A Bayesian adaptive singular control problem arising from corporate finance by J.-P. Décamp and S. Villeneuve

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Problem

This is the classical set-up with cash reserves

$$dX_t = \mu dt + \sigma dB_t - dL_t,$$

where B Brownian motion and L_t cumulative dividend process. L is the only control available and one wants to maximize

$$L \text{ non-decreasing} \quad \rightarrow \quad \mathbb{E}\left[\int_0^\tau e^{-rt} dL_t\right],$$

where τ is the time of bankruptcy. With μ and σ constant this is a simple singular control problem and can be solved explicitly.
The new feature here is the uncertainty about the drift μ.

Bayesian framework

We assume that μ is a random variable independent of the Brownian motion with values$[\underline{y}, \overline{y}]$, where $\underline{y} < 0 < \overline{y}$ are two known constants. Then, the probability distribution of this random variable given the observations of the cash reserves is a scalar quantity. One may use the following conditional expectation to track it,

$$Y_t := \mathbb{E}\left[\ \mu \mid \mathcal{F}_t\ \right],$$

where $\mathcal{F}$ is the observed filtration, i.e., the one generated by X or equivalently by $dR = \mu dt + \sigma dB_t$.

Filtering

Recall

$$dX_t = \mu dt + \sigma dB_t - dL_t, \quad \text{and} \quad Y_t := \mathbb{E}[\ \mu \mid \mathcal{F}_t\ ].$$

Also μ and B are independent. Then,

$$\begin{aligned} dX_t &= Y_t dt + \sigma dW_t - dL_t \\ dY_t &= \frac{1}{\sigma}(Y_t - \underline{y})(\overline{y} - Y_t)\ dW_t, \end{aligned}$$

where W is a Brownian motion in the observed filtration.

Control problem

$$V^*(x,y) := \sup_L \ \mathbb{E}\left[\int_0^\tau e^{-rt} dL_t\right],$$

$$\begin{aligned} dX_t &= Y_t dt + \sigma dW_t - dL_t \\ dY_t &= \frac{1}{\sigma}(Y_t - \underline{y})(\overline{y} - Y_t)\ dW_t =: \sigma\ F(Y_t) dW_t, \end{aligned}$$

L is non-decreasing process,

τ is the bankruptcy time,

W is a Brownian motion in the observed filtration $\mathcal{F}$,

everything needs to be adapted to $\mathcal{F}$.

Dynamic Programming Equation

This is a singular, degenerate, two-dimensional optimal control problem : for $x > 0$ and $y \in (\underline{y}, \overline{y})$,

$$\min \{ rV^*(x,y) - \mathcal{L}V^*(x,y) \ , \ V^*_x(x,y) - 1 \ \} = 0,$$

together with the boundary conditions, $V^*(0,y) = 0$, and the functions $V^*(x,\underline{y})$ and $V^*(x,\overline{y})$ are computed explicitly as the Y_t is a constant in each case. And

$$\mathcal{L}v(x,y) = yv_x + \frac{\sigma^2}{2}\left(v_{xx} + F^2(y)v_{yy} + 2F(y)v_{xy}\right).$$

An upper bound

$$dX_t = Y_t dt + \sigma dW_t - dL_t \quad \text{and} \quad X_t \geq 0,$$

imply that

$$\begin{aligned} \mathbb{E}\left[\int_0^T e^{-rt} dL_t\right] &= \mathbb{E}\left[\int_0^T e^{-rt} Y_t dt\right] - \mathbb{E}\left[\int_0^\tau e^{-rt} dX_t\right] \\ &\leq \mathbb{E}\left[\int_0^T e^{-rt} Y_t dt\right] + x. \end{aligned}$$

So

$$V^*(x,y) \leq \hat{V}(x,y) := x + \sup_\tau \mathbb{E}\left[\int_0^\tau e^{-rt} Y_t dt\right].$$

Properties - threshold

The upper bound can be solved explicitly, in particular there exists a threshold $y^* \in (\underline{y}, \overline{y})$ so that

$$\hat{V}(x,y) = x, \qquad \forall\ y \leq y^*.$$

Hence, if our estimate of the drift based on our observations is less than y^*, then the optimal strategy is to pay all cash reserves as dividend and go bankrupt.

Properties - concavity

$$x \rightarrow V^*(x,y), \quad \text{is concave,}$$

$$y \rightarrow V^*(x,y), \quad \text{is convex.}$$

Proofs are control theoretic.

Properties - explicit solution

In the symmetric case,

$$\underline{y} = -\overline{y},$$

V^* is computed explicitly.

Properties - regularity / verification

- Under the assumption of regularity, the value function V^* is proved to be the unique classical solution of the dynamic programming equation.
- In fact, more is assumed and more is proved.
- I also think that V^* is the unique viscosity solution among the class of continuous solutions that are convex in y and concave in x. This uniqueness would be important to ensure that the numerical studies are giving approximation soy the value function.

Concluding

- Uncertainty in the drift is a very appropriate extension of the previous theory.
- Addition of issuance would be quite interesting as the optimal issuance level could be non-zero for certain values of y.
- However, this would be hard to analyze explicitly. Instead one can do numerical studies.
- A study in this direction was done by Akyildirim, Güney, Rochet and Soner with uncertain interest rate with issuance.