $X_{T_0}^4 = 0$  (no default by  $T_0$ ). The recovery rate is one, that is,  $r = 0$  and  $\rho = 0$ . The remaining parameters are according to (2.14).

## 4 Swaption Pricing by Affine Approximation

In this section we sketch a method for approximating swaption prices as proposed by Singleton and Umantsev [22]. Consider a call option (swaption) on the above CDS with strike rate  $K$  and expiry date  $T_0$ . Its payoff at  $T_0$  is

$$(S_{T_0} - KB_{T_0})^+ = S_{T_0} 1_{\{C(T_0) > K\}} - KB_{T_0} 1_{\{C(T_0) > K\}}$$

and the price at time  $t < T_0$ , accordingly,

$$\begin{aligned}
P_{\text{swpt}}(t) &= \mathbb{E} \left[ e^{-\int_t^{T_0} X_s^0 ds} S_{T_0} 1_{\{C(T_0) > K\}} \mid \mathcal{F}_t \right] \\
&\quad - K \mathbb{E} \left[ e^{-\int_t^{T_0} X_s^0 ds} B_{T_0} 1_{\{C(T_0) > K\}} \mid \mathcal{F}_t \right].
\end{aligned}$$

Note that  $S_{T_0}$  and  $B_{T_0}$  are sums of exponential-affine functions in  $X_{T_0}$ . Define the CDS spread function

$$c(x) := C(T_0, X_{T_0} = x),$$

see (3.3). The idea is to approximate the exercise boundary  $\partial D(K) := \{c = K\}$  by a hyperplane in  $\mathbb{R}^7$ , hence to linearize  $c$ . That is, for a fixed average strike rate  $K^*$ , we write

$$c(x) \approx K^* + \langle \nabla c(x^*), x - x^* \rangle \quad (4.1)$$

for some  $x^* \in \partial D(K^*)$ . The exercise domain  $D(K) := \{c > K\}$  is accordingly replaced by the half-space

$$\{x \mid \langle \nabla c(x^*), x \rangle > K - K^* + \langle \nabla c(x^*), x^* \rangle\}.$$

The computation of  $P_{\text{swap}}(t)$  then boils down to the Fourier-inversion methods for conditional distributions with Laplace transforms of the form (2.2) with  $\delta = 1$  and  $T = T_0$ , as discussed in [12].

To illustrate the effectiveness of this approach, we show in Figure 6 the level sets for different levels,  $K$ , on the two-dimensional cross-sectional surface  $(x_0, x_1) \mapsto c(x_0, x_1, 0, \dots, 0)$ . The cross-sections of the corresponding exercise boundaries,  $\partial D(K)$ , are visible in the  $(x_0, x_1)$ -plane. The model parameters are as at the end of Section 3. Figure 6 suggests that the linear approximation (4.1) will yield accurate swaption prices. A more detailed empirical study is left for future research.

## 5 Extensions

### 5.1 Large Credit Portfolios

In this section, we briefly discuss some issues regarding the extension of the original three firm model to characterize default correlations for a large portfolio (e.g.,  $n=100$ ).

As already mentioned at the beginning of Section 2, it is straightforward to extend the preceding  $(1 + 6)$ -dimensional factor process,  $X = (X^0, \dots, X^7)$  (1-factor interest rates and 3 firms) to the  $(1 + m + 2n)$ -dimensional analog,

$$X = (X^0, \dots, X^m, X^{m+1}, \dots, X^{m+n}, X^{m+n+1}, \dots, X^{m+2n})$$

( $1+m$  common macroeconomic factors including the short rate and  $n$  firms) with  $(X^{m+i}, X^{m+n+i})$  describing the credit state of firm  $i = 1, \dots, n$ . Additional, e.g. industry specific, factors can easily be built in.

A special case is  $m = 1$  with generator of the process  $X = (X^0, \dots, X^{1+2n})$  given by

$$\begin{aligned} \mathcal{A}f(x) &= \sum_{i=0}^{1+n} \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=0}^{1+n} (b_i + \langle \beta_i, x \rangle) \partial_{x_i} f(x) \\ &+ \sum_{i=2+n}^{1+2n} (f(x + e_i) - f(x)) (\lambda_{i,1} x_1 + \lambda_{i,2} x_{i-n}). \end{aligned} \quad (5.1)$$

The model (5.1) accommodates the setup proposed in [11] for modeling collateralized debt obligations (CDOs), where  $X^1$  serves as a common default factor that influences all the firms in the market.

Extending the analysis of Section 2.2 one can derive explicit formulas for the joint distribution of  $n$  firms' default times in terms of a system of Riccati equations. In particular, it is straightforward to derive prices of first-to-default products. Experience shows that numerically solving a system of 100 generalized Riccati equations is quite efficient and takes about a minute.

## 5.2 Alternative Dynamics

We now sketch a possible change of the characteristics of the factor process  $X$ , replacing or extending the continuous diffusion parts by jumps. For the mathematical justification of what follows we refer, again, to [10]. The basic observation is that, for  $\theta \in (1, 2)$ ,

$$\mu_\theta(d\xi) := \frac{\theta(\theta-1)}{\Gamma(2-\theta)} \frac{d\xi}{\xi^{1+\theta}}$$

satisfies  $\int_{\mathbb{R}_{++}} (\xi \wedge \xi^2) \mu_\theta(d\xi) < \infty$  and

$$\int_{\mathbb{R}_{++}} (e^{v\xi} - 1 - v\xi) \mu_\theta(d\xi) = (-v)^\theta, \quad v \in \mathbb{R}_-. \quad (5.2)$$

The diffusion part

$$\alpha_i x_i \partial_{x_i}^2 f(x)$$

in (2.1) can now selectively for  $i \in \{0, \dots, 3\}$  be replaced (or extended) by

$$\alpha_i x_i \int_{\mathbb{R}_{++}} (f(x + \xi e_i) - f(x) - \partial_{x_i} f(x) \xi) \mu_{\theta_i}(d\xi) \quad (5.3)$$

for  $\theta_i \in (1, 2)$ . The equation for  $\psi_i$  in (2.3) accordingly changes to

$$\partial_t \psi_i = \alpha_i (-\psi_i)^{\theta_i} + \sum_{k=0}^3 \beta_{ki} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p},i} (e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1) - \delta 1_{\{i=0\}}. \quad (5.4)$$

Replacing the diffusion part of  $X^i$  by jumps (5.3) leads to a heavier tail distribution of  $X_t^i$  in general. Indeed, since the right hand side of (5.4) is monotonic increasing in  $\theta_i$  for  $-\psi_i = |\psi_i|$  large enough, a comparison argument for ODEs (see e.g. [3]) yields that  $-\psi_i(t, v)$  is monotonic decreasing in  $\theta_i$  for  $-v$  large enough. That is, the smaller  $\theta_i \in (1, 2)$ , the smaller  $\mathbb{E}[e^{-sX_t^i}] = e^{\phi(t, -se_i) + \langle \psi(t, -se_i), X_0 \rangle}$ , for  $s > 0$  large enough, indicating that there is more weight in the tail of  $X_t^i$ .

Finally, note that the limit case  $\theta \rightarrow 2$  corresponds to the diffusion setup (2.1).

## 6 Conclusion

This paper provides some basic and efficient techniques for valuating credit derivatives in an affine intensity based framework. In particular, the usual doubly stochastic methods (see e.g. [9]) are replaced by Laplace transformations and ODEs. This results in an analytically tractable framework which is flexible enough to capture counterparty credit risk and dependence structures of large credit portfolios.

The state of a firm is characterized by its credit index and default indicator process. The joint evolution of risk-free rates and multiple firm's state processes can incorporate complex dependence structures. Due to a simple mathematical trick, which allows to replace indicator variables by exponential-affine functions of the state process, we obtain closed form expressions for the conditional expectations of a variety of joint credit events.

We demonstrate the efficiency of this approach by explicitly calculating the joint distribution and density (provided it exists) of default times, default correlations, and CDS spreads in the presence of counter-party default risk. Also we sketch the pricing of swaptions by using an affine approximation technique, as proposed by Singleton and Umantsev [22].

Our results, for simplicity of exposure, are based on affine diffusion and simple point processes for a three-firm model. An extension towards large credit portfolios and more general affine jump-diffusion processes, including multi-factor interest rate models and additional industry specific factors, is straightforward and an alternative affine regime is sketched in this paper. It remains future research to compare their empirical performances.

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## A Proof of Proposition 2.4

By dominated convergence, the left hand side of (2.5) equals

$$\begin{aligned} & \lim_{k \rightarrow \infty} \mathbb{E} \left[ e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} e^{-k(p_4 X_T^4 + p_5 X_T^5 + p_6 X_T^6)} \mid \mathcal{F}_t \right] \\ &= \lim_{k \rightarrow \infty} e^{\phi(T-t, v - k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta) + \langle \psi(T-t, v - k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta), X_t \rangle}. \end{aligned}$$

Let  $j \in \{4, 5, 6\}$ . Since  $\partial_t \psi_j \leq 0$ , we have that

$$\begin{aligned} & \psi_j(t, v - k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta) \\ & \leq \psi_j(0, v - k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta) = v_j - k \mathbf{1}_{\{p_j=1\}}, \quad \forall t \geq 0. \end{aligned}$$

By classical results on inhomogeneous ODEs (see e.g. [3]), we conclude that the right hand sides of the GREs (2.3) which correspond to  $\phi$ ,  $i = 0, \dots, 3$  and



$j \in J_0(\mathbf{p})$  converge uniformly on compacts in  $t$ ,  $\phi$ ,  $\psi_i$ ,  $i = 0, \dots, 3$ , and  $\psi_j$ ,  $j \in J_0(\mathbf{p})$ , to the respective right hand sides of (2.6), for  $k \rightarrow \infty$ . This proves the proposition.

## B Detailed Expressions for (3.2)

Taking into account Remark 2.5, we derive

$$\begin{aligned}
S_t^{1k} &= \mathbb{E} \left[ e^{-\int_t^{T_k} X_s^0 ds} \lim_{l, m \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5) - mX_{T_k}^6} \mid \mathcal{F}_t \right] \\
&= \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[ e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} \lim_{m \rightarrow \infty} e^{-mX_{T_k}^6} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 0; 0, 1)} \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i=0}^5 \Psi_i(\Delta, 0; 1; 0; 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 0; 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 0; 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 0; 0, 1) e_i; 1; 1, 1, 1)} X_t^j \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}},
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
S_t^{2k} &= \mathbb{E} \left[ e^{-\int_t^{T_k} X_s^0 ds} e^{r + \langle \rho, X_{T_k} \rangle} \lim_{l, m \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5) - mX_{T_k}^6} \mid \mathcal{F}_t \right] \\
&= e^r \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[ e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} e^{\langle \rho, X_{T_k} \rangle} \lim_{m \rightarrow \infty} e^{-mX_{T_k}^6} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5)} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 0; 0, 1)} \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i=0}^5 \Psi_i(\Delta, \rho; 1; 0; 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 0; 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 0; 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 0; 0, 1) e_i; 1; 1, 1, 1)} X_t^j \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}},
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
S_t^{3k} &= \mathbb{E} \left[ e^{-\int_t^{T_k} X_s^0 ds} \lim_{l, m \rightarrow \infty} e^{-lX_{T_{k-1}}^5 - m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_t \right] \\
&= \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[ e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} \lim_{m \rightarrow \infty} e^{-m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-lX_{T_{k-1}}^5} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 1, 0, 1)} \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i \in \{0, \dots, 3, 5\}} \Psi_i(\Delta, 0; 1; 1, 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 1, 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 1, 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 1, 0, 1) e_i; 1; 1, 1, 1) X_t^j} 1_{\{X_t^4 = X_t^5 = X_t^6 = 0\}},
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
S_t^{4k} &= \mathbb{E} \left[ e^{-\int_t^{T_k} X_s^0 ds} e^{r + \langle \rho, X_{T_k} \rangle} \lim_{l, m \rightarrow \infty} e^{-lX_{T_{k-1}}^5 - m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_t \right] \\
&= e^r \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[ e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} e^{\langle \rho, X_{T_k} \rangle} \lim_{m \rightarrow \infty} e^{-m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-lX_{T_{k-1}}^5} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 1, 0, 1)} \mathbb{E} \left[ e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i \in \{0, \dots, 3, 5\}} \Psi_i(\Delta, \rho; 1; 1, 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 1, 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 1, 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 1, 0, 1) e_i; 1; 1, 1, 1) X_t^j} 1_{\{X_t^4 = X_t^5 = X_t^6 = 0\}}.
\end{aligned} \tag{B.4}$$

## References

- [1] P. Artzner and F. Delbaen, Default risk and incomplete insurance markets, *Mathematical Finance* **5** (1995) 187–195.
- [2] T.R. Bielecki and M. Rutkowski, *Credit Risk: Modeling, Valuation and Hedging*, Springer (2002).
- [3] G. Birkhoff and G. C. Rota, *Ordinary differential equations*, fourth edition, John Wiley & Sons Inc., New York (1989).
- [4] L. Chen and D. Filipović, A simple model for credit migration and spread curves, *Finance and Stochastics* **9** (2005) 211–231.

- [5] P. Cheridito, D. Filipović and M. Yor, Equivalent and absolutely continuous measure changes for jump-diffusion processes, *The Annals of Applied Probability* **15** (2005) 1713–1732.
- [6] P. Cheridito, D. Filipović and R. Kimmel, Market price of risk specifications for affine models: theory and evidence, forthcoming in *Journal of Financial Economics*.
- [7] P. J. Crosbie and J. R. Bohn, Modeling default risk, working paper, Moody's and KMV (2003).
- [8] M. Davis and V. Lo, Infectious defaults, *Quantitative Finance* **1** (2001) 382–386.
- [9] D. Duffie, *Credit Risk Modeling with Affine Processes*, Cattedra Galileiana, Scuola Normale Superiore, Pisa (2004).
- [10] D. Duffie, D. Filipović, and W. Schachermayer, Affine processes and applications in finance, *The Annals of Applied Probability* **13** (2003) 984–1053.
- [11] D. Duffie and N. Gârleanu, Risk and valuation of collateralized debt obligations, *Financial Analysts Journal* **57** (2001) 41–59.
- [12] D. Duffie, J. Pan, and K. Singleton, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* **68** (2000) 1343–1376.
- [13] D. Duffie and K. Singleton, *Credit Risk: Pricing, Measurement, and Management*, Princeton University Press (2003).
- [14] J. Hull and A. White, Valuing Credit Default Swaps I: No Counterparty Default Risk, *The Journal of Derivatives* **1** (2000) 29–40.
- [15] J. Hull and A. White, Valuing Credit Default Swaps II: Modeling Default Correlations, *The Journal of Derivatives* **8** (2001) 12–22.
- [16] R. Jarrow and Y. Yildirim, Valuing Default Swaps Under Market and Credit Risk Correlation, *The Journal of Fixed Income* **11** (2002) 7–19.
- [17] R. Jarrow and F. Yu, Counterparty Risk and the Pricing of Defaultable Securities, *Journal of Finance* **56** (2001) 1765–1799.
- [18] D. Lando, On Cox processes and credit-risky securities, *Review of Derivatives Research* **2** (1998) 99–120.
- [19] D. Lando, *Credit Risk Modeling : Theory and Applications*, Princeton University Press (2004).
- [20] P. Schönbucher, *Credit Derivatives Pricing Models: Model, Pricing and Implementation*, John Wiley & Sons (2003).
- [21] P. Schönbucher and D. Schubert, Copula-dependent default risk in intensity models, working paper, Bonn University (2001).

- [22] K. Singleton and L. Umantsev, Pricing coupon-bond options and swaptions in affine term structure models, *Mathematical Finance* **12** (2002) 427–446.
- [23] F. Yu, Correlated defaults in reduced-form models, *Journal of Investment Management* **3** (2005) 33–42.

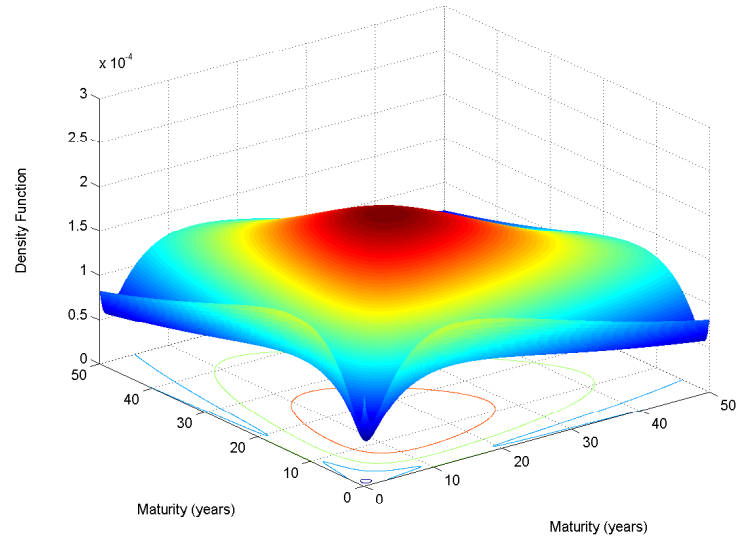


Figure 1: Density function of  $(\tau_1, \tau_2)$  for  $\lambda_{(1,0,0),2} = 0$ .

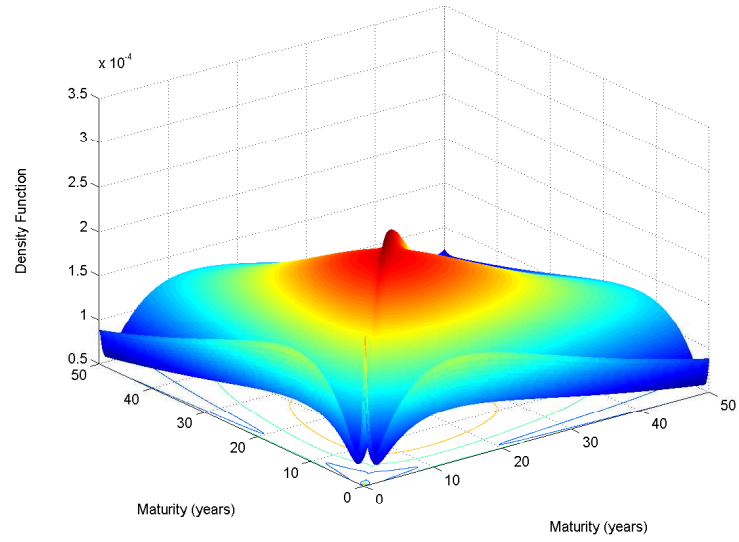


Figure 2: Density function of  $(\tau_1, \tau_2)$  for  $\lambda_{(1,0,0),2} = 0.01$ .

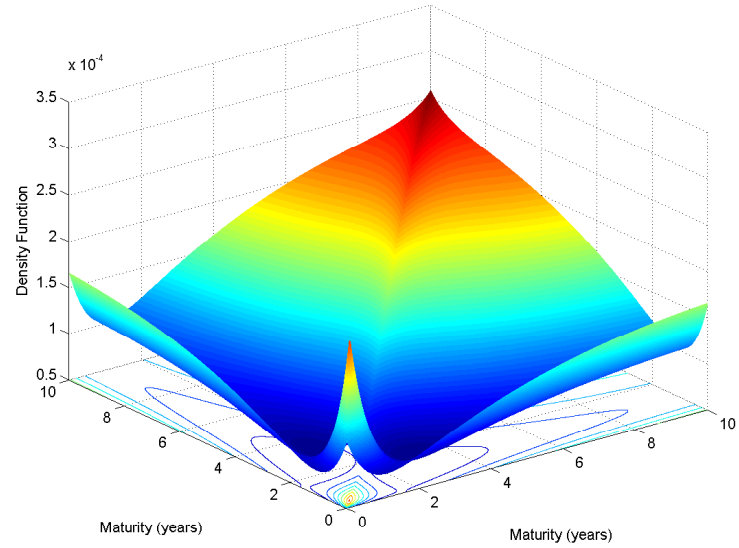


Figure 3: Density function of  $(\tau_1, \tau_2)$  for  $\lambda_{(1,0,0),2} = 0.01$  (zoomed).

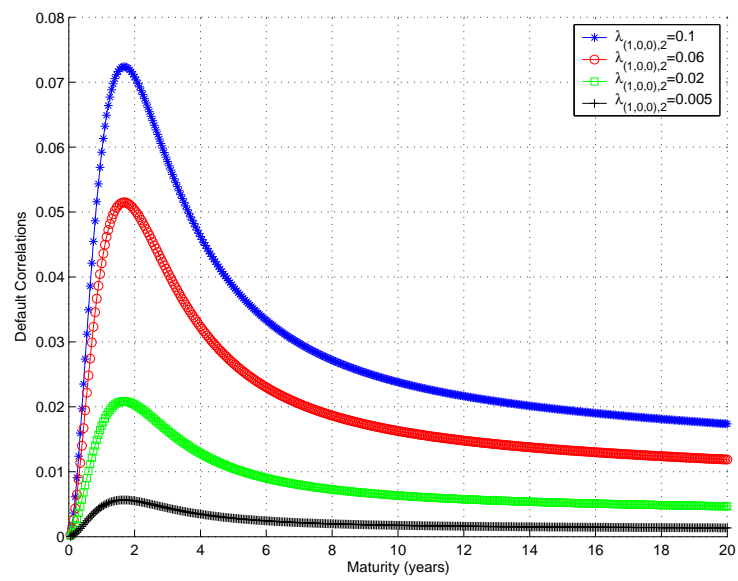


Figure 4: Term structure  $T \mapsto corr(T)$  of default correlations (2.15) between two 'Aaa' rated firms for the model (2.11), (2.12) and (2.14) and different values for  $\lambda_{(1,0,0),2}$ .

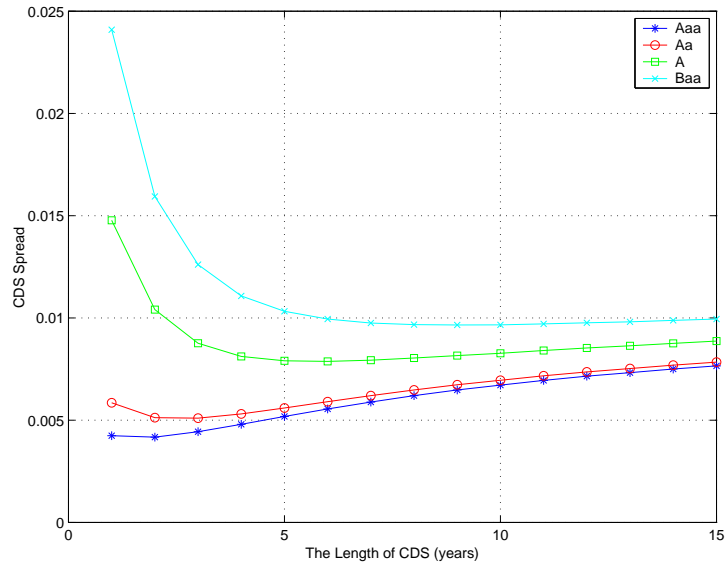


Figure 5: CDS spreads with single-party risk (reference entity)

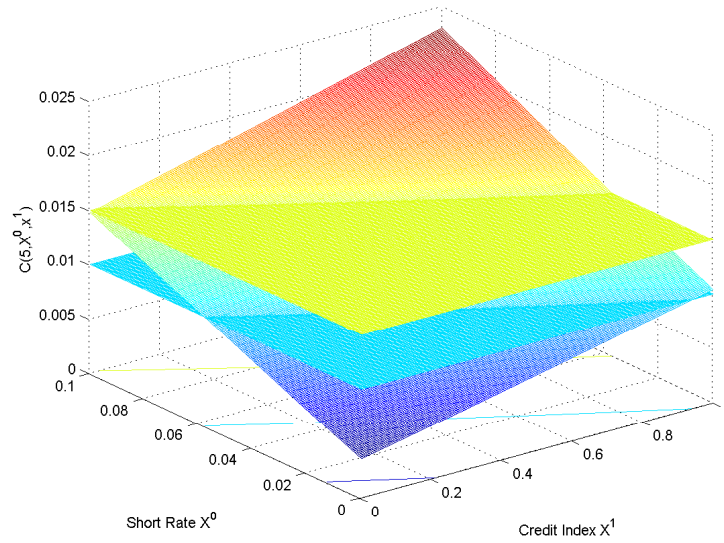


Figure 6: Exercising boundaries (dotted lines in the  $(X^0, X^1)$ -plane) of a default swaption with maturity 5 years.