A Bayesian Adaptive Singular Control Problem arising from Corporate Finance

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Introduction

- In Corporate Finance, Liquidity Risk refers to the difficulty to meet its operational needs.
- *Liquidity risk models take cash reserve as state variable and dividend as control variable* $\rightarrow$ Singular control problem.
- Solvency risk refers to the unability to honour its debt commitment. This means that the insolvent company owes more than it owns.
- *Solvency risk models take firm cash-flows as state variable*. This leads to determine the optimal time to liquidate $\rightarrow$ Optimal Stopping problem
- How to merge these two risks in a tractable model?

Solvency Risk

- Solvency risk models focus on firm profitability.
- But they often assume costless external financing $\rightarrow$ Cash reserves do not matter.
- As long as the firm asset is higher than the firm liability, shareholders inject cash at no costs to meet the operational needs.
- The liquidation of the firm asset is endogeneous.

The dominant paradigm: deep pocket shareholders (Leland).

- Cash-flows process (EBIT)= regular diffusion or Lévy process X_t
- The shareholders' decision is to select the default policy that maximizes the value of their share.

$$E(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_x \left[\int_0^{\tau} e^{-rt} X_t \, dt \right]$$

- The entire after tax cash flow is distributed as dividends.
- Optimal default policy

$$\tau_{x^*} = \inf\{t \geq 0 \, : \, X_t \leq x^*\} \quad x^* < 0$$

- Equity-holders are ready to inject cash to maintain the firm's activity when $x^* < X_t < 0$.

Liquidity Risk

- Liquidity risk models focus on optimal cash management (dividend and issuance policy) when external financing is costly.
- Financial frictions generate risk aversion even for well-diversified firm. Too little cash leads to liquidation of productive assets but holding too much cash reduces profitability.
- Optimal policy: firms have target cash levels (cash in excess of certain threshold is returned to shareholders).

The dominant paradigm: No external Funding (Jeanblanc-Shiryaev)

- *Cumulative cash flow process* $dR_t = \mu\, dt + \sigma\, dB_t$
- *Strong assumption* External financing is too costly $\rightarrow$ liquidation when the cash reserves hit zero.

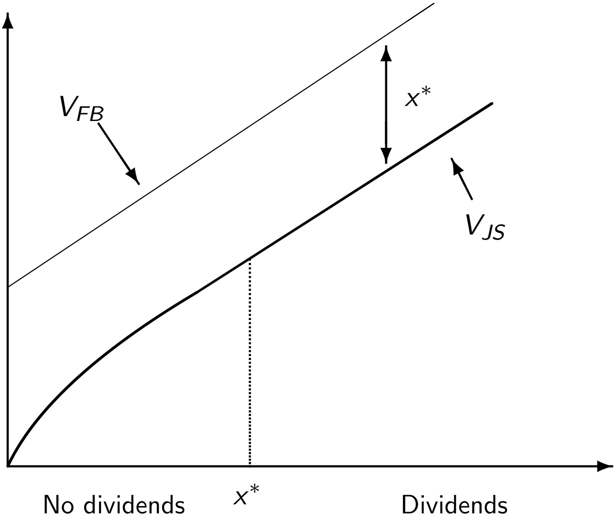
- *Cash reserves process* $dM_t = (r - \lambda) M_t\, dt + dR_t - dZ_t$
- *Liquidation time* $\tau_0 \equiv \inf\{t \geq 0 : M_t = 0\}$

- *Firm value* $V^*(m) = \sup_Z \mathbb{E}_m \left[\int_0^{\tau_0} e^{-rs} dZ_s \right]$
- *First best value* $V_{FB}(m) = m + \frac{\mu}{r}$.

The dominant paradigm: No external funding (Jeanblanc-Shiryaev)

- The value of the firm is an increasing and concave function of the level m of its cash reserves
- Any excess above m^* is paid to equity-holders
- The marginal value of cash is strictly greater than 1 for $m \in (0, m^*)$, where $m^* = \inf\{m > 0 \,|\, V^{*'}(m) = 1\}$, and equal to 1 for $m \in [m^*, \infty)$.
- The volatility of the firm value decreases with the level of cash

Optimal Dividend policy.



Literature.

- Literature on Solvency Risk: Leland (1994), Leland and Toft (1996), Hilberink and Rogers (2002), Chen and Kou, S. (2009), Décamps, JP., and Villeneuve, S. (2012).
- Literature on optimal liquidity management policies: Jeanblanc and Shiryaev (1995); Asmussen, Højgaard and Taksar (1999); Sethi and Taksar (2002); Choulli, Taksar and Zhou (2003);Lokka and Zervos, (2005); Cadenillas, Choulli, Taksar and Zhang (2006), Décamps, Mariotti, Rochet and Villeneuve (2007) .

Merging Liquidity and Solvency Risks

- Liquidity and solvency concerns are recognized to be driving forces behind management decisions.
- Finance literature has mainly focused on each type of financial distress independently.
- Standard trade-off models initiated by Leland (94) put aside the role of cash balances and focus only on endogeneous default triggered by shareholders. There is no financial constraints on external financing.
- Literature on liquidity management initiated by Radner and Shepp (95) considers costly external financing but neglects the optimal choice of leverage.

Merging Liquidity and Solvency Risks

- Davydenko (2007) summarized *"Neither solvency nor liquidity concerns alone can fully explain the observed corporate decisions"*.
- Following Gryglewicz (2011, JFE), the contribution of this study is the integration of liquidity and solvency concerns in a dynamic model.
- Understand liquidity management as a means to avoid inefficient default as documented by Lins, Servaes and Tufano (2008).

The model

Dynamic model for a cash-constrained firm with uncertainty about the profitability of its project

- Cumulative cash-flow process: $R = (R_t)_{t \geq 0}$ follows an arithmetic Brownian motion with unknown drift Y

$$dR_t = Y\,dt + \sigma\,dB_t$$

  Brownian motion B is independent of Y.
- Firm's profitability Y takes either of the two values $\underline{y} < 0 < \overline{y}$.

Shareholders' belief.

- Let us define $Y_t = \mathbb{E}[Y \mid \mathcal{F}_t^R]$ the shareholders' belief , Filtering theory (Lipster-Shiryaev)

$$dY_t = \frac{1}{\sigma}(Y_t - \underline{y})(\overline{y} - Y_t)dW_t$$

where $W = (W_t)_{t \geq 0}$ is a $\mathcal{F}^R$-Brownian motion called *innovation*

$$dW_t = \frac{1}{\sigma}(dR_t - Y_t dt).$$

- Itô's formula gives

$$dR_t \ = \ d\phi(Y_t) \ + \ \frac{1}{2}(\overline{y} + \underline{y})\, dt$$

where

$$\phi(y) = \frac{\sigma^2}{\overline{y} - \underline{y}} \ln\left(\frac{y - \underline{y}}{\overline{y} - y}\right).$$

The control problem.

- The cash reserves $X = (X_t)_{t \geq 0}$ of the firm evolve according to

$$dX_t = dR_t - dL_t \tag{1}$$

- The process $L = (L_t)_{t \geq 0}$ is $\mathcal{F}^R$ adapted and right-continuous and nondecreasing processes with $L_{0-} = 0$.
- The firm ceases its activity for two possible reasons:
  (i) it cannot meet its short-term operating costs by drawing cash from its reserves *liquidity problem*
  (ii) the firm's management decides to default for profitabilty reasons because the belief that the drift parameter Y is $\underline{y} < 0$ is strong *solvency problem*.
- Equation (1) represents the dynamics of the cash reserve up to the time τ_0 where $\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}$

The control problem.

- A control policy $\pi = (L_t; t \geq 0)$ is admissible if

$$X_t^\pi \geq 0, e^{-rt} X_t^\pi \text{ integrable and } \lim_{t \to \infty} e^{-rt} X_t^\pi = 0 \text{ a.s and in } L^1.$$

- For a given control π we define the firm value for all $(x, y) \in [0, \infty) \times (\underline{y}, \overline{y})$.

$$V_\pi(x, y) = \mathbb{E}_{(x,y)} \left[\int_0^{\tau_0} e^{-rt} dL_t \right],$$

where $\Delta L_{\tau_0} = \max(X_{\tau_0^-}, 0)$. The case $\Delta L_{\tau_0} = X_{\tau_0^-}$ corresponds to a strategic default.

- The objective is to find the optimal value function

$$V^*(x, y) = \sup_\pi V(x, y).$$

Benchmark : deep pocket shareholders.

- Itô's formula gives

$$V^*(x,y) \;=\; x + \sup_{\pi} \mathbb{E}_{(x,y)} \left[\int_0^{\tau_0} e^{-rs} \left(-rX_s^{\pi} + Y_s\right) \, ds \right].$$

- For all $(x,y) \in [0,\infty) \times (\underline{y}, \bar{y})$, $V^*(x,y) \leq \bar{V}(x,y)$ where

$$\bar{V}(x,y) \equiv x + \sup_{\tau \in \mathcal{T}^R} \mathbb{E}_y \left[\int_0^{\tau} e^{-rs} Y_s ds \right]. \qquad (2)$$

- Optimal stopping for (2)

$$\tau_{y^*} = \inf\{t \geq 0 \; : \; Y_t \leq y^*\} \quad y^* < 0.$$

Properties of the value function

Proposition

- *The mapping* $(x, y) \longrightarrow V^*(x, y)$ *is continuous on* $[0, \infty) \times (-1, 1)$.
- *For any* $x \in [0, \infty)$, *the mapping* $y \longrightarrow V^*(x, y)$ *is increasing and convex on* $(\underline{y}, \overline{y})$. *Positive value of information*
- *For any* $y \in (\underline{y}, \overline{y})$, *the mapping* $x \longrightarrow V^*(x, y)$ *is increasing and concave on* $[0, \infty)$.*Precautionary role of cash reserves*

Properties of the value function

$V^*(x,.)$ can be continuously extended at $y = \overline{y}$:

Let us consider the standard Jeanblanc-Shiryaev problem with known drift $\overline{y}$:

- Controlled cash reserves $dX_t = \overline{y}dt + \sigma dB_t - dL_t$
- $V_{JS}(x) \equiv \sup_L \mathbb{E}_x[\int_0^{\tau_0} e^{-rt} dL_t]$

Proposition

The following holds

$$\text{For every } x > 0 \lim_{y \longrightarrow \overline{y}} V^*(x,y) = V_{JS}(x).$$

HJB Equation and Verification Theorem

Proposition

Assume there exists a twice continuously differentiable function with bounded first derivatives V *defined on* $[0,\infty) \times (\underline{y}, \overline{y})$ *that satisfies* $V(0,.) = 0$, $V(.,y)$ *concave and*

$$\max(\mathcal{A}V - rV, 1 - V_x) \leq 0,$$

where

$$\mathcal{A}V(x,y) = \frac{1}{2\sigma^2}(y-\underline{y})^2(\overline{y}-y)^2 V_{yy} + \frac{1}{2}\sigma^2 V_{xx} + (1-y^2)V_{xy} + yV_x.$$

then $V \geq V^*$.

Heuristics about the target cash level

- Let $g(y) = \inf\{x,\ V_x(x,y)=1\}$ the target cash level.
- Concavity of $x \longrightarrow V(x,y)$: $V_x(x,y) > 1$ for $x \in (0, g(y))$, and $V_x(x,y) = 1$ for $x \geq g(y)$.
- For all $y \in (\underline{y}, \overline{y})$, $g(y) \leq \bar{x}$.

  where $\bar{x}$ is the dividend boundary associated to the JS problem with cash reserves $dX_t = \overline{y}dt + \sigma dB_t - dL_t$.

Free-boundary problem

Goal: Find a concave function V and a boundary g that solve the free-boundary problem

$$
\begin{aligned}
g(\overline{y}) &= \bar{x} \\
V(0,y) &= 0 \quad \forall y \in [0,1), \\
\mathcal{A}V(x,y) - rV(x,y) &= 0 \text{ on } \{(x,y),\ 0 < x < g(y)\}, \\
V_x(g(y),y) &= 1, \\
V_{xy}(g(y),y) &= 0.
\end{aligned}
$$

Solving the free-boundary problem

- $\overline{y} = -\underline{y} = 1$.
- Deterministic time-independent relationship between the cumulative cash-flow process and the belief process.

$$dR_t = d\phi(Y_t).$$

- Cash reserve process

$$X_t = \phi(Y_t) - \phi(y) + x - L_t$$

where $X_0 = x$ and $Y_0 = y$.

Solving the free-boundary problem... and the control problem

Idea: to consider the change of variable $Z_t = X_t - \phi(Y_t)$, to define $U(z,y) \equiv V(\phi(y) + z, y)$, to re-state the free-boundary problem in the (z,y)-space.

Equation $\mathcal{A}V(x,y) - rV(x,y) = 0$ takes the form

$$\frac{1}{2\sigma^2}(y+1)^2(1-y)^2 U_{yy}(z,y) - rU(z,y) = 0.$$

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The boundary g is an increasing function defined as the unique solution to $g'(y) = f(y, g(y))$ with terminal condition $g(1) = x_1$.

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The optimal time to default is the hitting time of y^{**} where $g(y^{**}) = 0$.

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Attainability: $(V^*, x^*) = (V, g)$, two dimensional version of the Skohorod lemma *to* a diffusion reflected at the boundary in horizontal directions. (Burdzy and Toby (1995), *Annals of Proba.*)

Conclusion

- We develop a simple dynamic model of a firm facing both solvency and liquidity risks.
- 1.5-dimentional stochastic control problem that we solve quasi-explicitly.
- The dividend boundary x^* is continuously increasing in the belief about the profitability of the firm and converges to the JS dividend boundary.
- Next: Costly external funding $\rightarrow$ optimal issuance.