Robustness of regulatory risk measures in aggregation

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Outline

Based on joint work with Paul Embrechts and Bin Wang

Risk Measures

Risk measures

A risk measure calculates the amount of capital the financial institution needs to reserve in order to undertake a risk (random loss) X in a fixed period.

- What is a good risk measure to use?
- Regulator's perspective and manager's perspective can be different (or even conflicting)

Risk Measures

A risk measure is a functional $\rho : \mathcal{X} \to [-\infty, \infty]$.

- $\mathcal{X}$ is a convex cone of random variables, $\mathcal{X} \supset L^\infty$.
- Typically one requires $\rho(L^\infty) \subset \mathbb{R}$ for an obvious reason.
- In this talk, $X \in \mathcal{X}$ represents loss/profit

A law-determined risk measure can be treated as a functional $\rho : \mathcal{D} \to [-\infty, \infty]$.

- $\mathcal{D}$ is the set of distributions of random variables in $\mathcal{X}$.

Value-at-Risk

$p \in (0,1)$, $X \sim F$.

Value-at-Risk (VaR)

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}.$$

Expected Shortfall

$p \in (0,1)$.

Expected Shortfall (ES, or TVaR, CVaR, CTE, AVaR)

$\mathrm{ES}_p : L^0 \to (-\infty, \infty]$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F\text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_p(X)\right].$$

Distortion Risk Measures

Distortion risk measure

A distortion risk measure is defined as

$$\rho(X) = \int_{\mathbb{R}} x \mathrm{d}h(F(x)), \; X \in \mathcal{X}, \; X \sim F$$

where h is an increasing function on $[0,1]$ with $h(0) = 0$ and $h(1) = 1$. h is called a distortion function.

- See Yaari (1987 Econometrika):
  distortion risk measure $\Leftrightarrow$ law-determined and comonotonic additive monetary risk measure.
- ES and VaR are special cases of distortion risk measures.

Regulatory Documents

From the **Basel Committee on Banking Supervision**:

R1: Consultative Document, May 2012,
Fundamental review of the trading book

R2: Consultative Document, October 2013,
Fundamental review of the trading book: A revised market risk framework

R3: Consultative Document, December 2014,
Fundamental review of the trading book: Outstanding issues

From the **International Association of Insurance Supervisors**:

R4: Consultation Document, December 2014,
Risk-based global insurance capital standard

Questions from Regulation

R1, Page 41, *Question 8*:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

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"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

- Some argue: Backtesting ES is difficult, whereas backtesting VaR is straightforward (Gneiting 2011 JASA)
  - Recent papers: Ziegel (2015 MF); Acerbi-Szekely (2014 Risk); Kou-Peng (2014 wp); Fissler-Ziegel-Gneiting (2015 wp)
- Some argue: ES is not robust, whereas VaR is (Cont-Deguest-Scandolo 2010 QF; Kou-Peng-Heyde 2013 MOR).

Questions from Regulation

R4, Page 43. *Question 42*:

"Which risk measure - VaR, Tail-VaR or another - is most appropriate for ICS [insurance capital standard] capital requirement purposes? Why?"

VaR versus ES

A summary of the current situation (mid 2015):

- VaR is still globally dominating banking and insurance regulation at the moment.
- ES and VaR co-exist; ES is proposed to replace VaR in many places.
- The search for alternative risk measures to VaR and ES is on going (mainly academic).

VaR versus ES

Features/Risk measure	VaR	Tail-VaR
Frequency captured?	Yes	Yes
Severity captured?	No	Yes
Sub-additive?	Not always	Always
Diversification captured?	Issues	Yes
Back-testing?	Straight-forward	Issues
Estimation?	Feasible	Issues with data limitation
Model uncertainty?	Sensitive to aggregation	Sensitive to tail modelling
Robustness I (with respect to "Lévy metric[33]")?	Almost, only minor issues	No
Robustness II (with respect to "Wasserstein metric[34]")?	Yes	Yes

From R4, International Association of Insurance Supervisors Consultation Document (December 2014).

VaR versus ES

Centers of discussion:

- backtesting: Is my (previously) estimated risk measure wrong based on incoming data?
- estimation: What if data are sparse or compromised?
- model uncertainty: What if the model is inappropriate?
- robustness: What if there is a small error in my model or data?

All refer to uncertainty

Robust Statistics

Robustness addresses the question of ”what if the data is compromised with small error?” (e.g. outlier)

- Originally robustness was defined on estimators (of a quantity T)
- Would the estimation be ruined if an outlier is added to the sample?
  - Think about VaR and ES

Robust Statistics

Classic qualitative robustness:

- Hampel (1971 AoMS): the robustness of estimators is equivalent to the continuity of T with respect to underlying distributions
- General reference: Huber and Ronchetti, 2007 book

When we talk about the robustness of a statistical functional, (Huber-Hampel's) robustness typically refers to some sense of continuity.

Robustness of Risk Measures

Consider the continuity of $\rho : \mathcal{X} \to \mathbb{R}$.

- The strongest sense of continuity is w.r.t. weak convergence.
  - $X_n \to X$ weakly, then $\rho(X_n) \to \rho(X)$.
- This is quite restrictive.

Robustness of Risk Measures

With respect to weak convergence:

- VaR_p is continuous at distributions whose quantile is continuous at p. VaR_p is argued as being generally robust.
- ES_p is not continuous
- A distortion risk measure is continuous if its distortion function has a (left and right) derivative which vanishes at neighbourhoods of 0 and 1 (classic property of L-statistics; see Cont-Deguest-Scandolo 2010 QF).

Robustness of Risk Measures

Is robustness w.r.t. weak convergence necessarily a good thing?

- Toy example: Let $X_n = n^2 \mathrm{I}_{\{U \le 1/n\}}$ for some U[0,1] random variable U (think about a credit default risk). Clearly $X_n \to 0$ a.s. but X_n is getting more "dangerous" in many senses. If ρ preserves weak convergence, that is,

$$\rho(X_n) \to \rho(0) \quad (= 0 \quad \text{typically}),$$

  it might not be a good thing.
- $\mathrm{VaR}_{0.999}(X_{10000}) = 0$!
- $\mathrm{ES}_{0.999}(X_{10000}) = 10^7$!

Robustness of Risk Measures

- On the other hand, this type of risks are very difficult to capture statistically (accuracy is impossible); asking some sort of continuity is indeed reasonable.

Robustness of Risk Measures

UK House of Lords/House of Commons, June 12, 2013

Changing banking for good. Volume II, page 119:

- *Output of a "stress test" excercise, from HBOS*:
  "We actually got an external advisor [to assess how frequently a particular event might happen] and they came out with one in 100,000 years and we said "no", and I think we submitted one in 10,000 years. But that was a year and a half before it happened. It doesn't mean to say it was wrong: it was just unfortunate that the 10,000th year was so near."

Robustness of Risk Measures

- A coherent or convex risk measure (such as ES_p) is not continuous w.r.t. weak convergence for general choices of $\mathcal{D}$ (Bäuerle-Müller 2006 IME)
- ES_p is continuous w.r.t. some other (stronger) metric, e.g. L^1 metric, or the Wasserstein metric (Stahl-Zheng-Kiesel-Rühlicke 2012 wp) (Krätschmer-Schied-Zähle 2014 FS)

Uncertainty in Risk Aggregation

Consider an aggregate risk model:

$$S = X_1 + \cdots + X_n,$$

$X_i \sim F_i$. We are interested in $\rho(S)$.

There are two levels of statistical uncertainty:

- at the level of marginal distributions F_i;
- at the level of the copula of $(X_1, \ldots, X_n)$.

Both $\mathrm{VaR}_p(S)$ and $\mathrm{ES}_p(S)$ depend on both levels of modeling.

The second level of uncertainty is arguably more challenging due to data, computation and modeling limitations.

Uncertainty in Risk Aggregation

To assess the uncertainty at the copula level, there is an active research field of dependence uncertainty in risk aggregation.

- Some recent papers (many more not listed)
  - Embrechts-Puccetti-Rüschendorf (2013 JBF)
  - W.-Peng-Yang (2013 FS)
  - Bernard-Jiang-W. (2014 IME)
  - Aas-Puccetti (2014 Extremes)
  - Bernard-Vanduffel (2015 JBF)
  - Embrechts-Wang-W. (2015 FS)
  - W.-Bignozzi-Tsanakas (2015 SIFIN)
  - Bernard-Vanduffel-Rüschendorf (2015 JRI)
- A book containing related problems: Rüschendorf (2013).

Aggregation-robustness (or Σ-robustness)

- In many practical cases, margins and dependence are modelled separately.
- The robustness at the first level is well addressed in classic statistical literature.
- Certainty of model at the second level is unrealistic.
- We are interested in robustness at the second level.

Aggregation-robustness

For given $F_1, \ldots, F_n \in \mathcal{D}$, denote

$$\mathcal{S}_n(F_1, \ldots, F_n) = \{X_1 + \cdots + X_n : X_i \in \mathcal{X},\ X_i \sim F_i,\ i = 1, \ldots, n\}.$$

$\mathcal{S}_n$ is the set of possible aggregate risks with unknown dependence. In fact we only care about distributions of random variables in $\mathcal{S}_n$.

Definition 1

$\rho : \mathcal{X} \to \mathbb{R}$ is aggregation-robust on $\mathcal{X}$ if ρ is continuous w.r.t. weak convergence in each $\mathcal{S}_n(F_1, \ldots, F_n)$.

Aggregation-robustness

Some interpretation:

- If ρ is continuous with respect to weak convergence on $\mathcal{D}$, then it is aggregation-robust.
- Given $X_1 \sim F_1, \ldots, X_n \sim F_n$, if ρ is continuous with respect to the copula of $(X_1, \ldots, X_n)$, then it is aggregation-robust.

Aggregation-robustness is weaker than the classic robustness (w.r.t. weak convergence)

Aggregation-robustness

Theorem 2

A distortion risk measure is aggregation-robust on L^∞ if and only if its distortion function is continuous on $[0, 1]$.

Some conclusions:

(i) A coherent distortion risk measures with the Lebesgue property is aggregation-robust on L^∞;

(ii) ES_p, $p \in (0, 1)$ is aggregation-robust on L^1;

(iii) VaR_p, $p \in (0, 1)$ is not aggregation-robust on L^∞. Since VaR is "almost" continuous with respect to weak convergence, it is also "almost" aggregation-robust; this should create no trouble for the use of VaR in practice.

Uncertainty Spread

Suppose that the marginal distributions $F_1, \ldots, F_n$ are known. If we have no knowledge about the copula of $(X_1, \ldots, X_n)$ (complete uncertainty), then the value of $\rho(X_1 + \cdots + X_n)$ lies in a range. How large is this range?

- We are more interested in the cases for VaR and ES.

Write $\mathcal{S}_n = \mathcal{S}_n(F_1, \ldots, F_n)$ and denote

$$\overline{\mathrm{VaR}}_p(\mathcal{S}_n) = \sup\{\mathrm{VaR}_p(S) : S \in \mathcal{S}_n\},$$

$$\underline{\mathrm{VaR}}_p(\mathcal{S}_n) = \inf\{\mathrm{VaR}_p(S) : S \in \mathcal{S}_n\}$$

similarly for $\overline{\mathrm{ES}}_p$ and $\underline{\mathrm{ES}}_p$.

Uncertainty Spread

We are interested in the intervals

$$[\underline{\mathrm{VaR}}_p(\mathcal{S}_n), \overline{\mathrm{VaR}}_p(\mathcal{S}_n)] \text{ and } [\underline{\mathrm{ES}}_q(\mathcal{S}_n), \overline{\mathrm{ES}}_q(\mathcal{S}_n)].$$

Calculation of $\overline{\mathrm{VaR}}_p$, $\underline{\mathrm{VaR}}_p$, $\underline{\mathrm{ES}}_p$ are generally unavailable.

- Numerical methods are available in Embrechts-Puccetti-Rüschendorf (2013 JBF).
- When we make comparison, typically $p \geq q$. In recent Consultative Documents of the Basel Committee, $\mathrm{VaR}_{0.99}$ is compared with $\mathrm{ES}_{0.975}$: $p = 0.975$ and $q = 0.99$.
- We are able to come to a conclusion in the case of large n.

Uncertainty Spread

Theorem 3

Take $1 > p \geq q > 0$. Under some moment conditions,

$$\liminf_{n\to\infty} \frac{\overline{\mathrm{VaR}}_p(\mathcal{S}_n) - \underline{\mathrm{VaR}}_p(\mathcal{S}_n)}{\overline{\mathrm{ES}}_q(\mathcal{S}_n) - \underline{\mathrm{ES}}_q(\mathcal{S}_n)} \geq 1.$$

- The uncertainty spread of VaR is generally bigger than that of ES.

Dependence-uncertainty spread

ES and VaR of $S = X_1 + \cdots + X_n$, where

- $X_i \sim \text{Pareto}(2 + 0.1i),\ i = 1, \ldots, 5;$
- $X_i \sim \text{Exp}(i - 5),\ i = 6, \ldots, 10;$
- $X_i \sim \text{Log–Normal}(0, (0.1(i - 10))^2),\ i = 11, \ldots, 20.$

	$n = 5$			$n = 20$		
	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\text{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\text{VaR}_{0.9875}$	12.06	56.21	44.16	22.12	126.63	104.51
$\text{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\text{ES}}_{0.975}}{\overline{\text{VaR}}_{0.975}}$		1.08			1.02	

Discussion

Some general message:

- Uncertainty is a key in future research
  - Every probabilistic problem is a special case of a uncertainty problem
- Is it always better that a risk measure is robust? Be careful about what to require
- ES has clear comparative advantages against VaR in risk aggregation
- Our results support the use of coherent risk measures (consistent with Basel III), of course always with caution

Discussion

- It is also important that an estimation for a good risk measure is simple and reliable
- We focused on the (Huber-Hampel) robustness of a risk measure as a functional. Robustness issues in data handling, modeling, estimation and optimization: important and different topics.
- Discussion papers:
  - Embrechts et al. (2014 Risks)
  - Emmer-Kratz-Tasche (2014 wp)
  - On uncertainty: Tsanakas-Beck-Thompson (2016 ASTIN Bulletin)

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Thank you

Thank you for your kind attendance

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