# Linear-Rational Term-Structure Models

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# Near-zero short-term interest rates

# Contribution

- Existing models that respect zero lower bound (ZLB) on interest rates face limitations:
  - Shadow-rate models do not capture volatility dynamics
  - Multi-factor CIR and quadratic models do not easily accommodate unspanned factors and swaption pricing
- We develop a new class of **linear-rational** term structure models
  - Respects ZLB on interest rates
  - Easily accommodates unspanned factors affecting volatility and risk premia
  - Admits semi-analytical solutions to swaptions
- Extensive empirical analysis
  - Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
  - Captures many features of term structure, volatility, and risk premia dynamics.

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# Outline

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The linear-rational framework

The Linear-Rational Square-Root (LRSQ) model

Empirical analysis

# Linear-rational framework and bond pricing

- **State-price density**, $\zeta_t$

$$\Pi(t, T) = \frac{1}{\zeta_t}\mathbb{E}_t[\zeta_T C_T]$$

- $m$-dimensional **factor process**, $Z_t$, with linear drift given by

$$\mathrm{d}Z_t = \kappa(\theta - Z_t)\mathrm{d}t + \mathrm{d}M_t,$$

  for some $\kappa \in \mathbb{R}^{m\times m}$, $\theta \in \mathbb{R}^m$, and some martingale $M_t$

- $\zeta_t$ given by

$$\zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \psi^\top Z_t\right),$$

  for some $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that $\phi + \psi^\top z > 0$ for all $z \in E$, and some $\alpha \in \mathbb{R}$

- Conditional expectations:

$$\mathbb{E}_t[Z_T] = \theta + \mathrm{e}^{-\kappa(T-t)}(Z_t - \theta)$$

- Price of **zero-coupon bond**:

$$P(t, t+\tau) = \frac{(\phi + \psi^\top\theta)\mathrm{e}^{-\alpha\tau} + \psi^\top \mathrm{e}^{-(\alpha+\kappa)\tau}(Z_t - \theta)}{\phi + \psi^\top Z_t}$$

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# Interest rates and the zero lower bound

- **Short rate**:

$$r_t = -\partial_T \log P(t, T)|_{T=t} = \alpha - \frac{\psi^\top \kappa(\theta - Z_t)}{\phi + \psi^\top Z_t}$$

- Define

$$\alpha^* = \sup_z \frac{\psi^\top \kappa(\theta - z)}{\phi + \psi^\top z} \quad \text{and} \quad \alpha_* = \inf_z \frac{\psi^\top \kappa(\theta - z)}{\phi + \psi^\top z}$$

- Set $\alpha = \alpha^*$ so that

$$r_t \in [0, \alpha^* - \alpha_*]$$

- $\alpha^*$ and $\alpha_*$ are finite if $z \in \mathbb{R}^d_+$ and all components of $\psi$ are strictly positive
- Range is parameter dependent, verify that range is wide enough
- If eigenvalues of $\kappa$ have nonnegative real part then $\alpha$ is the infinite-maturity ZCB yield

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# Interest rate swaps

- Exchange a stream of fixed-rate for floating-rate payments
- Consider a tenor structure

$$T_0 < T_1 < \cdots < T_n, \quad T_i - T_{i-1} \equiv \Delta$$

- At $T_i$, $i = 1 \dots n$:
  - pay $\Delta k$, for fixed rate $k$
  - receive floating LIBOR $\Delta L(T_{i-1}, T_i) = \frac{1}{P(T_{i-1}, T_i)} - 1$
- Value of **payer swap** at $t \leq T_0$

$$\Pi_t^{\mathrm{swap}} = \underbrace{P(t, T_0) - P(t, T_n)}_{\text{floating leg}} - \underbrace{\Delta k \sum_{i=1}^{n} P(t, T_i)}_{\text{fixed leg}}$$

- Forward swap rate $S_t = \frac{P(t,T_0) - P(t,T_n)}{\Delta \sum_{i=1}^{n} P(t,T_i)}$

# Swaptions

- **Payer swaption** = option to enter the swap at $T_0$ paying fixed, receiving floating
- Payoff at expiry $T_0$ of the form

$$C_{T_0} = \left(\Pi_{T_0}^{\mathrm{swap}}\right)^+ = \left(\sum_{i=0}^{n} c_i P(T_0, T_i)\right)^+ = \frac{1}{\zeta_{T_0}} p_{\mathrm{swap}}(Z_{T_0})^+$$

  for the explicit linear function

$$p_{\mathrm{swap}}(z) = \sum_{i=0}^{n} c_i \mathrm{e}^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top \mathrm{e}^{-\kappa(T_i - T_0)}(z - \theta)\right)$$

- Swaption price at $t \leq T_0$ is given by

$$\Pi_t^{\mathrm{swaption}} = \frac{1}{\zeta_t} \mathbb{E}[\zeta_{T_0} C_{T_0} \mid \mathcal{F}_t] = \frac{1}{\zeta_t} \mathbb{E}_t\left[p_{\mathrm{swap}}(Z_{T_0})^+\right]$$

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# Fourier transform

- Define
$$\widehat{q}(x) = \mathbb{E}_t \left[\exp\left(x\, p_{\mathrm{swap}}(Z_{T_0})\right)\right]$$
for every $x \in \mathbb{C}$ such that the conditional expectation is well-defined
- Then
$$\Pi_t^{\mathrm{swaption}} = \frac{1}{\zeta_t \pi} \int_0^\infty \mathrm{Re}\left[\frac{\widehat{q}(\mu + \mathrm{i}\lambda)}{(\mu + \mathrm{i}\lambda)^2}\right] d\lambda$$
for any $\mu > 0$ with $\widehat{q}(\mu) < \infty$
- $\widehat{q}(x)$ has semi-analytical solution in LRSQ model

# Outline

# Linear-Rational Square-Root (LRSQ) model

- Objective: A model with joint factor process $(Z_t, U_t)$, where
  - $Z_t$: $m$ term structure factors
  - $U_t$: $n \leq m$ USV factors
- Denoted *LRSQ(m,n)*
- Based on a $(m+n)$-dimensional square-root diffusion process $X_t$ taking values in $\mathbb{R}_+^{m+n}$ of the form

$$\mathrm{d}X_t = (b - \beta X_t)\,\mathrm{d}t + \mathrm{Diag}\left(\sigma_1\sqrt{X_{1t}}, \ldots, \sigma_{m+n}\sqrt{X_{m+n,t}}\right)\mathrm{d}B_t,$$

- Define $(Z_t, U_t) = SX_t$ as linear transform of $X_t$ with state space $\mathcal{E} = S(\mathbb{R}_+^{m+n})$
- Need to specify a $(m+n) \times (m+n)$-matrix $S$ such that
  - the implied term structure state space is $E = \mathbb{R}_+^m$
  - the drift of $Z_t$ does not depend on $U_t$, while $U_t$ feeds into the martingale part of $Z_t$

# Linear-Rational Square-Root (LRSQ) model (cont.)

- $S$ given by

$$S = \begin{pmatrix} \mathrm{Id}_m & A \\ 0 & \mathrm{Id}_n \end{pmatrix} \quad \text{with } A = \begin{pmatrix} \mathrm{Id}_n \\ 0 \end{pmatrix}.$$

- $\beta$ chosen upper block-triangular of the form

$$\beta = S^{-1} \begin{pmatrix} \kappa & 0 \\ 0 & A^\top \kappa A \end{pmatrix} S = \begin{pmatrix} \kappa & \kappa A - AA^\top \kappa A \\ 0 & A^\top \kappa A \end{pmatrix}$$

for some $\kappa \in \mathbb{R}^{m \times m}$

- $b$ given by

$$b = \beta S^{-1} \begin{pmatrix} \theta \\ \theta_U \end{pmatrix} = \begin{pmatrix} \kappa\theta - AA^\top \kappa A \theta_U \\ A^\top \kappa A \theta_U \end{pmatrix}$$

for some $\theta \in \mathbb{R}^m$ and $\theta_U \in \mathbb{R}^n$.

# Linear-Rational Square-Root (LRSQ) model (cont.)

- Resulting joint factor process $(Z_t, U_t)$:

$$\mathrm{d}Z_t = \kappa\,(\theta - Z_t)\,\mathrm{d}t + \sigma(Z_t, U_t)\mathrm{d}B_t$$
$$\mathrm{d}U_t = A^\top \kappa A\,(\theta_U - U_t)\,\mathrm{d}t + \mathrm{Diag}\left(\sigma_{m+1}\sqrt{U_{1t}}\,\mathrm{d}B_{m+1,t}, \ldots, \sigma_{m+n}\sqrt{U_{nt}}\,\mathrm{d}B_{m+n,t}\right),$$

  with dispersion function of $Z_t$ given by
  $\sigma(z,u) = (\mathrm{Id}_m, A)\,\mathrm{Diag}\left(\sigma_1\sqrt{z_1 - u_1}, \ldots, \sigma_{m+n}\sqrt{u_n}\right)$.

- Example: *LRSQ(1,1)*

$$\mathrm{d}Z_{1t} = \kappa_{11}\,(\theta_1 + \theta_2 - Z_{1t})\,\mathrm{d}t + \sigma_1\sqrt{Z_{1t} - U_{1t}}\mathrm{d}B_{1t} + \sigma_2\sqrt{U_{1t}}\mathrm{d}B_{2t}$$
$$\mathrm{d}U_{1t} = \kappa_{22}\,(\theta_2 - U_{1t})\,\mathrm{d}t + \sigma_2\sqrt{U_{1t}}\mathrm{d}B_{2t}$$

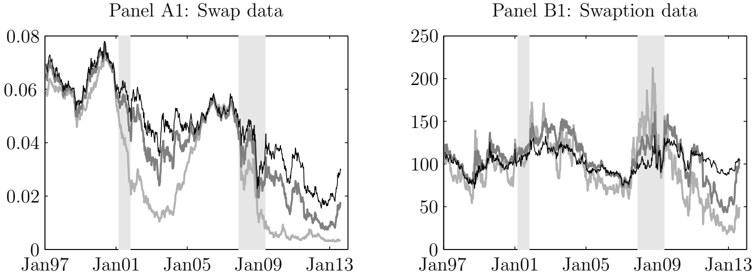
# Linear-rational vs. exponential-affine framework

| | Exponential-affine | Linear-rational |
|---|---|---|
| Short rate | affine | LR |
| ZCB price | exponential-affine | LR |
| ZCB yield | affine | log of LR |
| Coupon bond price | sum of exponential-affines | LR |
| Swap rate | ratio of sums of exponential-affines | LR |
| ZLB | (✓) | ✓ |
| USV | (✓) | ✓ |
| Cap/floor valuation | semi-analytical | semi-analytical |
| Swaption valuation | approximate | semi-analytical |
| Linear state inversion | ZCB yields | bond prices or swap rates |

# Outline

# Data and estimation approach

- Panel data set of swaps and swaptions
- Swap maturities: 1Y, 2Y, 3Y, 5Y, 7Y, 10Y
- Swaptions expiries: 3M, 1Y, 2Y, 5Y
- 866 weekly observations, Jan 29, 1997 – Aug 28, 2013
- Estimation approach: Quasi-maximum likelihood in conjunction with the unscented Kalman Filter



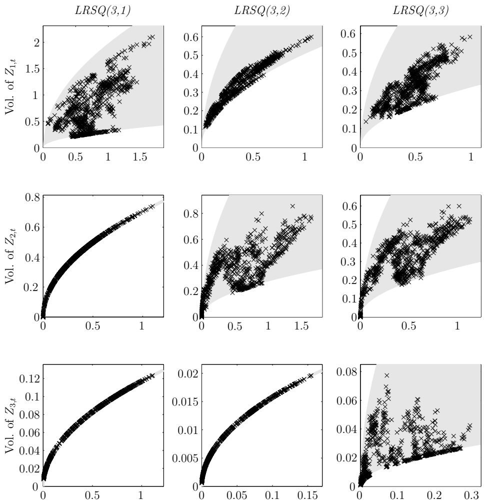
# Model specifications

- Model specifications (always 3 term structure factors)
  - *LRSQ(3,1)*: volatility of $Z_{1t}$ containing an unspanned component
  - *LRSQ(3,2)*: volatility of $Z_{1t}$ and $Z_{2t}$ containing unspanned components
  - *LRSQ(3,3)*: volatility of term structure factors containing unspanned components
- $\alpha = \alpha^*$ and range of $r_t$:

| | *LRSQ(3,1)* | *LRSQ(3,2)* | *LRSQ(3,3)* |
|---|---|---|---|
| Long ZCB yield $\alpha$ | 7.46% | 6.88% | 5.66% |
| Upper bound on $r_t$ | 20% | 146% | 72% |

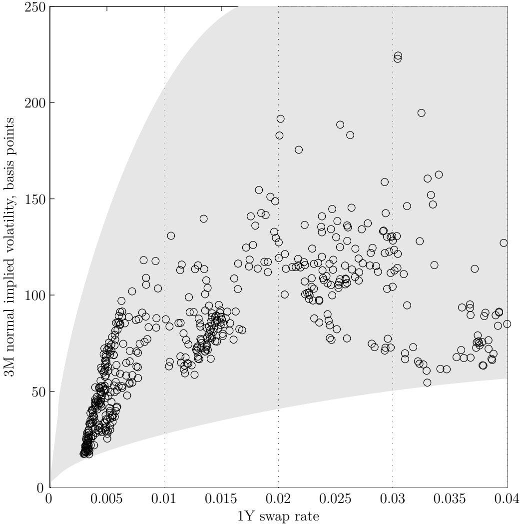
# Level-dependence in factor volatilities

- Volatility of $Z_{it}$ with USV: $\sqrt{\sigma_i^2 Z_{it} + (\sigma_{i+3}^2 - \sigma_i^2) U_{it}}$
- Volatility of $Z_{it}$ without USV: $\sigma_i \sqrt{Z_{it}}$



# Volatility dynamics near the ZLB

- Level-dependence in volatility, 3M/1Y swaption IV vs. 1Y swap rate



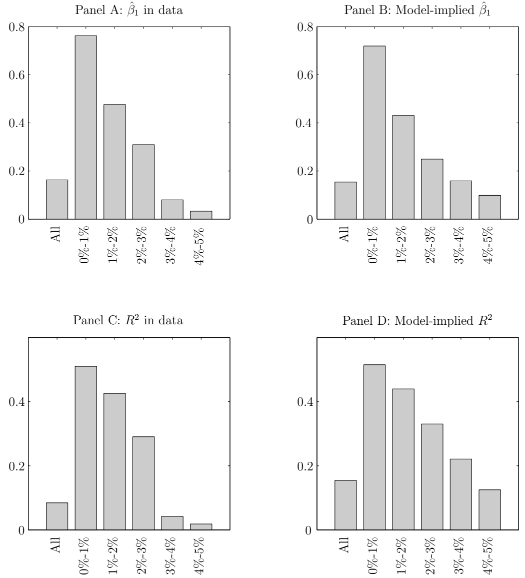
# Level-dependence in volatility

- Regress weekly changes in the 3M swaption IV on weekly changes in the underlying swap rate

$$\Delta\sigma_{N,t} = \beta_0 + \beta_1 \Delta S_t + \epsilon_t$$

| | 1 yr | 2 yrs | 3 yrs | 5 yrs | 7 yrs | 10 yrs | Mean |
|---|---|---|---|---|---|---|---|
| *Panel A:* $\hat{\beta}_1$ | | | | | | | |
| All | $0.18^{**}$ (2.38) | $0.16^{***}$ (2.88) | $0.16^{***}$ (3.31) | $0.16^{***}$ (4.12) | $0.16^{***}$ (4.59) | $0.16^{***}$ (4.97) | 0.16 |
| 0%-1% | $1.20^{***}$ (8.03) | $0.74^{***}$ (8.79) | $0.62^{***}$ (8.19) | $0.48^{***}$ (7.83) | | | 0.76 |
| 1%-2% | $0.54^{***}$ (2.70) | $0.64^{***}$ (6.21) | $0.46^{***}$ (6.77) | $0.52^{***}$ (5.02) | $0.45^{***}$ (5.23) | $0.26^{***}$ (8.24) | 0.48 |
| 2%-3% | $0.28^{***}$ (3.10) | $0.11^{**}$ (1.97) | $0.30^{***}$ (3.77) | $0.36^{***}$ (5.08) | $0.40^{***}$ (5.62) | $0.40^{***}$ (4.93) | 0.31 |
| 3%-4% | −0.02 (−0.22) | 0.11 (1.21) | 0.06 (0.92) | 0.05 (0.80) | $0.11^{*}$ (1.82) | $0.17^{*}$ (1.96) | 0.08 |
| 4%-5% | 0.04 (0.31) | −0.07 (−0.82) | 0.01 (0.08) | 0.08 (1.59) | $0.07^{*}$ (1.76) | $0.07^{*}$ (1.65) | 0.03 |
| *Panel B:* $R^2$ | | | | | | | |
| All | 0.05 | 0.06 | 0.08 | 0.10 | 0.11 | 0.10 | 0.08 |
| 0%-1% | 0.52 | 0.54 | 0.54 | 0.44 | | | 0.51 |
| 1%-2% | 0.25 | 0.49 | 0.45 | 0.55 | 0.55 | 0.27 | 0.43 |
| 2%-3% | 0.16 | 0.06 | 0.28 | 0.37 | 0.44 | 0.45 | 0.29 |
| 3%-4% | 0.00 | 0.03 | 0.01 | 0.01 | 0.07 | 0.12 | 0.04 |
| 4%-5% | 0.00 | 0.01 | 0.00 | 0.03 | 0.03 | 0.03 | 0.02 |

# Level-dependence in volatility, *LRSQ(3,3)*



# Conclusion

- Key features of framework:
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  - Easily accommodates unspanned factors affecting volatility and risk premia
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