Consistent Market Extensions under the Benchmark Approach*

Damir Filipović[†] Eckhard Platen[‡]

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Abstract

The existence of the growth optimal portfolio (GOP), also known as the Kelly portfolio, is vital for a financial market to be meaningful. The GOP, if it exists, is uniquely determined by the market parameters of the primary security accounts. However, markets may develop and new security accounts become tradable. What happens to the GOP if the original market is extended? In this paper we provide a complete characterization of market extensions which are consistent with the existence of a GOP. We show that a three fund separation theorem applies for the extended GOP. This includes, in particular, the introduction of a locally risk free security, the savings account. We give necessary and sufficient conditions for a consistent exogenous specification of the prevailing short rates.

Key words: growth optimal portfolio, market extension, three fund separation theorem

1 Introduction

In Kelly [5] an important portfolio, the growth optimal portfolio (GOP), also known as the Kelly portfolio, has been discovered. It maximizes expected logarithmic utility from terminal wealth, see Karatzas and Shreve [4]. Long [6] pointed out that the GOP is the numeraire portfolio that when used as numeraire leads to the real world probability measure as pricing measure. As discussed in Platen and Heath [8], the GOP plays a central role in finance. Its existence is vital for a financial market to be meaningful. The GOP, if it exists, is uniquely determined by the market parameters of the primary security accounts. However, markets may develop and new security accounts become

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†Department of Mathematics, University of Munich, Theresienstrasse 39, 80333 München, Germany. Email: filipo@math.lmu.de

‡School of Finance and Economics, University of Technology Sydney, GPO Box 123, Broadway, NSW, 2007, Australia. Email: eckhard.platen@uts.edu.au

tradable. What happens to the GOP if the original market is extended? In this paper we provide a complete characterization of market extensions which are consistent with the existence of a GOP. We show that a three fund separation theorem applies for the composition of the extended GOP: it consists of the original GOP and a position in the new security account, balanced by a position in the portfolio formed by the original market which optimally replicates the new security account. Our discussion includes, in particular, the introduction of a locally risk free security, the savings account. We give necessary and sufficient conditions for a consistent exogenous specification of the prevailing short rates.

The remainder of the paper is as follows. In Section 2 we introduce the stochastic financial market model. The GOP is defined and characterized in Section 3. In Section 4 we elaborate on the, so called, minimal variance portfolio (MVP). Necessary and sufficient conditions are given for the MVP to be locally risk free. In Section 5 we link the GOP to the numeraire portfolio (NP). We infer that the GOP is currency invariant. In Section 6 we clarify the relationship between the existence of the GOP and the existence of an equivalent risk neutral measure. In particular, we link the GOP to the minimal martingale measure. Section 7 contains our main result: a three fund separation theorem for the extended GOP. We then discuss several special cases: fair valued and locally risk free security accounts, respectively. In particular, we provide necessary and sufficient conditions on the original market which allow a free exogenous specification of the short rate process. A simple example further illustrates our findings. We conclude by Section 8. For the sake of readability, the proof of our main theorem is postponed to the Appendix.

2 Financial Market Model

The uncertainty in the financial market is driven by an n-dimensional Brownian motion $W_t = (W_t^1, \dots, W_t^n)^T$ defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a finite time horizon $\tau > 0$ and $\mathcal{F} = \mathcal{F}_\tau$.

For matrices x and y, we write $x \cdot y$ for the matrix product of x and y, and x^T, $\mathrm{im}(x)$ and $\ker(x)$ for the transpose, image and kernel of x, respectively, see any textbook on linear algebra, e.g. [3]. We denote $\mathbf{1} = (1, \dots, 1)^T$ and write 0 for the zero matrix, where the dimension follows from the context.

We consider m *primary security accounts* with value processes $S_t = (S_t^i)$, $i = 1, \dots, m$, given as

$$\frac{dS_t}{S_t} = a_t\,dt + b_t \cdot dW_t. \tag{1}$$

Here we write dS_t/S_t for the m-vector of stochastic differentials (dS_t^i/S_t^i), $i = 1, \dots, m$. To avoid technicalities, we assume throughout that the processes of *appreciation rates* $a_t = (a_t^i)$ and *volatilities* $b_t = (b_t^{ij})$, for $i = 1, \dots, m$, $j = 1, \dots, n$, satisfy the necessary measurability and integrability conditions such that the following formal manipulations and statements are meaningful[1]. For details we refer to [7] and [8].

A positive self-financing portfolio is described by its positive initial value and the *fractions* of wealth $\pi_t = (\pi_t^i)$, $i = 1, \ldots, m$, invested in the primary security accounts. Its value process S_t^π accordingly satisfies

$$\frac{dS_t^\pi}{S_t^\pi} = \pi_t^T \cdot \frac{dS_t}{S_t} = \pi_t^T \cdot a_t \, dt + \pi_t^T \cdot b_t \cdot dW_t \tag{2}$$

while the self-financing condition $\pi_t^T \cdot \mathbf{1} = 1$ holds.

3 Growth Optimal Portfolio

A *growth optimal portfolio (GOP)* is a positive self-financing portfolio S^π which maximizes the portfolio growth rate, that is, the drift of its logarithm

$$d \ln S_t^\pi = \left(\pi_t^T \cdot a_t - \frac{1}{2} \pi_t^T \cdot b_t \cdot b_t^T \cdot \pi_t \right) dt + \pi_t^T \cdot b_t \cdot dW_t. \tag{3}$$

This leads to the constrained m-dimensional quadratic optimization problem

$$\max \left\{ \pi_t^T \cdot a_t - \frac{1}{2} \pi_t^T \cdot b_t \cdot b_t^T \cdot \pi_t \mid \pi_t \in \mathbb{R}^m, \ \pi_t^T \cdot \mathbf{1} = 1 \right\}. \tag{4}$$

The portfolio strategy π_t at time t is a solution for (4) if and only if it satisfies the first order conditions

$$a_t - b_t \cdot b_t^T \cdot \pi_t - \lambda_t \mathbf{1} = 0 \tag{5}$$

$$\pi_t^T \cdot \mathbf{1} = 1 \tag{6}$$

for some Lagrange multiplier λ_t.

In matrix notation, (5)–(6) read

$$M_t \cdot \begin{pmatrix} \pi_t \\ \lambda_t \end{pmatrix} = \begin{pmatrix} a_t \\ 1 \end{pmatrix} \tag{7}$$

for the symmetric $(m+1) \times (m+1)$-matrix

$$M_t := \begin{pmatrix} b_t \cdot b_t^T & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}.$$

Hence (4) has a solution if and only if

$$\begin{pmatrix} a_t \\ 1 \end{pmatrix} \in \operatorname{im}(M_t). \tag{8}$$

The following lemma gives a sufficient condition for (8) to be satisfied:

[1]For instance, in (3) it is required that π_t satisfies

$$\int_0^\tau \left(\|\pi_t^T \cdot a_t\| + \|\pi_t^T \cdot b_t\|^2 \right) dt < \infty \quad \text{a.s.}$$

This is implied for the optimal π_t via inverting the matrix equation (7) under the appropriate measurability and integrability conditions on a_t and b_t.

Lemma 3.1. *The matrix M_t is non-singular if $b_t \cdot b_t^T$ is non-singular. In fact,*

$$\ker(M_t) = \ker(b_t \cdot b_t^T) \cap \ker(\mathbf{1}^T) \oplus \{0\} = \ker(b_t^T) \cap \ker(\mathbf{1}^T) \oplus \{0\}. \tag{9}$$

Proof. Indeed, $\pi \in \ker(b_t \cdot b_t^T) \cap \ker(\mathbf{1}^T)$ implies $(\pi^T, 0)^T \in \ker(M_t)$. Conversely, let $(\pi^T, \lambda)^T \in \ker(M_t)$. Then $b_t \cdot b_t^T \cdot \pi + \lambda \mathbf{1} = 0$ and $\pi^T \cdot \mathbf{1} = 0$. Multiplying the first equation by π^T and combining this with the second yields $\pi^T \cdot b_t \cdot b_t^T \cdot \pi = 0$, hence $\pi \in \ker(b_t^T)$, and $\lambda = 0$. Recall the fact, which can be found in any textbook on linear algebra, e.g. [3], that $\ker(b_t^T) \oplus \mathrm{im}(b_t) = \mathbb{R}^m$ and hence $\ker(b_t \cdot b_t^T) = \ker(b_t^T)$. It follows that $\pi \in \ker(b_t \cdot b_t^T) \cap \ker(\mathbf{1}^T)$, and (9) is proved. □

Now suppose (8) holds, and let (π_t^*, λ_t^*) be a solution of (7). There may be other solutions of (7), but in view of (9), λ_t^* and

$$\theta_t := b_t^T \cdot \pi_t^* \tag{10}$$

are unambiguously determined through a_t and b_t. In fact, by (5), the appreciation rates of the primary security accounts satisfy

$$a_t = \lambda_t^* \mathbf{1} + b_t \cdot \theta_t. \tag{11}$$

Hence their value processes (1) can be represented as

$$\frac{dS_t}{S_t} = \lambda_t^* \mathbf{1}\, dt + b_t \cdot (\theta_t\, dt + dW_t)$$

and (2) takes the form

$$\frac{dS_t^\pi}{S_t^\pi} = \lambda_t^*\, dt + \pi_t^T \cdot b_t \cdot (\theta_t\, dt + dW_t). \tag{12}$$

In summary, we arrive at the following result:

Theorem 3.2. *A GOP exists if and only if* (8) *holds for all t. In this case, albeit the GOP strategy π^* may not be unique, its value process $S^* := S^{\pi^*}$ is unique, for some fixed initial value $S_0^* > 0$, and of the form*

$$\frac{dS_t^*}{S_t^*} = \lambda_t^*\, dt + \theta_t^T \cdot (\theta_t\, dt + dW_t). \tag{13}$$

Henceforth, we identify the GOP with its unique value process, for some fixed initial value $S_0^* > 0$.

4 Minimal Variance and Locally Risk Free Portfolio

A *minimal variance portfolio (MVP)* is a positive self-financing portfolio S^π which minimizes the instantaneous conditional variance, or the derivate of the quadratic variation, $\pi_t^T \cdot b_t \cdot b_t^T \cdot \pi_t$, of its logarithm (3). This leads to the constrained m-dimensional quadratic optimization problem

$$\min\left\{\pi_t^T \cdot b_t \cdot b_t^T \cdot \pi_t \mid \pi_t \in \mathbb{R}^m,\ \pi_t^T \cdot \mathbf{1} = 1\right\}. \tag{14}$$

Obviously, (14) is equivalent to (4) with a_t set equal to zero. Hence $\hat{\pi}_t$ is a solution of (14) if and only if

$$M_t \cdot \begin{pmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{15}$$

for some Lagrange multiplier $\hat{\lambda}_t$. Even though $\hat{\pi}_t$ may not be unique, in view of (9) and (15), we see that $\hat{\lambda}_t$ and $\hat{\pi}_t^T \cdot b_t$ are unambiguously determined through b_t. The value process $S^0 := S^{\hat{\pi}}$ of a MVP, if it exists, has thus unique volatility $\hat{\pi}_t^T \cdot b_t$. A *locally risk free portfolio* is a MVP with zero volatility and initial value one.

Lemma 4.1. *There exists a locally risk free portfolio* S^0 *if and only if*

$$\ker(b_t^T) \not\subset \ker(\mathbf{1}^T) \quad \textit{for all } t. \tag{16}$$

In this case,

$$S_t^0 = \exp\left(\int_0^t r_s\, ds\right) \tag{17}$$

where $r_t = \hat{\pi}_t^T \cdot a_t$ *represents the prevailing* short rate *for this financial market at time* t*, for some* $\hat{\pi}_t \in \ker(b_t^T)$ *with* $\hat{\pi}_t^T \cdot \mathbf{1} = 1$.

Proof. This follows from (9) and (15). □

We can say more if the GOP exists:

Theorem 4.2. *Suppose the GOP* S^* *given by* (13) *and a locally risk free portfolio* S^0 *exist. Then its value process* (17) *is uniquely determined by* $r_t = \lambda_t^*$ *for all* t.

Proof. This follows from (12) and Lemma 4.1. □

We shall see in Corollary 7.4 below that the existence of a solution for (15) and the negation of (16) is necessary for a consistent exogenous specification of the short rates via market extension.

5 Numeraire Portfolio

A *numeraire portfolio (NP)* is a positive self-financing portfolio S^π such that

$$\text{the } \textit{benchmarked} \text{ primary security accounts } \frac{S_t}{S_t^\pi} \text{ are local martingales,} \tag{18}$$

see Long [6] and Becherer [1]. We emphasize that we do not assume the existence of an equivalent risk neutral measure for any of the markets considered, see also Remark 6.2 below.

Let S^π be a positive self-financing portfolio. Straightforward Itô calculus yields

$$a_t - b_t \cdot b_t^T \cdot \pi_t - (\pi_t^T \cdot a_t - \pi_t^T \cdot b_t \cdot b_t^T \cdot \pi_t)\mathbf{1} \tag{19}$$

for the drift part of the m-vector of stochastic differentials $d(S_t/S_t^\pi)/(S_t/S_t^\pi)$. Hence (18) holds if and only if (19) is zero for all t. But (19) is zero if and only if π_t satisfies the first order condition (5) with $\lambda_t = \pi_t^T \cdot a_t - \pi_t^T \cdot b_t \cdot b_t^T \cdot \pi_t$. We have thus shown:

Theorem 5.1. *A NP exists if and only if the GOP exists. In this case, the GOP is the unique NP with the same initial value.*

It is an obvious but fundamental remark that the NP property (18) is currency invariant: suppose all security account values are expressed in dollar and let ξ_t denote the prevailing exchange rate for dollar against euro (1 dollar = ξ_t euro). Then $\xi_t S_t$ are the primary security account values in euro. The respective euro denominated value of any positive self-financing portfolio strategy π_t is $\xi_t S_t^\pi$. From (18) we thus see that S_t^π is the dollar denominated NP if and only if $\xi_t S_t^\pi$ is the euro denominated NP. Theorem 5.1 yields:

Corollary 5.2. *The GOP is currency invariant: π^* is a dollar denominated GOP strategy if and only if π^* is a euro denominated GOP strategy.*

Remark 5.3. The existence of a NP is equivalent to the absence of some form of "strong" arbitrage (see [7]). For any market model to be meaningful, the existence of the GOP is thus vital.

6 Equivalent Risk Neutral Measures

In this section we assume that a locally risk free portfolio S^0 of the form (17) exists, see Lemma 4.1.

An *equivalent risk neutral measure* is an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the *discounted* primary security accounts S_t/S_t^0 are $\mathbb{Q}$-local martingales. Obviously, there is a relationship between the existence of the GOP and the existence of an equivalent risk neutral measure:

Theorem 6.1. *Suppose the GOP S^* given by* (13) *exists and the benchmarked locally risk free portfolio S_t^0/S_t^* is a martingale. Then*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = S_0^* \frac{S_\tau^0}{S_\tau^*} = \exp\left(-\int_0^\tau \theta_t^T \cdot dW_t - \frac{1}{2}\int_0^\tau \theta_t^T \cdot \theta_t \, dt\right) \tag{20}$$

defines an equivalent risk neutral measure.

Conversely, if an equivalent risk neutral measure of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^\tau \gamma_t^T \cdot dW_t - \frac{1}{2}\int_0^\tau \gamma_t^T \cdot \gamma_t \, dt\right) \tag{21}$$

exists, then the GOP (13) *exists and* $b_t \cdot \theta_t = b_t \cdot \gamma_t$.

Proof. Suppose the GOP (13) exists. Then the second equality in (20) follows from Theorem 4.2. Theorem 5.1 together with Bayes' rule yields the claim.

Conversely, assuming that (21) defines an equivalent risk neutral measure, Girsanov's theorem implies that $dW_t + \gamma_t\, dt$ is a $\mathbb{Q}$-Brownian motion. Hence the drift part of the m-vector of stochastic differentials $d(S_t/S_t^0)/(S_t/S_t^0)$ under $\mathbb{Q}$ becomes

$$a_t - b_t \cdot \gamma_t - r_t \mathbf{1} = 0 \quad \text{for all } t.$$

Denote by γ_t^p the orthogonal projection of γ_t onto $\operatorname{im}(b_t^T)$ (recall that $\mathbb{R}^n = \ker(b_t) \oplus \operatorname{im}(b_t^T)$, see e.g. [3]). Then $b_t \cdot \gamma_t = b_t \cdot \gamma_t^p$, and in view of (16) there exists some $\pi_t^* \in \mathbb{R}^m$ with $(\pi_t^*)^T \cdot \mathbf{1} = 1$ and $b_t^T \cdot \pi_t^* = \gamma_t^p$. In particular thus $b_t \cdot \gamma_t = b_t \cdot b_t^T \cdot \pi_t^*$. Hence π_t^* and $\lambda_t^* = r_t$ solve (7) and Theorem 3.2 yields the existence of the GOP (13). □

Remark 6.2. *The measure* $\mathbb{Q}$ *in* (20) *is known as the* minimal martingale measure *introduced by Föllmer and Schweizer [2], see also [8, Section 11.5]. It does not always exist, that is,* S^0/S^* *may fail to be a true martingale. An example is the, so called, minimal market model in [7].*

7 Market Extensions

In this section we consider what happens to the GOP if the original market, consisting of the primary security accounts (1), is extended by a new security account with value process

$$\frac{d\Sigma_t}{\Sigma_t} = \alpha_t\, dt + \beta_t^T \cdot dW_t \tag{22}$$

and some initial value $\Sigma_0 > 0$.

Our main result is the following three fund separation theorem, the proof of which we postpone to Section A.

Theorem 7.1. *Suppose the GOP* S^* *given by* (13) *for the original market exists. The GOP* $\widetilde{S}^*$ *for the extended market with primary security accounts* $S^1, \ldots, S^m, \Sigma$ *exists if and only if for all* t *at least one of the following two conditions holds:*

$$\alpha_t = \lambda_t^* + \beta_t^T \cdot \theta_t \tag{23}$$

or

$$\begin{pmatrix} b_t \cdot \beta_t \\ 1 \end{pmatrix} \in \operatorname{im}(M_t). \tag{24}$$

In this case, an extended GOP strategy is given by the three fund separation

$$\widetilde{\pi}_t^* = \begin{pmatrix} \pi_t^* \\ 0 \end{pmatrix} + p_t^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} - p_t^* \begin{pmatrix} x_t^* \\ 0 \end{pmatrix} \tag{25}$$

with unique extended GOP value process

$$\frac{d\widetilde{S}_t^*}{\widetilde{S}_t^*} = \widetilde{\lambda}_t^*\, dt + \widetilde{\theta}_t^T \cdot (\widetilde{\theta}_t\, dt + dW_t) \tag{26}$$

where

$$\widetilde{\lambda}_t^* = \lambda_t^* - p_t^*(\beta_t - b_t^T \cdot x_t^*)^T \cdot b_t^T \cdot x_t^* \tag{27}$$

$$\widetilde{\theta}_t = \theta_t + p_t^*(\beta_t - b_t^T \cdot x_t^*) \tag{28}$$

and $(x_t^{*T}, p_t^*) \in \mathbb{R}^{m+1}$ *are uniquely determined by the market parameters* a_t*,* b_t*,* α_t *and* β_t*. In fact, if* (23) *holds then*

$$(x_t^{*T}, p_t^*) = 0, \tag{29}$$

and if (24) *holds then* x_t^* *is a solution of the well posed minimization problem*

$$\min\left\{\|\beta_t - b_t^T \cdot x_t\|^2 \mid x_t \in \mathbb{R}^m,\ x_t^T \cdot \mathbf{1} = 1\right\}, \tag{30}$$

with first order conditions

$$M_t \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} b_t \cdot \beta_t \\ 1 \end{pmatrix}, \tag{31}$$

and p_t^* *is determined by*

$$p_t^* \|\beta_t - b_t^T \cdot x_t^*\|^2 = \alpha_t - \lambda_t^* - \beta_t^T \cdot \theta_t. \tag{32}$$

Hence $\beta_t = b_t^T \cdot x_t^*$ *necessitates* (23).

Remark 7.2. The economic interpretation of the three fund separation (25) is as follows: suppose (24) holds. Then there exists a positive self-financing portfolio S^{x^*} in the original market which optimally replicates the new security account Σ in the sense that it minimizes the instantaneous conditional variance $\|\beta_t - b_t^T \cdot x_t^*\|^2$ of its unhedgeable return component $d\Sigma_t/\Sigma_t - dS_t^{x^*}/S_t^{x^*}$, see (30). The extended GOP is then obtained by investing in the original GOP and holding a long (short) position p_t^* in the new security account Σ, balanced by a short (long) position $-p_t^*$ in the portfolio S^{x^*}.

The case where (23) holds is degenerate in the sense that then the new security account Σ_t does not contribute to the growth rate of the GOP (see Corollary 7.3 below). Consequently, for forming the extended GOP no investment in Σ_t is needed, whence $p_t^* = 0$.

For further illustrations of Theorem 7.1 we discuss two special cases and an example in Sections 7.1–7.3 below.

7.1 Special Case: Fair Valuation

Suppose the GOP S^* given by (13) for the original market exists. The benchmarked value process Σ_t/S_t^* satisfies

$$\frac{d(\Sigma_t/S_t^*)}{\Sigma_t/S_t^*} = (\alpha_t - \lambda_t^* - \theta_t^T \cdot \beta_t)\, dt + (\beta_t^T - \theta_t^T) \cdot dW_t. \tag{33}$$

Combining this with Theorem 7.1 immediately yields the following special result:

Corollary 7.3. *The benchmarked value process Σ_t/S_t^* is a local martingale if and only if*

$$\alpha_t - \lambda_t^* - \theta_t^T \cdot \beta_t = 0 \quad \textit{for all } t. \tag{34}$$

In this case, the GOP remains the same for the extended market with primary security accounts $S^1, \ldots, S^m, \Sigma$.

Economically speaking, any additional security account Σ satisfying (34) does not improve the performance of the GOP.

As an example for Corollary 7.3 we consider an $\mathcal{F}_\tau$-measurable claim $H \geq 0$ due at date τ satisfying

$$\mathbb{E}\left[\frac{H}{S_\tau^{\pi^*}}\right] < \infty. \tag{35}$$

A consistent value at time t, denoted by Σ_t, is then given by the fair valuation formula (see [7])

$$\frac{\Sigma_t}{S_t^{\pi^*}} = \mathbb{E}\left[\frac{H}{S_\tau^{\pi^*}} \mid \mathcal{F}_t\right]. \tag{36}$$

If this positive martingale can be written as stochastic integral (e.g. if the filtration $\mathcal{F}_t$ is generated by the Brownian motion W),

$$\frac{d(\Sigma_t/S_t^*)}{\Sigma_t/S_t^*} = (\beta_t - \theta_t)^T \cdot dW_t, \tag{37}$$

for some n-vector process β_t, then we are in the situation of Corollary 7.3. Hence a market extension by fair valued derivatives is indeed consistent with the original GOP framework.

In fact, it can be shown that – even if no minimal martingale measure exists (see Remark 6.2) – the fair valuation formula (36) yields the minimal hedge portfolio for the hedgeable part of the claim H and minimizes the variance of the benchmarked profit and loss for the unhedgeable part of H, see [8, Section 11.5]. Under fair valuation expected benchmarked values of unhedgeable parts of claims have zero mean.

The fair valuation formula (36) is further justified in [8, Section 11.4] where it is demonstrated that it is consistent with utility indifference pricing.

7.2 Special Case: Locally Risk Free Account

As above suppose the GOP S^* given by (13) for the original market exists. Theorem 7.1 implies another special result:

Corollary 7.4. *Suppose Σ_t is locally risk free, i.e. $\beta_t = 0$ for all t, so that*

$$\frac{d\Sigma_t}{\Sigma_t} = \alpha_t \, dt.$$

Then the extended GOP (26) *exists if and only if*

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathrm{im}(M_t). \tag{38}$$

In this case, the prevailing short rate can be exogenously set to any arbitrary level $\widetilde{\lambda}_t^* = \alpha_t$ different from λ_t^* if and only if

$$\ker(b_t^T) \subset \ker(\mathbf{1}^T). \tag{39}$$

Proof. Only (39) needs some explanation. But this readily follows from (27) and (32). □

Note that (39) is just the negation of the necessary and sufficient condition (16) for the existence of a locally risk free portfolio in the original market in Lemma 4.1. On the other hand, (38) and (39) are in line with Lemma 4.1 applied to the extended market $S^1, \ldots, S^m, \Sigma$ with $S^0 = \Sigma$.

We remark that the benchmarked value process Σ_t / S_t^* may be a strict local martingale, see also Remark 6.2. An example is the, so called, minimal market model in [7].

Corollary 7.4 emphasizes the conditions under which a Central Bank is free to set the short rate to any level that is economically appropriate without generating any arbitrage. This also means when modelling a short rate process one has to mimic the actions of the Central Bank with respect to the changing financial and economic conditions.

7.3 Example

We consider a financial market with a locally risk free and a risky primary security account

$$\frac{dS_t^1}{S_t^1} = r \, dt$$
$$\frac{dS_t^2}{S_t^2} = \mu \, dt + \sigma \, dW_t^1$$

for some constants $r, \mu, \sigma \in \mathbb{R}$ with $\sigma \neq 0$. Assuming $n = 2$ underlying independent Brownian motions, $W_t = (W_t^1, W_t^2)^T$, we thus have in line with (1)

$$a_t = \begin{pmatrix} r \\ \mu \end{pmatrix}, \quad b_t = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}, \quad b_t \cdot b_t^T = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}.$$

The GOP is given by the solution of (7),

$$\pi_t^* = \begin{pmatrix} 1-(\mu-r)/\sigma^2 \\ (\mu-r)/\sigma^2 \end{pmatrix}, \quad \lambda_t^* = r,$$

that is, $\theta_t = b_t^T \cdot \pi_t^* = ((\mu-r)/\sigma, 0)^T$ and

$$\frac{dS_t^*}{S_t^*} = \left(r + \frac{(\mu-r)^2}{\sigma^2}\right) dt + \frac{\mu-r}{\sigma}\, dW_t^1, \tag{40}$$

as shown in (10) and Theorem 3.2.

Now introduce the new security account

$$\frac{d\Sigma_t}{\Sigma_t} = \alpha\, dt + \rho\, dW_t^2,$$

for some constants $\alpha, \rho \in \mathbb{R}$. In line with (22), this reads $\beta_t = (0,\rho)^T$. Hence $b_t \cdot \beta_t = 0$. The extended GOP is given via the unique solution $x_t^* = (1,0)^T$, $y_t^* = 0$ of (31), which now reads

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \sigma^2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_t^1 \\ x_t^2 \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{41}$$

Thus $b_t^T \cdot x_t^* = 0$ and in view of (32), $p_t^* = (\alpha - r)/\rho^2$. Note that $\rho = 0$ necessitates $\alpha = r$ and thus $p_t^* = 0$. The extended GOP strategy (25) is

$$\widetilde{\pi}_t^* = \begin{pmatrix} 1-(\mu-r)/\sigma^2 \\ (\mu-r)/\sigma^2 \\ 0 \end{pmatrix} + (\alpha-r)/\rho^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - (\alpha-r)/\rho^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

From (27) and (28), we obtain $\widetilde{\lambda}_t^* = \lambda_t^* = r$ and $\widetilde{\theta}_t = ((\mu-r)/\sigma, (\alpha-r)/\rho)^T$. Hence the extended GOP value process is

$$\frac{d\widetilde{S}_t^*}{\widetilde{S}_t^*} = \left(r + \frac{(\mu-r)^2}{\sigma^2} + \frac{(\alpha-r)^2}{\rho^2}\right) dt + \frac{\mu-r}{\sigma}\, dW_t^1 + \frac{\alpha-r}{\rho}\, dW_t^2. \tag{42}$$

This example further illustrates the preceding special results:

(i) For $\alpha = r$, we are in the situation of (34). Indeed, it is obvious from (40) and (42) that $S_t^* = \widetilde{S}_t^*$ in this case. Hence the introduction of new traded noise into the market does not yet necessarily change the GOP.

(ii) From (41) we see that (38) is satisfied here. But α has no impact on the prevailing short rates $\widetilde{\lambda}_t^* = r$ in the extended market. This is in line with the failure of (39).

(iii) We could have started with S^2 and Σ as the two primary security accounts, assuming $\sigma \neq 0$ and $\rho \neq 0$. Straightforward calculations, following (7), (10) and Theorem 3.2, give

$$\lambda_t^* = \frac{\alpha\sigma^2 + \mu\rho^2 - \sigma^2\rho^2}{\sigma^2 + \rho^2}$$

and the GOP

$$\frac{dS_t^*}{S_t^*} = \left(\lambda_t^* + \frac{(\mu - \lambda_t^*)^2}{\sigma^2} + \frac{(\alpha - \lambda_t^*)^2}{\rho^2}\right) dt + \frac{\mu - \lambda_t^*}{\sigma}\, dW_t^1 + \frac{\alpha - \lambda_t^*}{\rho}\, dW_t^2. \tag{43}$$

Since (38) and (39) are satisfied for this market, we know from Corollary 7.4 that the prevailing short rate can be exogenously set to any arbitrary level r. Indeed, this fact becomes obvious in our example by comparing (43) with (42), where the latter is just the new GOP for the original market, S^2 and Σ, extended by the locally risk free account S^1.

8 Conclusion

In this paper we have elaborated on the sensitivity of the growth optimal portfolio (GOP) with respect to market extensions. We provided a complete characterization of markets which can actually be extended in a consistent way. Our results are normative as we provided a three fund separation for the extended GOP: it consists of holding the original GOP and a position in the new security account, balanced by some portfolio formed by the original market which optimally replicates the new security account. A special result allows Central Banks to assess their possibilities of setting the short rate to any level that is economically appropriate without generating any arbitrage.

A Proof of Theorem 7.1

From Theorem 3.2 we know that the extended GOP exists if and only if

$$\begin{pmatrix} a_t \\ \alpha_t \\ 1 \end{pmatrix} \in \operatorname{im}(\widetilde{M}_t), \tag{44}$$

for the symmetric $(m+2) \times (m+2)$-matrix

$$\widetilde{M}_t := \begin{pmatrix} b_t \cdot b_t^T & b_t \cdot \beta_t & \mathbf{1} \\ \beta_t^T \cdot b_t^T & \beta_t^T \cdot \beta_t & 1 \\ \mathbf{1}^T & 1 & 0 \end{pmatrix}.$$

The extended GOP strategy in (25) is then given as solution of the $(m+2) \times (m+2)$-system of equations

$$\widetilde{M}_t \cdot \left(\begin{pmatrix} \pi_t^* \\ 0 \\ \lambda_t^* \end{pmatrix} + p_t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - p_t \begin{pmatrix} x_t \\ 0 \\ y_t \end{pmatrix} \right) = \begin{pmatrix} a_t \\ \alpha_t \\ 1 \end{pmatrix}.$$

with corresponding Lagrange multiplier $\widetilde{\lambda}_t^* = \lambda_t^* - p_t y_t$. Subtracting $\widetilde{M}_t \cdot (\pi^{*T}, 0, \lambda_t^*)^T$ on both sides, using (10) and (11), yields the equivalent system of equations

$$p_t \begin{pmatrix} b_t \cdot \beta_t \\ \beta_t^T \cdot \beta_t \\ 1 \end{pmatrix} - p_t \widetilde{M}_t \cdot \begin{pmatrix} x_t \\ 0 \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_t - \lambda_t^* - \beta_t^T \cdot \theta_t \\ 0 \end{pmatrix}. \tag{45}$$

The $(m+1)$-th equation in (45) reads

$$p_t \left(\beta_t^T \cdot \beta_t - \beta_t^T \cdot b_t^T \cdot x_t - y_t \right) = \alpha_t - \lambda_t^* - \beta_t^T \cdot \theta_t. \tag{46}$$

Omitting the $(m+1)$-th equation in (45) leaves us with the $(m+1)$-system of equations

$$p_t M_t \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix} = p_t \begin{pmatrix} b_t \cdot \beta_t \\ 1 \end{pmatrix}. \tag{47}$$

Now suppose that (45) has a solution (x_t^{*T}, p_t^*, y_t^*). If $p_t^* = 0$, then (45) implies (23). If $p_t^* \neq 0$, then (47) implies (24).

Conversely, if (23) holds then $(x_t^{*T}, p_t^*, y_t^*) = 0$ is a solution of (45). If (24) holds then there exists a solution (x_t^*, y_t^*) of (47) with arbitrary p_t. It follows by inspection that (47) with $p_t = 1$, or (31), are just the first order conditions for (30). Hence x_t^* is also solution of (30). Moreover, we have from (47) that $x^{*T} \cdot \mathbf{1} = 1$ and thus

$$y_t^* = x_t^{*T} \cdot (y_t^* \mathbf{1}) = x_t^{*T} \cdot b_t \cdot \beta_t - x_t^{*T} \cdot b_t \cdot b_t^T \cdot x_t^*. \tag{48}$$

Plugging (48) in (46) gives (32), which determines p_t^*. Note that $b_t^T \cdot x_t^* = \beta_t$ necessitates (23), whence $(x_t^{*T}, p_t^*, y_t^*) = 0$ is a solution of (45), as shown above. From (45), we derive $\widetilde{\lambda}_t^* = \lambda_t^* - p_t^* y_t^*$, which combined with (48) proves (27). Finally, (28) follows from (10). Hence Theorem 7.1 is proved.

References

[1] Becherer, D. (2001), "The numeraire portfolio for unbounded semimartingales," *Finance and Stochastics*, 5, 327–341.

[2] Föllmer, H. and Schweizer, M. (1991), "Hedging of contingent claims under incomplete information," *in* M. H. A. Davis and R. J. Elliott (eds), *Applied Stochastic Analysis*, Vol. 5 of *Stochastics Monogr.*, Gordon and Breach, London/New York, 389–414.

[3] Golub, G.H. and Van Loan, C.F. (1996), *Matrix Computations*, Third Edition, John Hopkins University Press.

[4] Karatzas, I. and Shreve, S.E. (1998), *Methods of Mathematical Finance*, Vol. 39 of *Appl. Math.*, Springer.

[5] Kelly, J.R. (1956), "A new interpretation of information rate," *Bell. Syst. Techn. J.*, 35, 917–926.

[6] Long, J.B. (1990), "The numeraire portfolio," *J. Financial Economics*, 26, 29–69.

[7] Platen, E. (2006), "A benchmark approach to finance," *Mathematical Finance*, 16, 131–151.

[8] Platen, E. and Heath, D. (2006), *A Benchmark Approach to Quantitative Finance*, Springer.