Infinite-dimensional methods for path-dependent equations

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Based on

Flandoli F., Zanco G. - *An infinite-dimensional approach to path-dependent Kolmogorov equations*, to be published on AoP

Flandoli F., Russo F., Zanco G. - *Infinite-dimensional calculus under weak spatial regularity of the process*, in preparation

Motivation

Models with general (adapted) dependance on the past

- Delay in state and/or in control variable
- Pricing and hedging of path-dependent options
- Pricing and hedging of options written on stocks described by equations that depend on (at least part of) the previous history of the stocks
- Analysis of incentives to innovation in firms with an established market, optimal investment strategies depending on knowledge and R&D investment

$$\begin{cases} \mathrm{d}X(t) &= b_t(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W(t) \quad \text{for } t \in [0,T], \\ X(0) &= x \end{cases}$$

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$$X_t = \{X(s)\}_{s\in[0,t]} \,,$$

$$b = \{b_t\}_{t\in[0,T]} \,,\ b_t : D([0,t];\mathbb{R}^d) \to \mathbb{R}^d$$

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SDEs: classical. Tsirelson’s example.

Calculus&PDEs: very recent, Dupire ’09, Cont-Fournié ’10&’13, Peng ’11, Cosso ’12, Ekren-Keller-Touzi-Zhang ’13&’14, Di Girolami-Russo ’10&’14, Cosso-Russo ’15, Cosso-Federico-Gozzi-Rosestolato-Touzi ’15...

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Example

$$b_t\left(\gamma_t\right) = \int_0^t g\left(\gamma(t), \gamma(s)\right)\, \mathrm{d}s$$

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Example

$$b_t(\gamma_t) = q(\gamma(t), \gamma(t-\tau))$$

SDEs and Kolmogorov equation

Delay equations framework

SDEs and Kolmogorov equation

Delay equations framework

$$\mathcal{L}^p := \mathbb{R}^d \times \mathrm{L}^p((-T,0);\mathbb{R}^d),\ p \geq 2,$$
$$\mathcal{D} := \mathbb{R}^d \times D_b([-T,0);\mathbb{R}^d),$$
$$\mathcal{C} := \mathbb{R}^d \times \left\{ \varphi \in C_b([-T,0);\mathbb{R}^d) : \exists \lim_{s\uparrow 0} \varphi(s) \right\},$$
$$\overset{\curvearrowleft}{\mathcal{C}} := \left\{ y = \binom{x}{\varphi} \in \mathcal{C} \text{ s.t. } x = \lim_{s\uparrow 0} \varphi(s) \right\}.$$

backward extension operator

$$L_t : D([0,t);\mathbb{R}^d) \longrightarrow D([-T,0);\mathbb{R}^d)$$
$$L_t(\gamma)(s) = \gamma(0)\mathbb{1}_{[-T,-t)}(s) + \gamma(t+s)\mathbb{1}_{[-t,0)}(s), \quad s \in [-T,0)$$

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restriction operator

$$M_t : D([-T,0);\mathbb{R}^d) \longrightarrow D([0,t);\mathbb{R}^d)$$
$$M_t(\varphi)(s) = \varphi(s-t), \quad s \in [0,t)$$

$$\widetilde{M}_t \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix} \right)(s) = \begin{cases} M_t\varphi(s) & s \in [0,t) \\ x & s = t \end{cases}$$

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$$M_t L_t \gamma = \gamma \ , \quad L_t M_t \varphi \neq \varphi$$

$$\hat{b}\left(t, \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\right) = \hat{b}(t, x, \varphi) := b_t\left(\widetilde{M}_t \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\right); \quad b_t(\gamma) := \hat{b}(t, \gamma(t), L_t\gamma)$$

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$$Y(t) = \begin{pmatrix} X(t) \\ \{X(t+s)\}_{s\in[-T,0]} \end{pmatrix} = \begin{pmatrix} X(t) \\ X_{[t-T,t]} \end{pmatrix}$$

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$$\mathrm{Dom}(A) = \left\{\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in \overset{\curvearrowleft}{\mathcal{C}} \text{ s.t. } \varphi \in C^1\left([-T, 0); \mathbb{R}^d\right)\right\} \Rightarrow Y(t) \notin \mathrm{Dom}(A)$$

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A generates semigroup e^{tA}, C_0 in $\mathcal{L}^p$ (with the right domain) and in $\overset{\curvearrowleft}{\mathcal{C}}$, not C_0 in $\mathcal{C}$ and in $\mathcal{D}$

Assumption

$$B \in L^{\infty}\left(0, T; C_b^{2,\alpha}(E, E)\right), \qquad \Sigma \textit{ constant}$$

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$$Y^{t_0,y}(\omega, t) = e^{(t-t_0)A}y + \int_{t_0}^{t} e^{(t-s)A} B(s, Y^{t_0,y}(\omega, s))\, \mathrm{d}s + \int_{t_0}^{t} e^{(t-s)A} \Sigma\, \mathrm{d}\beta(\omega, s)$$

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- Existence and uniqueness of solutions

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- Two times Fréchet differentiable w.r.t. initial data with α-Hölder second differential

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- Continuous if $E = \mathcal{L}^p$, L^∞ if $E = \mathcal{D}$
- Two times Fréchet differentiable w.r.t. initial data with α-Hölder second differential
- Continuous $\Longrightarrow$ Markov

$$\frac{\partial u}{\partial t}(t,y)+\langle Du(t,y), Ay + B\left(t,y\right)\rangle+\frac{1}{2}\operatorname{Tr}\left(\Sigma\Sigma^{*}D^{2}u(t,y)\right)=0, \qquad u(T,\cdot)=\Phi,$$

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$$u(t,y)=\mathbb{E}\left[\Phi\left(Y^{t,y}(T)\right)\right]$$

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$$\operatorname{Tr}\left(\Sigma\Sigma^* D^2 u(t,y)\right) = \sum_{j=1}^{\infty} \sigma_j^2 D^2 u(t,y)(e_j, e_j)$$

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$$u\left(t,y\right)-\Phi\left(y\right)=\int_{t}^{T}\left\langle Du\left(s,y\right),Ay+B\left(s,y\right)\right\rangle\,\mathrm{d}s+$$
$$+\frac{1}{2}\int_{t}^{T}\sum_{j=1}^{d}\sigma_j^2 D^2u(s,y)(e_j,e_j)\,\mathrm{d}s$$

$$u(t,y) - \Phi(y) = \int_t^T \langle Du(s,y), Ay + B(s,y) \rangle \, \mathrm{d}s + \\ + \frac{1}{2} \int_t^T \sum_{j=1}^d \sigma_j^2 D^2 u(s,y)(e_j, e_j) \, \mathrm{d}s$$

Definition

Given $\Phi \in C_b^{2,\alpha}(\mathcal{D}, \mathbb{R})$, we say that $u : [0,T] \times \mathcal{D} \to \mathbb{R}$ is a classical solution of the Kolmogorov equation with final condition Φ if

$$u \in L^\infty\left(0, T; C_b^{2,\alpha'}(\mathcal{D}, \mathbb{R})\right) \cap C([0,T] \times \mathcal{D}, \mathbb{R})$$

for some $\alpha' \in (0,1)$, and satisfies the above identity for every $t \in [0,T]$ and $y \in \mathrm{Dom}(A)$, with the duality terms understood with respect to the topology of $\mathcal{D}$.

Theorem

Let $\Phi \in C^{2,\alpha}(\mathcal{D},\mathbb{R})$ *be given and let* $B \in L^\infty\left(0,T;C_b^{2,\alpha}(\mathcal{D},\mathcal{D})\right)$. *The function* $u:[0,T]\times\mathcal{D}\to\mathbb{R}$ *given by*

$$u(t,y)=\mathbb{E}\left[\Phi\left(Y^{t,y}(T)\right)\right],$$

is a solution of the Kolmogorov equation with final condition Φ *under the assumption that for any* $s\in[-T,0]$, *any* $r\geq s$, *any* $y\in\overset{\frown}{\mathcal{C}}$ *and for almost every* $a\in[-T,0]$ *the following hold:*

$$\langle D\Phi(y), J_n\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix}\rangle \longrightarrow \langle D\Phi(y), \begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix}\rangle;$$

$$D^2\Phi(y)\Big(J_n\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix}-\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix},\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix}\Big)\longrightarrow 0;$$

$$D^2\Phi(y)\Big(\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix},J_n\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix}-\begin{pmatrix}1\\ \mathbb{1}_{[a,0]}\end{pmatrix}\Big)\longrightarrow 0;$$

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and idem for B.

Theorem

If $\Phi : \mathcal{L}^p \to \mathbb{R}$ *is* $C_b^{2,\alpha}$ *and* $B \in L^\infty\left(0, T; C_b^{2,\alpha}\left(\mathcal{L}^p, \mathcal{L}^p\right)\right)$ *then the function*

$$u\left(t, y\right) := \mathbb{E}\left[\Phi\left(Y^{t,y}\left(T\right)\right)\right], \qquad \left(t, y\right) \in [0, T] \times \mathcal{L}^p,$$

is a solution of the Kolmogorov equation in $\mathcal{L}^p$ *with final condition* Φ.

Theorem

If $\Phi : \mathcal{L}^p \to \mathbb{R}$ *is* $C_b^{2,\alpha}$ *and* $B \in L^\infty\left(0, T; C_b^{2,\alpha}\left(\mathcal{L}^p, \mathcal{L}^p\right)\right)$ *then the function*

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is a solution of the Kolmogorov equation in $\mathcal{L}^p$ *with final condition* Φ*.*

+ approximating procedure using smoothing only of the past

Change of variables formula

What about uniqueness?

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What about uniqueness? Need Ito formula

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$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + B(t, Y(t))\,\mathrm{d}t + \Sigma(t, Y(t))\,\mathrm{d}\beta(t) \in \mathcal{C}\ ,$$

Change of variables formula

What about uniqueness? Need Ito formula

$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + B(t, Y(t))\,\mathrm{d}t + \Sigma(t, Y(t))\,\mathrm{d}\beta(t) \in \mathcal{C}\,, \quad F : [0,T] \times \overset{\curvearrowleft}{\mathcal{C}} \longrightarrow \mathbb{R}$$

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$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + B(t,Y(t))\,\mathrm{d}t + \Sigma(t,Y(t))\,\mathrm{d}\beta(t) \in \mathcal{C}\,, \quad F : [0,T] \times \overset{\curvearrowleft}{\mathcal{C}} \longrightarrow \mathbb{R}$$

$$\begin{aligned} F(t,Y(t)) - F(0,Y(0)) &= \int_0^t \partial_t F(s,Y(s))\,\mathrm{d}s \\ + \int_0^t \langle AY(s) + B(s,Y(s), DF(s,Y(s))\rangle\,\mathrm{d}s &+ \int_0^t \langle DF(s,Y(s)), \Sigma(s,Y(s))\,\mathrm{d}\beta(s)\rangle \\ &+ \int_0^t \frac{1}{2}\,\mathrm{Tr}_{\mathbb{R}^d}\left[\Sigma(s,Y(s))\Sigma(s,Y(s))^* D^2 F(s,Y(s))\right]\,\mathrm{d}s \end{aligned}$$

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$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + B(t, Y(t))\,\mathrm{d}t + \Sigma(t, Y(t))\,\mathrm{d}\beta(t) \in \mathcal{C}\,, \quad F : [0, T] \times \overset{\curvearrowleft}{\mathcal{C}} \longrightarrow \mathbb{R}$$

$$F(t, Y(t)) - F(0, Y(0)) = \int_0^t \partial_t F(s, Y(s))\,\mathrm{d}s$$

$$+ \int_0^t \langle AY(s) + B(s, Y(s)), DF(s, Y(s))\rangle\,\mathrm{d}s + \int_0^t \langle DF(s, Y(s)), \Sigma(s, Y(s))\,\mathrm{d}\beta(s)\rangle$$

$$+ \int_0^t \frac{1}{2}\,\mathrm{Tr}_{\mathbb{R}^d}\left[\Sigma(s, Y(s))\Sigma(s, Y(s))^* D^2 F(s, Y(s))\right]\,\mathrm{d}s$$

Problems: $Y(t) \notin \mathrm{Dom}(A)$ and usually also $\partial_t F(t, \cdot)$ makes sense only on $\mathrm{Dom}(A)$

Theorem

Let $F \in C([0,T]\times\mathcal{C})$ *be such that* $DF \in C([0,T]\times\mathcal{C};\mathcal{C}^*)$, $D^2F \in C([0,T]\times\mathcal{C}; L(\mathcal{C};\mathcal{C}^*))$, $\partial_t F \in C([0,T]\times\mathrm{Dom}(A);\mathbb{R})$. *Assume that there exists a continuous function* $G : [0,T]\times\overset{\curvearrowleft}{\mathcal{C}} \to \mathbb{R}$ *such that*

$$G(t,y) = \partial_t F(t,y) + \langle Ay, DF(t,y)\rangle \quad \text{on} \quad [0,T]\times\mathrm{Dom}(A).$$

Then

$$\begin{aligned} F(t,Y(t)) &= F(0,Y(0)) + \int_0^t G(s,Y(s))\,\mathrm{d}s \\ &+ \int_0^t \langle B(s,Y(s)), DF(s,Y(s))\rangle\,\mathrm{d}s + \int_0^t \langle DF(s,Y(s)), \Sigma(s,Y(s))\,\mathrm{d}\beta(s)\rangle \\ &+ \int_0^t \frac{1}{2}\mathrm{Tr}_{\mathbb{R}^d}\left[\Sigma(s,Y(s))\Sigma(s,Y(s))^* D^2F(s,Y(s))\right]\,\mathrm{d}s \end{aligned}$$

Theorem

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Theorem

If F, DF, D^2F are piecewise continuous, i.e. $\exists\, t_1, \dots, t_n$ s.t. $F \in C\left([t_j, t_{j+1}) \times E\right)$, $\forall\, y\ t \mapsto F(t,y)$ is càdlàg, $\forall t\ y \mapsto F(t,y)$ is continuous, $\partial_t F$ exists on $\mathcal{T} \times \mathrm{Dom}(A)$, then

$$
\begin{aligned}
F(t, Y(t)) = F(0, Y(0)) + \int_0^t \hat{G}(s, Y(s))\,\mathrm{d}s + \Delta F(Y; t_1, \dots, t_n) \\
+ \int_0^t \langle B(s, Y(s)), DF(s, Y(s))\rangle\,\mathrm{d}s + \int_0^t \langle DF(s, Y(s)), \Sigma(s, Y(s))\,\mathrm{d}\beta(s)\rangle \\
+ \int_0^t \frac{1}{2}\,\mathrm{Tr}_{\mathbb{R}^d}\left[\Sigma(s, Y(s))\Sigma(s, Y(s))^* D^2 F(s, Y(s))\right]\,\mathrm{d}s
\end{aligned}
$$

Example

$$f_t\left(\gamma_t\right) = \int_0^t g\left(\gamma(t), \gamma(s)\right)\, \mathrm{d}s \quad \Rightarrow \quad \frac{\partial F}{\partial t}(t, y) = g\left(y^{(1)}, y^{(2)}(-t)\right)$$

but

$$\frac{\partial F}{\partial t}(t, y) + \langle Ay, DF(t, y)\rangle = g\left(y^{(1)}, y^{(1)}\right)$$

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Example

$$f_t(\gamma_t) = h(\gamma(t), \gamma(t_0)) \Rightarrow \frac{\partial F}{\partial t}(t,y) = -\partial_2 h\left(y^{(1)}, y^{(2)}(t_0 - t)\right) \cdot \dot{y^{(2)}}(t_0 - t)$$

but

$$\frac{\partial F}{\partial t}(t,y) + \langle Ay, DF(t,y)\rangle = 0$$

Example

$$f_t\left(\gamma_t\right) = w\left(\gamma(t), \int_0^t p(s)\,\mathrm{d}\gamma(s)\right)$$

$$\begin{aligned}\frac{\partial F}{\partial t}(t,y) &= \partial_2 w\left(y^{(1)}, \int_0^t p(s)\,\mathrm{d}y^{(2)}(s-t)\right) \cdot \int_0^t \dot{p}(s)\,\mathrm{d}\dot{y}^{(2)}(s-t) \\ &= -\langle Ay, DF(t,y)\rangle\end{aligned}$$

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For the Kolmogorov equation the equation itself provides the extension:

$$G(t,y) = -\langle B(t,y), DF(t,y)\rangle - \frac{1}{2}\operatorname{Tr}_{\mathbb{R}^d}\left[\Sigma\Sigma^* D^2F(t,y)\right]$$

therefore uniqueness follows by standard arguments.

Example

For $f_t\left(\gamma_t\right) = q\left(\gamma(t), \gamma(t-\tau)\right)$ we have

$$
\begin{aligned}
f_t\left(W_t\right) - q(0,0) = & \int_0^t \partial_1\, q\left(W(s), W(s-\tau)\right) \\
& + \frac{1}{2}\int_0^t \partial^2_{11}\, q\left(W(s), W(s-\tau)\right)\, \mathrm{d}s + \frac{1}{2}\int_0^t \partial^2_{22}\, q\left(W(s), W(s-\tau)\right)\, \mathrm{d}s \\
& + \int_0^{t-\tau} \partial_2\, q\left(W(s+\tau), W(s)\right) \delta W(s) + \int_\tau^t \partial^2_{12}\, q\left(W(s), W(s-\tau)\right)\, \mathrm{d}s
\end{aligned}
$$

Comparison with functional Ito calculus

$$\nabla^i \nu_t(\gamma_t) = \lim_{h \to 0} \frac{\nu_t\left(\gamma_t^{he_i}\right) - \nu_t(\gamma_t)}{h} \qquad \mathscr{D}_t \nu\left(\gamma_t\right) = \lim_{h \to 0^+} \frac{\nu_{t+h}\left(\gamma_{t,h}\right) - \nu_t\left(\gamma_t\right)}{h}$$

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$$\begin{cases} \mathscr{D}_t \nu(\gamma_t) + b_t(\gamma_t)\cdot\nabla\nu_t(\gamma_t) + \frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^*\nabla^2\nu_t(\gamma_t)\right] = 0\,, \\ \nu_T(\gamma_T) = f(\gamma_T) \end{cases}$$

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Theorem (Dupire - 2009; Cont, Fournié - 2013)

(Under suitable assumptions)

$$\nu_t\left(X_t\right) - \nu_0\left(X_0\right) = \int_0^t \mathscr{D}_s \nu_s\left(X_s\right)\, \mathrm{d}s + \int_0^t \nabla \nu_s\left(X_s\right)\, \mathrm{d}X(s) + \\ + \frac{1}{2} \int_0^t \operatorname{Tr} \nabla^2 \nu_s\left(X_s\right)\, \mathrm{d}\left[X\right](s)$$

Theorem

If $\nu_t(\gamma) := u(t, \gamma(t), L_t\gamma)$ *or* $u(t, x, \varphi) := \nu_t\left(\widetilde{M}_t\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\right)$ *then*

$$\mathscr{D}_i\nu_t(\gamma) = \frac{\partial}{\partial x_i}u(t, x, L_t\gamma), \quad i = 1, \ldots, d\ .$$

If $\nu_t(\gamma) := u(t, \gamma(t), L_t\gamma)$ *and* Du *and* $\partial_t u$ *exists on* $\mathrm{Dom}(A)$*, then* $\mathscr{D}_t\nu$ *exists on* $\mathrm{Dom}(A)$ *and*

$$\mathscr{D}_t\nu(\gamma_t) = \frac{\partial u}{\partial t}(t, \gamma(t), L_t\gamma_t) + \langle Du(t, \gamma(t), L_t\gamma_t), A(L_t\gamma_t)\rangle.$$

If $u(t, x, \varphi) := \nu_t\left(\widetilde{M}_t\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\right)$*,* $\mathscr{D}_t\nu$ *exists everywhere and* Du*,* $\partial_t u$ *exist on* $\mathrm{Dom}(A)$ *then* $\mathscr{D}_t\nu$ *is the extension.*

Theorem

If the infinite dimensional liftings B *and* Φ *of* b_t *and* f *satisfy the assumptions of the previous theorems, then, for almost every* t*, the function*

$$\nu_t(\gamma_t) = \mathbb{E}\left[f\left(X^{\gamma_t}(T)\right)\right]$$

is a solution of the path dependent Kolmogorov equation.

Theorem

If the infinite dimensional liftings B and Φ of b_t and f satisfy the assumptions of the previous theorems, then, for almost every t, the function

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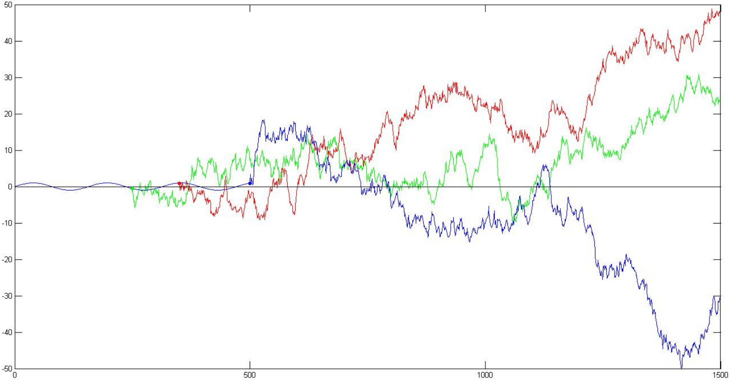
is a solution of the path dependent Kolmogorov equation.

Proof:

set $y = (\gamma(t), L_t\gamma_t) \in \overset{\curvearrowleft}{\mathcal{C}}$;

$$\begin{aligned}
\nu_t(\gamma_t) :&= u(t, \gamma(t), L_t\gamma_t) \\
&= u(t, y) \\
&= \mathbb{E}\left[\Phi\left(Y^{t,y}(T)\right)\right] \\
&= \mathbb{E}\left[f\left(\widetilde{M}\left(Y^{t,y}(T)\right)\right)\right] \\
&= \mathbb{E}\left[f\left(X^{\gamma_t}(T)\right)\right]
\end{aligned}$$



Thank you for your kind attention