Measures of Systemic Risk

Stefan Weber

Leibniz Universität Hannover

www.stochastik.uni-hannover.de

(joint work with Zachary Feinstein & Birgit Rudloff)

7th General AMaMeF and Swissquote Conference – 9 September 2015

Motivation

- Various financial crises have highlighted the paramount importance of systemic risk in the financial sector.
- The tremendous cost of systemic risk requires instruments for an efficient macroprudential regulation of financial institutions.
- **Goal of talk**:

  Novel approach to the measurement of systemic risk

  - Systemic risk measures that are based on macroprudential objectives,
  - but enable at the same time systemic risk measurement on the level of firms.

Outline

(i) Measures of systemic risk

- General definition and properties

(ii) Orthant risk measures

- Conservative simplification that excludes externalities of capital levels

(iii) Numerical examples

- Systemic risk aggregation as described in Chen, Iyengar & Moallemi (2013) and Kromer, Overbeck & Zilch (2015); network models as suggested by Eisenberg & Noe (2001) and Cifuentes, Shin & Ferrucci (2005), see also Awiszus & W. (2015)

Measures of Systemic Risk

The Basic Ingredients

Consider a one-period economy with l entities.

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(i) **Structure of the underlying system**

- $Y = (Y_k)_{k \in \mathbb{R}^l}$ non-decreasing random field
  - For each capital allocation $k = (k_i)_{i=1,2,\ldots,l}$ the random variable Y_k captures the relevant stochastic outcome
  - The topological vector space of suitable random variables is denoted by $\mathcal{X}$

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(ii) **Objectives of a financial regulator**

- $\mathcal{A} \subseteq \mathcal{X}$ set of random variables
  - Each element of $\mathcal{A}$ is acceptable from the point of view of a regulatory authority
  - Mathematically: an **acceptance set** of a scalar monetary risk measure

Systemic Risk Measures – Definition

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Definition 1

Letting $\mathcal{P}(\mathbb{R}^l;\mathbb{R}^l_+) := \{B \subseteq \mathbb{R}^l \mid B = B + \mathbb{R}^l_+\}$ *be the collection of upper sets with ordering cone* $\mathbb{R}^l_+$*, we call the function*

$$R : \mathcal{Y} \times \mathbb{R}^l \to \mathcal{P}(\mathbb{R}^l;\mathbb{R}^l_+)$$

a **systemic risk measure***, if for some acceptance set* $\mathcal{A} \subseteq \mathcal{X}$ *of a scalar monetary risk measure:*

$$\mathbf{R(Y;k) = \{m \in \mathbb{R}^l \mid Y_{k+m} \in \mathcal{A}\}.}$$

Systemic Risk Measures – Properties

(i) **Cash-invariance**:

$$R(Y;k) + m = R(Y;k-m)$$

(ii) **Monotonicity**:

$$(\forall k \in \mathbb{R}^l: \; Y_k \geq Z_k) \;\Rightarrow\; (\forall k \in \mathbb{R}^l: \; R(Y;k) \supseteq R(Z;k))$$

(iii) **Closed values**:

Suppose that $\mathbb{R}^l \to \mathcal{X}, k \mapsto Y_k$ is continuous. Then $R(Y;k)$ is a closed subset of $\mathbb{R}^l$.

Systemic Risk Measures – Properties (2)

(i) **Convex values**:

Suppose that $\mathcal{A}$ is convex and that $\mathbb{R}^l \to \mathcal{X}, k \mapsto Y_k$ is concave. Then $R(Y;k)$ is a convex subset of $\mathbb{R}^l$ for all $k \in \mathbb{R}^l$, i.e. $R(Y;\cdot)$ has convex values.

(ii) **Diversification and quasi-convexity**:

The required notion of diversification is slightly more complicated for random fields.

An appropriate construction is described in

Feinstein, Rudloff & W. (2015).

Examples of Random Fields

Examples of Non-Decreasing Random Fields

(i) Aggregation mechanism insensitive to capital levels

- Setting of Chen, Iyengar & Moallemi (2013)
- Based on axiomatic characterization of scalar systemic risk measures

(ii) Aggregation mechanism sensitive to capital levels

- Extension that allows for feedback effects

(iii) Financial networks with market clearing

- This includes network models as described in Eisenberg & Noe (2001), Cifuentes, Shin & Ferrucci (2005), Rogers & Veraart (2013), Awiszus & W. (2015)
- Essentially special case of example (ii)

Example (i): Aggregation mechanism insensitive to capital levels

- Interconnected financial economy of financial institutions

$$N = \{1, 2, \dots, n\}$$

- $X \in L^0(\mathbb{R}^n)$ future wealths of the agents in the financial sector
- $\Lambda : \mathbb{R}^n \to \mathbb{R}$ increasing aggregation function
- Random output is

$$\mathbf{Y_k := \Lambda(X) + \sum_{i=1}^{n} k_i, \quad k \in \mathbb{R}^n}$$

- In this case, $n = l$.

Example (i): Aggregation mechanism insensitive to capital levels (cont.)

- If one assumes that there are l **groups** with identical capital levels, one could alternatively consider

$$\mathbf{Y_k := \Lambda(X) + \sum_{i=1}^{n} g_i(k), \quad k \in \mathbb{R}^l}$$

with

$$g(k) = (\underbrace{k_1, \ldots, k_1}_{\text{Group } 1}, \underbrace{k_2, \ldots, k_2}_{\text{Group } 2}, \ldots, \underbrace{k_l, \ldots, k_l}_{\text{Group } l}),$$

i.e. $g : \mathbb{R}^l \to \mathbb{R}^n$ increasing.

Example (ii): Aggregation mechanism sensitive to capital levels

- Example (i) can be modified by setting

$$\mathbf{Y_k := \Lambda(X + k), \quad k \in \mathbb{R}^n.}$$

- The aggregation mechanism is sensitive to capital levels. In particular, feedback from capital levels to the final outputs is captured.

- As in Example (i), we could have groups with equal capital levels which can be encoded by a function $g : \mathbb{R}^l \to \mathbb{R}^n$:

$$Y_k := \Lambda(X + g(k)), \quad k \in \mathbb{R}^l.$$

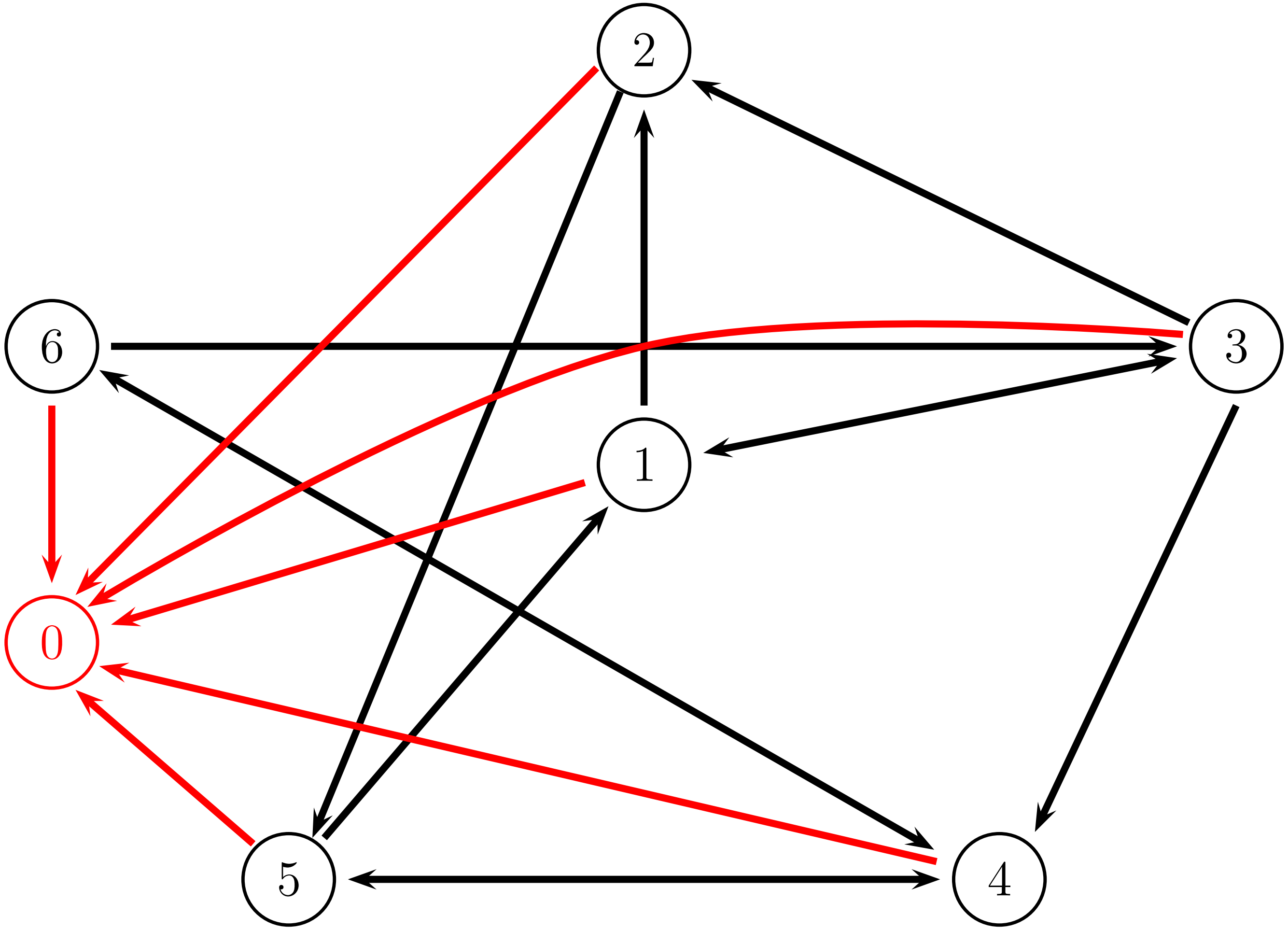
Example (iii): Financial networks

- Model of Eisenberg & Noe (2001), Cifuentes, Shin & Ferrucci (2005)
- Financial institutions $N = \{1, 2, \ldots, n\}$
- **Society** is additional node $\mathbf{0}$
- Nominal liability matrix: $(\bar{p}_{ij})_{i,j=0,1,2,\ldots,n}$, $\bar{p}_i = \sum_{j=0}^{n} \bar{p}_{ij}$
- Relative liabilities:

$$a_{ij} = \begin{cases} \frac{\bar{p}_{ij}}{\bar{p}_i}, & \bar{p}_i > 0, \\ 0, & \bar{p}_i = 0. \end{cases}$$

- Inverse demand function for illiquid asset: $f : \mathbb{R}_+ \to \mathbb{R}_{++}$

Example (iii): Financial networks (cont.)



Example (iii): Financial networks (cont.)

- Liquid positions: $x \in \mathbb{R}^n$
  - Obligations must be fulfilled via transfers of the liquid asset.
- Illiquid positions: $s \in \mathbb{R}^n$
  - If necessary, illiquid positions must be liquidated, but these are subject to price impact described by the inverse demand function.
- Equilibrium computed as unique fixed point:
  - Clearing vector: $p(x; s) \in \mathbb{R}_+^{n+1}$
  - Clearing price of illiquid asset: $\pi(x; s) \in \mathbb{R}_+$

Example (iii): Financial networks (cont.)

$$
\begin{aligned}
p_i(x;s) &= \bar{p}_i \wedge \left(\sum_{j=0}^{n} p_j(x;s) a_{ji} + x_i + \pi(x;s) s_i \right), \qquad i = 1, 2, \ldots, n \\
\pi(x;s) &= f \left[\sum_{i=1}^{n} \left(\frac{1}{\pi(x;s)} \left[\bar{p}_i - x_i - \sum_{j=0}^{n} a_{ji} p_j(x;s) \right]^+ \wedge s_i \right) \right] \\
p_0(x;s) &= \bar{p}_0
\end{aligned}
$$

$\Longrightarrow$

$$
e_i(x;s) = \sum_{j \neq i} p_j(x;s) a_{ji} + x_i + \pi(x;s) s_i - \bar{p}_i, \quad i = 0, 1, 2, \ldots, n
$$

Example (iii): Financial networks (cont.)

- Let $X, S \in L^0(\mathbb{R}^n)$ the random number of shares the agents hold at time $t = 1$ before market clearing

- If we focus at the wealth of the society, then the relevant stochastic outcome is provided by the random field

$$Y_k := e_0(X + k; S), \quad k \in \mathbb{R}^n.$$

- Special case of example (ii) with aggregation function

$$\Lambda(\cdot) := e_0(\cdot; S).$$

- Again, one can consider groups with identical capital

Orthant Risk Measures

A Pragmatic Approach

- Systemic risk measurements $R(Y;k)$ are sets of allocations of additional capital that lead to acceptable outcomes:
  - Financial firms cannot choose their capital independently of the other firms.
  - Risk measurements $R(Y;k)$ capture the essence of systemic risk, but are potentially difficult to communicate.
- A simple alternative consists in choosing a point k^* in the boundary of $R(Y;k)$ and to require firms to hold capital inside

$$k^* + \mathbb{R}^l_+$$

  - Construction is more conservative, without any externalities of the choices of capital levels, and easy to communicate.

A Pragmatic Approach (cont.)

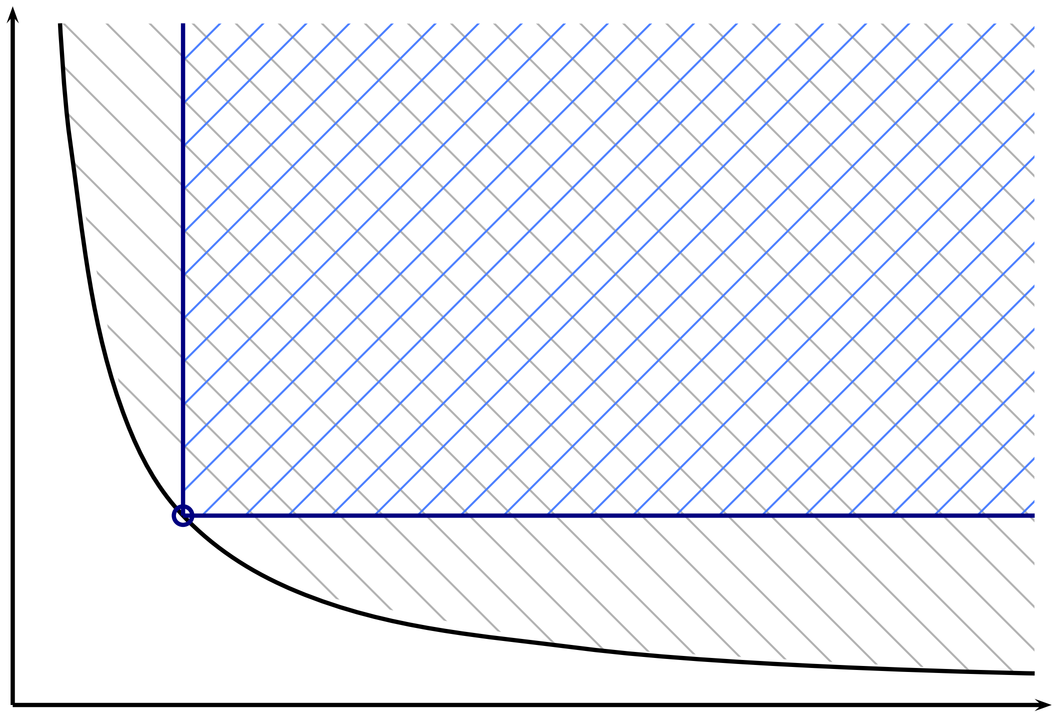


Figure 1: Illustration of a minimal point k^* of an upper set with the orthant $k^* + \mathbb{R}^2_+$ in blue.

Orthant Risk Measures – Definition

Definition 2 *Let* $\mathcal{P}(\mathbb{R}^l)$ *be the power set of* $\mathbb{R}^l$*. A mapping* $k^* : \mathcal{Y} \times \mathbb{R}^l \to \mathcal{P}(\mathbb{R}^l)$ *is called an* orthant risk measure *associated with a systemic risk measure* R*, if the following properties are satisfied:*

(i) **Minimal values***:*

$$k^*(Y;k) \subseteq \mathrm{Min} R(Y;k)$$

(ii) **Convex values***:*

$$k^1, k^2 \in k^*(Y;k) \;\Rightarrow\; \alpha k^1 + (1-\alpha)k^2 \in k^*(Y;k)$$

(iii) **Cash-invariance***:*

$$k^*(Y;k) + m = k^*(Y;k-m)$$

Orthant Risk Measures – Characterization

Lemma 1 *Let* $R : \mathcal{Y} \times \mathbb{R}^l \to \mathcal{P}(\mathbb{R}^l; \mathbb{R}^l_+)$ *be a systemic risk measure with convex values. For* $w : \mathcal{Y} \to \mathbb{R}^l_{++}$ *such that* $w(Y) \in \mathrm{recc} R(Y; 0)^+$*, the set-valued mapping*

$$\hat{k}(Y; k) = \arg\min \left\{ \sum_{i=1}^{l} w(Y)_i m_i \mid m \in R(Y; k) \right\} \qquad (1)$$

defines an orthant risk measure.

All orthant risk measures k^* *as defined above are included in orthant risk measures* $\hat{k}$ *of form* (1)*, i.e.* $k^*(Y; k) \subseteq \hat{k}(Y; k)$ *for all* $Y \in \mathcal{Y}$ *and* $k \in \mathbb{R}^l$*.*

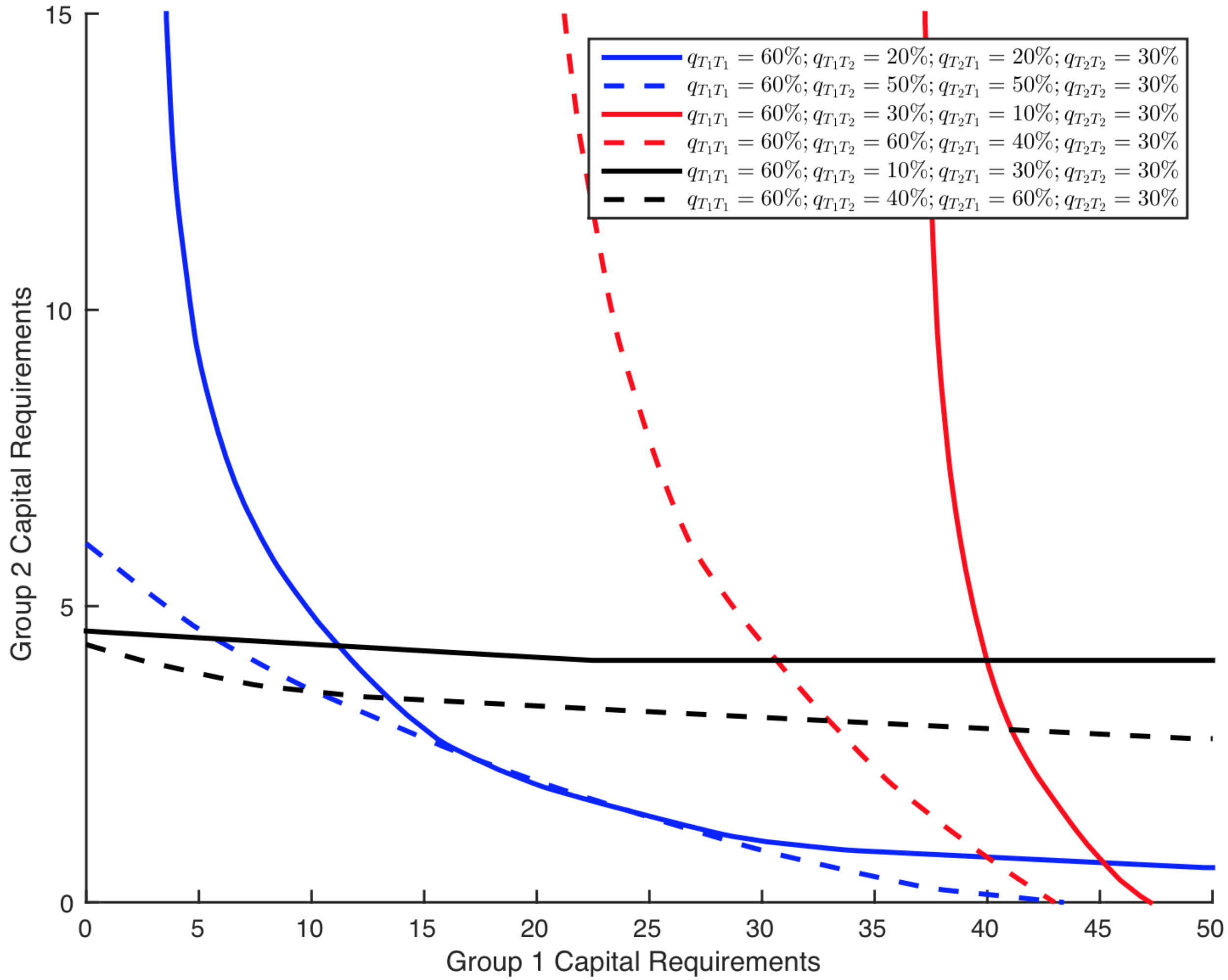
- The lemma provides examples of orthant risk measures via a specific choice of the "**regulatory price of capital**" $\mathbf{w}$.

Numerical Examples

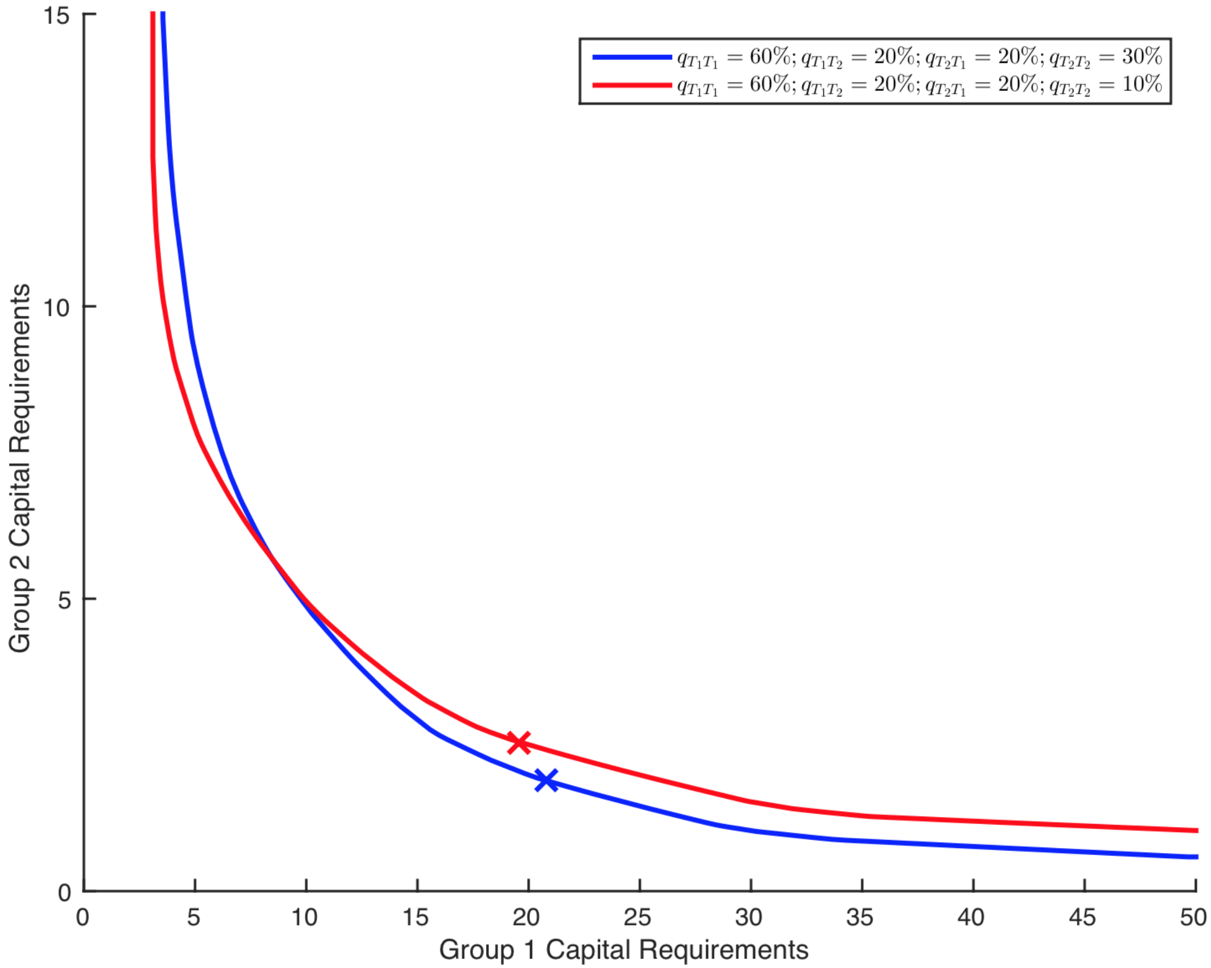
Case Study A

- Framework of Eisenberg & Noe (2001):
  only local interaction in the network
- Tiered graph:
  - Connections are randomly generated, probabilities within tiers and between tiers are fixed
  - Size of obligations within tiers and between tiers along connections are fixed
- 2 Tiers/Groups: few firms with large obligations, many firms with small obligations
- Further ingredients:
  random endowments, acceptance set defined by AV@R
- **Comparative statics: varying the degrees of connectedness**

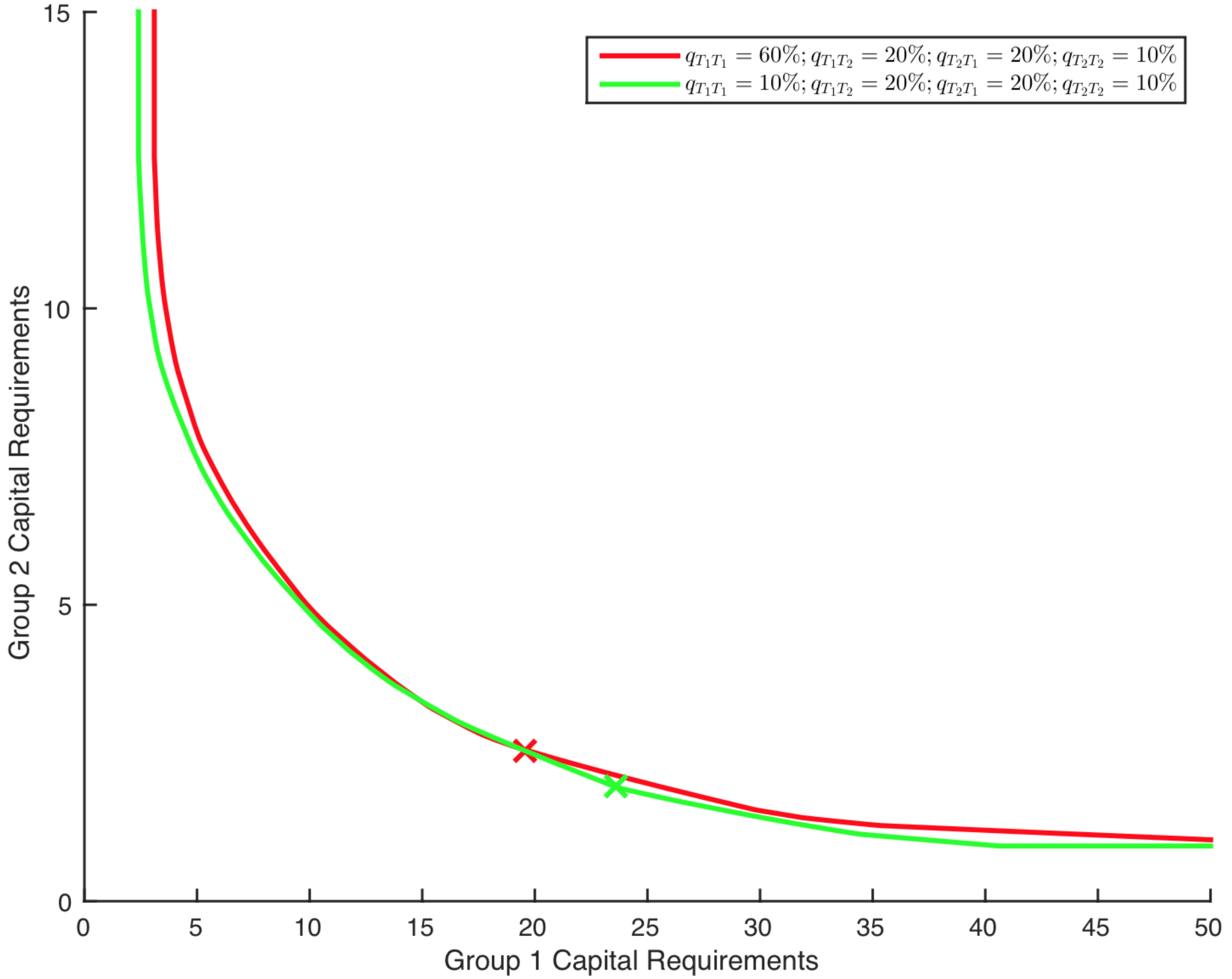
Fixed Intra-Group Connections



Fixed Inter-Group Connections



Fixed Inter-Group Connections

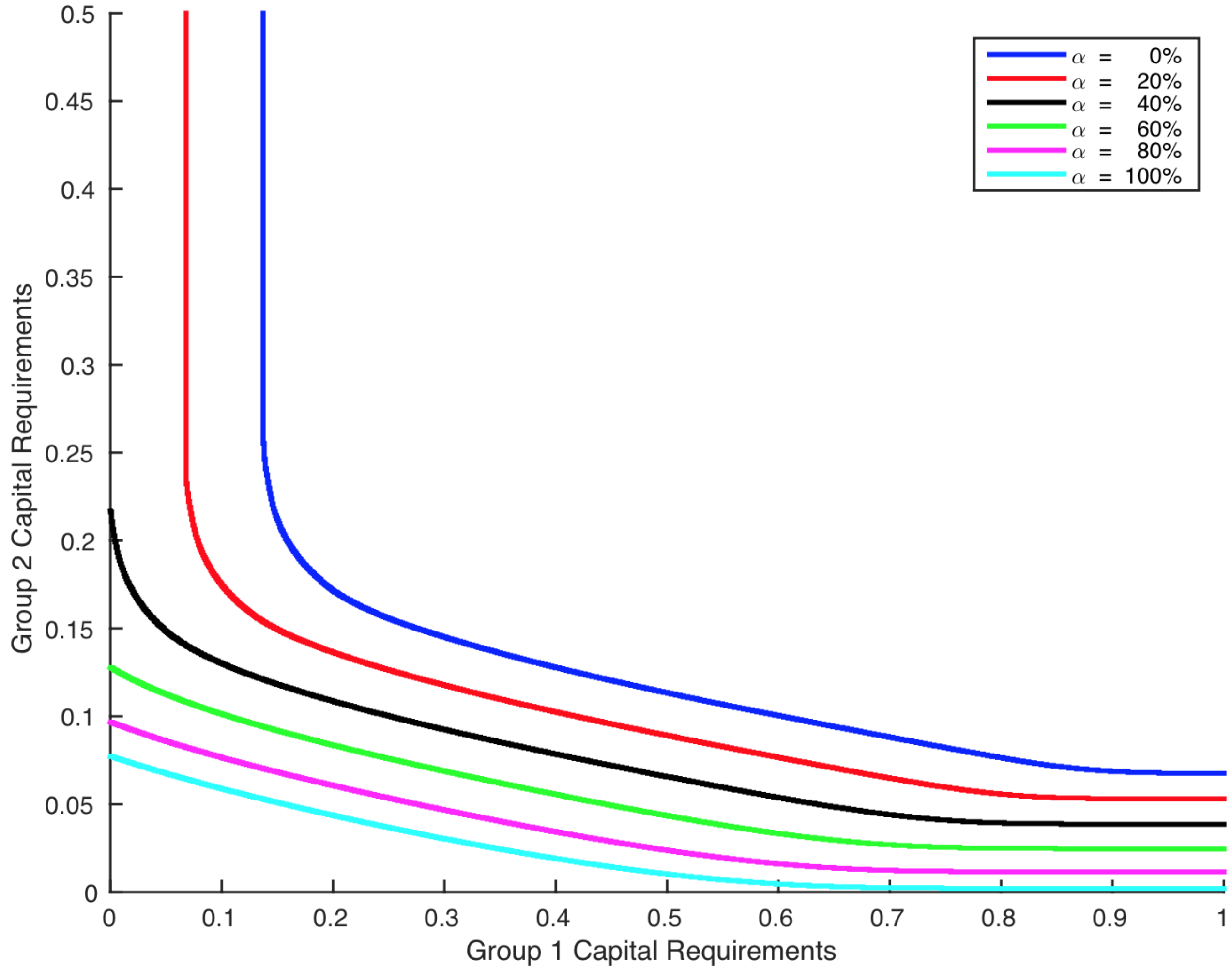


Case Study B

- Both local (network) and global (price impact) interaction
- Tiered graph:
  - Connections are randomly generated, probabilities within tiers and between tiers are fixed
  - Size of obligations within tiers and between tiers along connections are fixed
- 3 Tiers/Groups: few large, intermediate number of intermediate size, many small
- Further ingredients:

  random endowments, acceptance set defined by entropic risk measure
- **Comparative statics: varying fraction in illiquid asset**

Case Study B (2)



Conclusion

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(i) Multi-variate approach to systemic risk

- Integrates macroprudential objectives and systemic risk measurement on the level of the firms
- Applicable to general financial system models

(ii) Pragmatic approach for implementation in practice

- Conservative orthant risk measures derived from systemic risk measures via "regulatory price" of capital
- Includes previous contributions as special cases

(iii) Implementation

- Combination of Monte Carlo simulation and grid search
- Successfully implemented in case studies

References

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