Change-point analysis of volatility

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September 8, 2015

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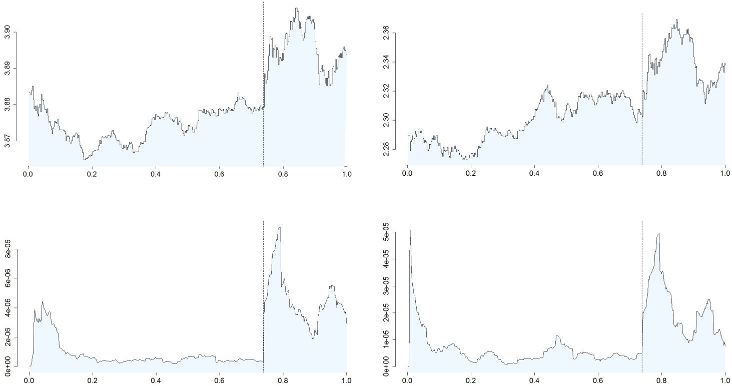
Motivating example

Illustration: Prices of MMM (left) and GE (right) on March 18th, 2009:

- Top line: Log-price intra-day evolutions
- Bottom line: Estimated spot squared volatilities



Natural questions:

- Jumps in price and/or volatility?
- Joint jumps in both assets?

Only few rigorous statistical papers. Exception: Jacod and Todorov (2010).

The model

Aim in this talk: Statistical inference on changes in the volatility of a (continuous) Itô semimartingale X, given by

$$X_t = X_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s \text{ (+ additional jumps)}$$

where

- a is a (locally) bounded drift process,
- W is a standard Brownian motion with some right-continuous volatility process $\sigma^2 > 0$,
- sometimes jumps are present, whose characteristics will be discussed later.

Null hypothesis: The volatility is Hölder continuous with regularity index $\mathfrak{a}$, i.e.

$$\sigma^2 \in \Sigma(\mathfrak{a}, L) = \left\{ (\sigma_t^2(\omega))_{t\in[0,1]} \Big| \sup_{s,t\in[0,1],|s-t|<\delta} \left|\sigma_t^2(\omega) - \sigma_s^2(\omega)\right| \le L\delta^{\mathfrak{a}} \right\}.$$

Possible change point alternatives:

- Is there a jump in the volatility, i.e. $\Delta\sigma_\theta^2 = \left(\sigma_\theta^2 - \lim_{s\uparrow\theta} \sigma_s^2\right) \neq 0$ for some $\theta \in (0,1)$?
- Does volatility get rougher in the sense of a regularity exponent $\mathfrak{a}' < \mathfrak{a}$ on $(\theta, 1]$?

The observations

We will work in a high frequency setting (one trading day) over the interval $[0,1]$, i.e. the data is recorded at discrete regular times $i\Delta_n$ with a mesh $\Delta_n \to 0$. Setting $n = \Delta_n^{-1} \in \mathbb{N}$, we thus have observations

$$X_{i\Delta_n}, \quad i = 0, \ldots, n,$$

or equivalently increments

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}, \quad i = 1, \ldots, n.$$

Then detection bounds shrinking with n become interesting, that is

- minimal jump sizes $b_n \to 0$ or
- minimal lengths of intervals $c_n \to 0$, over which volatility gets rougher,

such that these jumps or these changes in smoothness still can be detected.

Formally in case of jumps: Under the alternative, there is $\theta \in (0,1)$ such that

$$(\sigma_t^2(\omega))_{t\in[0,1]} \in \mathcal{S}_\theta^J(\mathfrak{a}, b_n, L) = \left\{ (v_t)_{t\in[0,1]} \middle| (v_t - \Delta v_t)_{t\in[0,1]} \in \Sigma(\mathfrak{a}, L)\,;\, |\Delta v_\theta| \geq b_n \right\}.$$

Note:

- θ plays the role of a change point here, but more than one jump (or even infinitely many) in the volatility is allowed as well.
- There is a clear separation between null hypothesis and alternative.

A toy example

Suppose that we work in the parametric model

$$X_t = X_0 + \sigma\, W_t\,.$$

Natural statistic for changes in the volatility: The cusum statistic given by

$$S_{n,\lfloor nt\rfloor} = \sqrt{n}\Big(\sum_{i=1}^{\lfloor nt\rfloor}\big(\Delta_i^n X\big)^2 - \frac{\lfloor nt\rfloor}{n}\sum_{j=1}^{n}\big(\Delta_j^n X\big)^2\Big),\ \ t\in[0,1]\,.$$

Result: If $\sigma_s = \sigma$ holds, then

$$S_{n,\lfloor nt\rfloor} \rightsquigarrow \gamma\big(B_t - tB_1\big)\,,$$

where $\gamma^2 = 2\sigma^4$. Self-normalizing version using e.g. the quarticity estimator

$$\hat{\gamma}^2 = (2n/3)\sum_{i=1}^{n}\big(\Delta_i^n X\big)^4\,.$$

Overall: We have functional weak convergence to a standard Brownian bridge, i.e.

$$\hat{\gamma}^{-1}\, S_{n,\lfloor nt\rfloor} \rightsquigarrow B_t - tB_1\,.$$

Then we obtain a test for jumps in the volatility based on

$$T_n = \sup_{m=1,\ldots,n}\left|\hat{\gamma}^{-1}\, S_{n,m}\right|,$$

which tends under the null to a Kolmogorov-Smirnov law and diverges under the alternative almost surely.

The general situation

Recall: We observe

$$X_t = X_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s \, .$$

With the volatility process being time-varying, the cusum-based test is not suitable to test for jumps in the volatility.

Solution: Use local versions of realized volatility instead. Let $k_n \to \infty$ with $n/k_n \to \infty$ be an auxiliary sequence of integers and set

$$X_{n,i} = \frac{n}{k_n} \sum_{j=1}^{k_n} \left(\Delta_{ik_n+j}^n X\right)^2 , \; i = 0, \ldots, \lfloor n/k_n \rfloor - 1 \, .$$

These variables are computed over $[ik_n\Delta_n, (i+1)k_n\Delta_n]$ and estimate a block-wise constant proxy of the spot volatility $\sigma^2_{ik_n\Delta_n}$ on the respective blocks.

Intuition:

- A large distance between $X_{n,i}$ and $X_{n,i+1}$ suggests the presence of a jump or unsmooth breaks in the volatility close to time $ik_n\Delta_n$.
- In order to obtain normalized statistics, we work with ratios instead of differences.

Statistics to test for jumps in the volatility

Two statistics:

- We compute

$$V_n = \max_{i=0,\dots,\lfloor n/k_n\rfloor - 2} |X_{n,i}/X_{n,i+1} - 1|$$

  over non-overlapping intervals.

- For

$$V_n^* = \max_{i=k_n,\dots,n-k_n} \left| \frac{\frac{n}{k_n}\sum_{j=i-k_n+1}^{i} \left(\Delta_j^n X\right)^2}{\frac{n}{k_n}\sum_{j=i+1}^{i+k_n} \left(\Delta_j^n X\right)^2} - 1 \right|$$

  we us all available ratios and work over overlapping blocks.

Conditions on k_n: We assume the growth condition

$$k_n^{-1}\Delta_n^{-\epsilon} + \sqrt{k_n}(k_n\Delta_n)^{\mathfrak{a}}\sqrt{\log(n)} \to 0\,,$$

for some $\epsilon > 0$ and with the smoothness index $\mathfrak{a} > 0$ as before.

Interpretation:

- $k_n \to \infty$ faster than some power of n is a mild lower bound on the growth of k_n. The blocks should not be too small in order to estimate spot volatility consistently.
- The second condition gives an upper bound related to the continuity of σ. Naturally, the smaller $\mathfrak{a}$ (and the less smooth σ), the smaller the size of the blocks.

Asymptotics

First main theorem under the null:

Set $m_n = \lfloor n/k_n \rfloor$ and $\gamma_{m_n} = [4\log(m_n) - 2\log(\log(m_n))]^{1/2}$. Under all previous conditions, we have under the null

$$\sqrt{\log(m_n)}\big(\big(k_n^{1/2}/\sqrt{2}\big)V_n - \gamma_{m_n}\big) \rightsquigarrow V,$$

$$\sqrt{\log(m_n)}\big(k_n^{1/2}/\sqrt{2}\big)V_n^* - 2\log(m_n) - \frac{1}{2}\log\log(m_n) - \log(3) \rightsquigarrow V,$$

where V follows an extreme value distribution with distribution function

$$P(V \leq x) = \exp(-\pi^{-1/2}\exp(-x)).$$

Comments:

- Limiting distribution is as in Wu and Zhao (2007).
- The only essential condition regards the granted smoothness $\mathfrak{a} > 0$. Note that less smooth paths require smaller block lengths k_n which reduces the power of the test.
- We can cope with standard models for σ. For a continuous semimartingale volatility, we have $\mathfrak{a} \approx 1/2$. In this case, we take $k_n \propto n^{1/2-\epsilon}$ for $\epsilon > 0$ small to preserve the highest possible power. Similarly, for a Lipschitz volatility, i.e. $\mathfrak{a} = 1$, one might choose $k_n \propto n^{2/3-\epsilon}$.

Jumps in the price process

More general model: Suppose we observe

$$X_t = X_0 + \int_0^t a_s\, ds + \int_0^t \sigma_s\, dW_s + \int_0^t \int_{\mathbb{R}} \kappa(\delta(s,x))(\mu - \nu)(ds, dx) + \int_0^t \int_{\mathbb{R}} \bar{\kappa}(\delta(s,x))\mu(ds, dx)\,.$$

Additional technical condition: Suppose $\sup_{\omega,x} |\delta(s,x)|/\gamma(x)$ is (locally) bounded by some deterministic non-negative function γ which satisfies for some $r < 2$:

$$\int_{\mathbb{R}} (1 \wedge \gamma^r(x))\lambda(dx) < \infty\,.$$

Alternative statistics: Based on truncated spot volatility estimators

$$X_{n,u_n,i} = \frac{n}{k_n} \sum_{j=1}^{k_n} \left(\Delta^n_{ik_n+j} X\right)^2 1_{\{|\Delta^n_{ik_n+j}X| \le u_n\}}\,,\ i = 0, \dots, \lfloor n/k_n \rfloor - 1\,,$$

with a truncation sequence $u_n \propto n^{-\tau}$, $\tau \in (0, 1/2)$. Set then

$$V_{n,u_n} = \max_{i=0,\dots,\lfloor n/k_n \rfloor - 2} |X_{n,u_n,i}/X_{n,u_n,i+1} - 1|,$$

$$V^*_{n,u_n} = \max_{i=k_n,\dots,n-k_n} \left| \frac{\frac{n}{k_n} \sum_{j=i-k_n+1}^{i} \left(\Delta^n_j X\right)^2 1_{\{|\Delta^n_j X| \le u_n\}}}{\frac{n}{k_n} \sum_{j=i+1}^{i+k_n} \left(\Delta^n_j X\right)^2 1_{\{|\Delta^n_j X| \le u_n\}}} - 1 \right|.$$

Asymptotics

Additional condition: Suppose $k_n \propto n^{\beta}$ for $0 < \beta < 1$ such that the previous growth conditions are satisfied. Furthermore,

$$r < \min\left(2\left(2 - \tau^{-1}(1 - \beta/2)\right), \left(\tau^{-1}\min(1/2, 1-\beta)\right), \left(2 - \tau^{-1}\beta/2\right)\right).$$

This is stronger than the usual $r < 1$ for realized volatility of Itô semimartingales.

Second main theorem under the null:

With $m_n = \lfloor n/k_n \rfloor$ and $\gamma_{m_n} = [4\log(m_n) - 2\log(\log(m_n))]^{1/2}$ as before, and if either $r = 0$ or the jump process is a time-inhomogeneous Lévy process, we have under the null the weak convergence

$$\sqrt{\log(m_n)}\left(\left(k_n^{1/2}/\sqrt{2}\right)V_{n,u_n} - \gamma_{m_n}\right) \rightsquigarrow V,$$

$$\sqrt{\log(m_n)}\left(k_n^{1/2}/\sqrt{2}\right)V^{*}_{n,u_n} - 2\log(m_n) - \frac{1}{2}\log\log(m_n) - \log(3) \rightsquigarrow V,$$

with the same extreme value distribution V as before.

Examples:

- Finite activity jumps: In this case, the only condition is $\tau > 1/2 - \beta/4$, and the threshold may not be chosen too small.
- For the typical case $\beta \approx 1/2$ and $\tau \approx 1/2$: $r < 1$.
- For other choices of β the condition is more restrictive, e.g. $r < 2/3$ for $\beta \approx 2/3$ under a Lipschitz volatility.

Simulations I

Model: $n = 10000$ observations of a continuous Itô semimartingale with

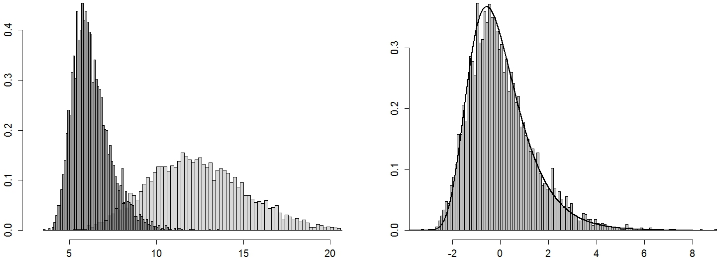
$$\sigma_t = 1 - 0.2 \sin\left(\tfrac{3}{4}\pi\, t\right), \;\; t \in [0,1],$$

and constant drift $a = 0.1$. σ mimics a realistic volatility shape with strong decrease after opening and slight increase before closing.

Under the alternative: Add one jump of size 0.2 at the fixed time $t = 2/3$ to σ_t. This means, the volatility jumps back at $t = 2/3$ to its maximum start value.

Results are shown only for the statistic V_n^* using all available blocks, which always performs best. Precisely,

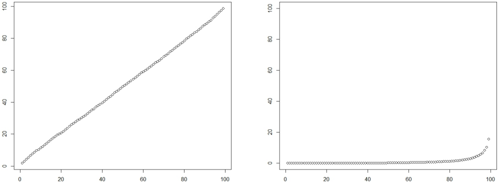
- $k_n = 500$, histograms under hypothesis (dark) and alternative (light),
- rescaled version comparing left hand side and limit law (density marked by solid line).



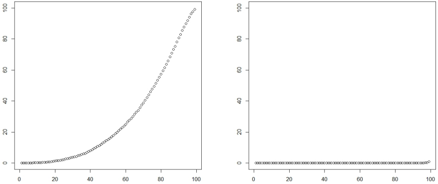
Simulations II

Finite-sample size and power of the test:

- Empirical percentiles of the limit law against the rescaled statistics.
- Size: left; power: right.



Problem: Same figure for $k_n = 1000$: Power great, but size unreliable.



Typical for convergence to extreme value distributions. Solution in practice: Bootstrap.

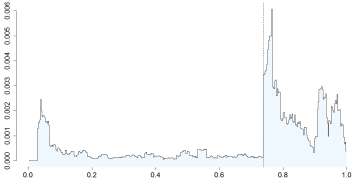
Simulations III

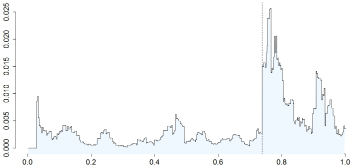
Findings carry over to more realistic setups involving

- jumps in the price (jointly with jumps in the volatility or not),
- stochastic volatility models,
- other sizes of volatility or jumps.

In all cases, V^*_{n,u_n} performs best, and for $k_n = 500$ it works very well, whereas $k_n = 1000$ is inaccurate in terms of size.

Initial example: Running local (truncated) volatility estimates for MMM (left) and GE (right).





In both cases:

- p-values are essentially zero,
- the maximum is attained at grid point 285, which corresponds to 02:15 p.m. EST as the estimated change-point.

Detection bounds

Under H_0: At stage n, we have $\sigma^2 \in \Sigma(\mathfrak{a}, L)$. All possible alternatives (e.g. jumps in the volatility) will depend on an auxiliary sequence b_n.

For a test ψ that maps a sample $(X_{i\Delta_n})_{0\leq i\leq n}$ to zero or one, consider

- the maximal type I error $\alpha_\psi\left(\mathfrak{a}\right) = \sup_{\sigma^2\in\Sigma(\mathfrak{a},L)} P_\sigma\left(\psi = 1\right)$,
- the maximal type II error $\beta_\psi\left(\mathfrak{a}, b_n\right) = \sup_{\sigma^2\in\mathcal{S}_\theta^J(\mathfrak{a},b_n,L)} P_\sigma\left(\psi = 0\right)$,
- the global testing error $\gamma_\psi\left(\mathfrak{a}, b_n\right) = \alpha_\psi\left(\mathfrak{a}\right) + \beta_\psi\left(\mathfrak{a}, b_n\right)$.

Aim: Find tests that minimize $\gamma_\psi\left(\mathfrak{a}, b_n\right)$, given the boundary b_n. In particular, find sequences of tests ψ_n and boundaries b_n with

$$\gamma_{\psi_n}\left(\mathfrak{a}, b_n\right) \to 0 \quad \text{as} \quad n \to \infty.$$

The smaller $b_n > 0$, the harder it is for a test to control the global testing error.

Natural question: Given $\mathfrak{a}$, what is the minimal size of $b_n > 0$ such that

$$\lim_{n\to\infty} \inf_{\psi} \gamma_\psi\left(\mathfrak{a}, b_n\right) = 0$$

holds? The optimal b_n^{opt} is called minimax distinguishable boundary, and a sequence (ψ_n) that satisfies this condition for all $b_n \geq b_n^{\text{opt}}$ is called minimax-optimal (cf. Ingster (1993)).

Alternatives

Jump alternative: For a given b_n, there exists $\theta \in (0,1)$ with

$$(\sigma_t^2(\omega))_{t\in[0,1]} \in \mathcal{S}_\theta^J(\mathfrak{a}, b_n, L) = \left\{ (v_t)_{t\in[0,1]} \middle| (v_t - \Delta v_t)_{t\in[0,1]} \in \Sigma(\mathfrak{a}, L)\, ;\, |\Delta v_\theta| \geq b_n \right\}.$$

Smoothness alternative: Set

$$\Delta_h^{\mathfrak{a}'} f_t = \frac{f_{t+h} - f_t}{|h|^{\mathfrak{a}'}},\ t \in [0,1],\ h \in [-1,1]\,.$$

Until some change-point $\theta \in (0,1)$, the process $(\sigma_{t\wedge\theta}^2)$ behaves as a process in $\Sigma(\mathfrak{a}, L)$. After θ, the regularity exponent drops to some $0 < \mathfrak{a}' < \mathfrak{a}$. Formally: Define $\mathcal{S}_\theta^{\mathrm{R}}\big(\mathfrak{a}, \mathfrak{a}', b_n, L, C\big)$ to be the set

$$\Big\{ \big(v_{t\wedge\theta}\big)_{t\in[0,1]} \in \Sigma(\mathfrak{a}, L) \Big| \inf_{0\le h\le b_n^{1/\mathfrak{a}}} \Delta_h^{\mathfrak{a}'} v_\theta > C \text{ or } \sup_{0\le h\le b_n^{1/\mathfrak{a}}} \Delta_h^{\mathfrak{a}'} v_\theta < -C \Big\},$$

for some $C > 0$.

Note:

- We need the difference in roughness to be exploited on a small interval.
- Our test is able to detect non change point alternatives as well.

Asymptotics I: lower bound

First main theorem on detection bounds:

Assume that $\mathfrak{a} > \mathfrak{a}' > 0$ and

$$\inf_t \sigma_t^2 \geq \sigma_-^2 > 0.$$

Consider either set of hypotheses $\{H_0, H_1^{\mathrm{J}}\}$ or $\{H_0, H_1^{\mathrm{R}}\}$. Then for

$$b_n \leq C_1(\sigma, \mathfrak{a})\big(n/\log(m_n)\big)^{-\frac{\mathfrak{a}}{2\mathfrak{a}+1}} L^{\frac{1}{2\mathfrak{a}+1}} \tag{1}$$

with a constant $C_1(\sigma, \mathfrak{a})$, we have $\lim_{n\to\infty} \inf_\psi \gamma_\psi\big(\mathfrak{a}, b_n\big) = 1$ in both cases.

Comments:

- This result gives a lower bound for the size of b_n in order for jumps or changes in roughness to be detected.
- This lower bound does not depend on $\mathfrak{a}'$, only the fact that $\mathfrak{a}' < \mathfrak{a}$ is relevant. This is an asymptotic result, though, and in practice the size of the difference $(\mathfrak{a} - \mathfrak{a}')$ may have a significant impact.

Asymptotics II

Based on V_n^*, we define the test $\psi^\diamond$ as follows:

$$\psi^\diamond\big((X_{i\Delta_n})_{0\le i\le n}\big):\ \text{reject } H_0 \text{ if } V_n^* \ge 2D^\diamond\sqrt{2\log(m_n^\diamond)/k_n^\diamond},$$

where $D^\diamond > 2$, $k_n^\diamond = \big(\sqrt{\log(m_n^\diamond)}n^{\mathfrak{a}}/L\big)^{\frac{2}{2\mathfrak{a}+1}}$ and $m_n^\diamond = \lfloor n/k_n^\diamond \rfloor$.

Second main theorem on detection bounds:

Let either $\inf_t \Delta\sigma_t > 0$, or the regularity drop to $0 < \mathfrak{a}' < \mathfrak{a} \le 1$ in the previous sense. If

$$b_n^\diamond > C_2(\sigma,\mathfrak{a},D^\diamond)\left(\sqrt{\log(m_n^\diamond)/k_n^\diamond} + L_n\,(k_n^\diamond\Delta_n)^{\mathfrak{a}}\right),$$

then $\lim_{n\to\infty}\gamma_{\psi^\diamond}\big(\mathfrak{a}, b_n^\diamond\big) = 0$.

Discussion: A simple calculation shows that

$$b_n^{\text{opt}} \propto \big(n/\log(n)\big)^{-\frac{\mathfrak{a}}{2\mathfrak{a}+1}} L^{1/(2\mathfrak{a}+1)}.$$

Thus bound is similar to Spokoiny (1998).