Time-Varying Credit Risk and Liquidity in Bond and CDS Markets

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Bond yield spreads and CDS premia

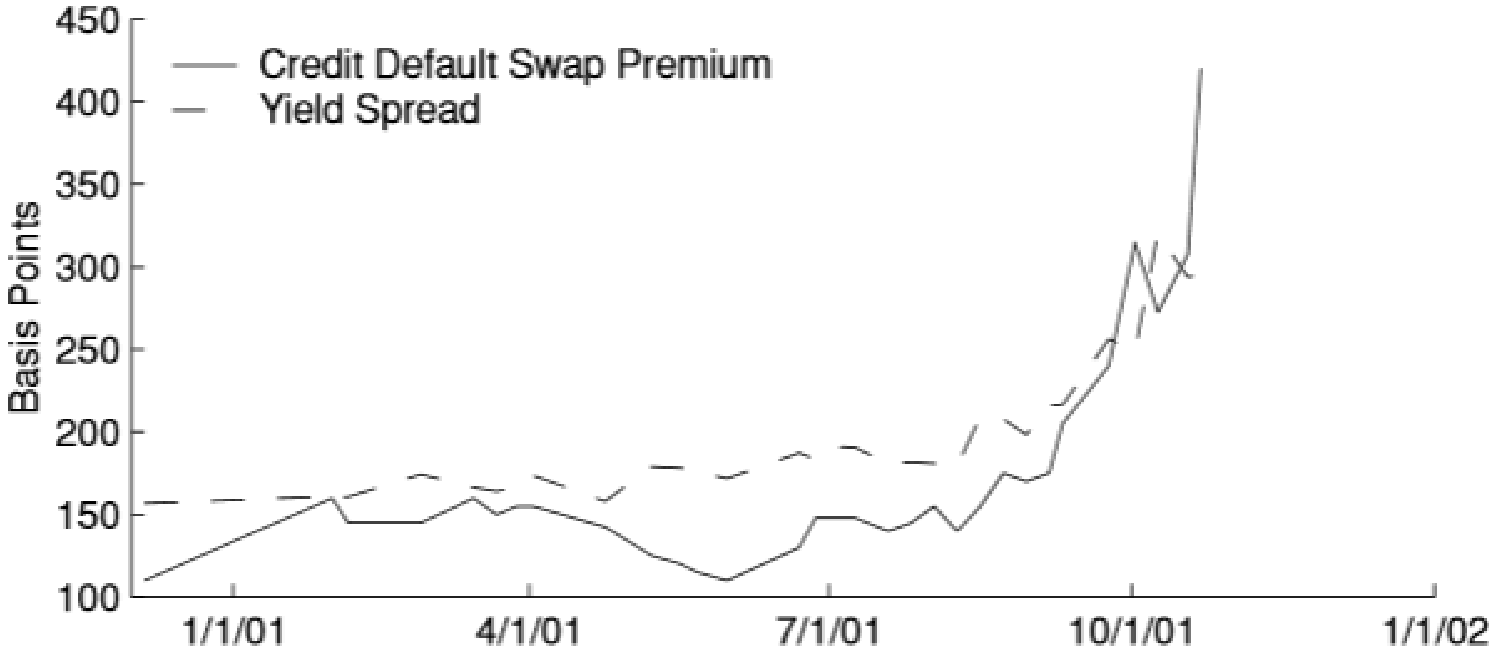

Figure 1 from Longstaff, Mithal, Neis (2005), p. 2224

- Enron bond yield spread and CDS premium (mid)
- Assumption: difference due to bond liquidity

Disaggregating bond yield spreads

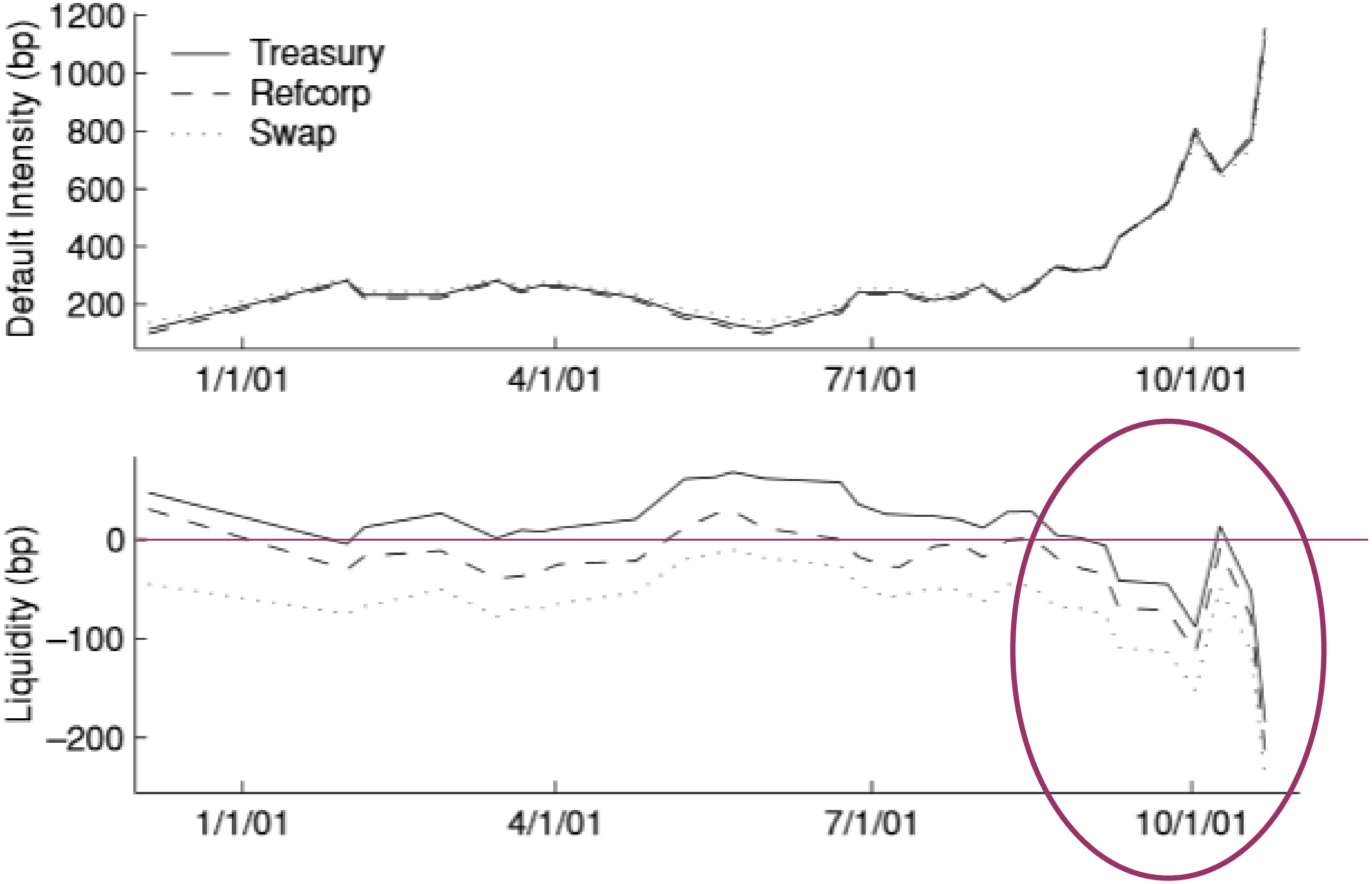

Figure 2 from Longstaff, Mithal, Neis (2005), p. 2226

Our approach

Explicit model for CDS liquidity

➢ Liquidity of CDS contracts affects CDS bid and ask quotes

➢ Effects in line with market microstructure concept, but using parsimonious reduced-form setting

⇒ Bond prices and CDS premia function of

- Risk-free interest rates
- Credit risk
- Liquidity
- Correlation factors

Consequences

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1. We can quantify the relative importance of credit risk, liquidity, and correlation between the two in bonds and CDS
2. We obtain a threefold link between bond and CDS markets
   - a) Common fundamental factor: credit risk
   - b) Impact of credit risk on liquidity and / or liquidity on credit risk (directional) (Vayanos 2004, Ericsson/Renault 2006, He/Xiong 2012, Dick-Nielsen et al. 2012, Friewald et al. 2012)
   - c) Liquidity link (independent of credit risk link) allows us to identify the predominant trading motive (hedging/speculation)

   $\Rightarrow$ Relation to studies on link between asset and derivatives markets (Garleanu 2009, Cetin et al. 2006, Sambalaibat 2015)

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    - Bond: 60% credit risk / 35% liquidity / 5% correlation
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  - Credit risk and liquidity link (2.b)
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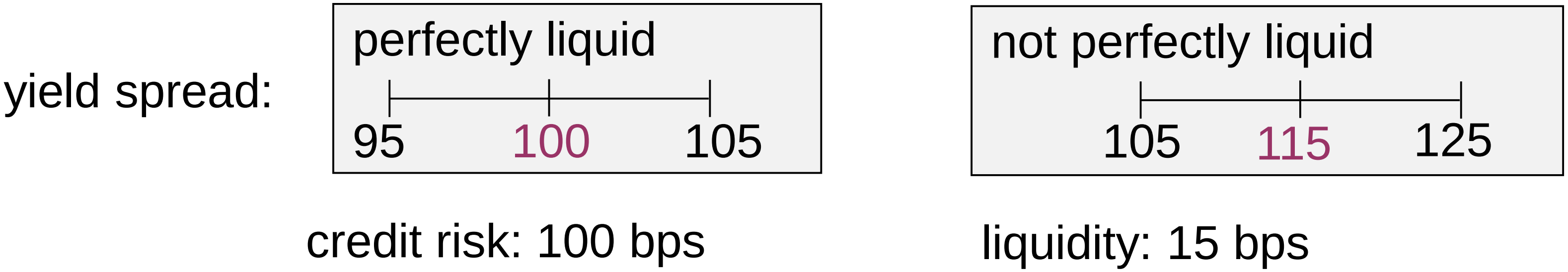
    - Bond liquidity dries up as credit risk increases (flight-to-liquidity)
    - CDS: widening bid-ask spread, ask quote more sensitive (inventory)
  - Liquidity link (2.c): inverse relation of bond and CDS liquidity premia, pointing to hedging demand

The liquidity concept

- Standard effect of illiquidity in bond market (positive net supply)



yield spread:

perfectly liquid

95 100 105

credit risk: 100 bps

not perfectly liquid

105 115 125

liquidity: 15 bps

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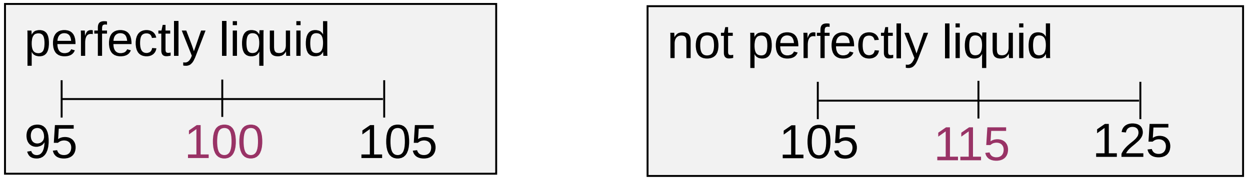

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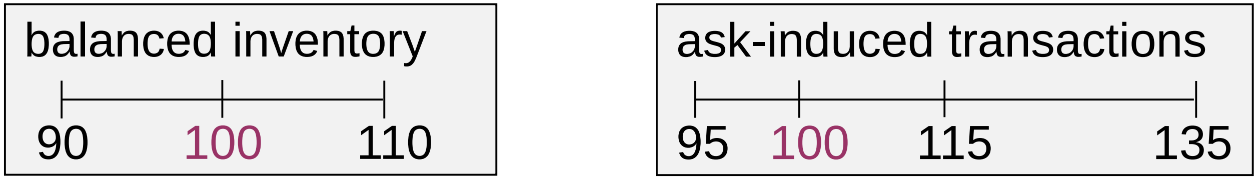

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ask-induced transactions

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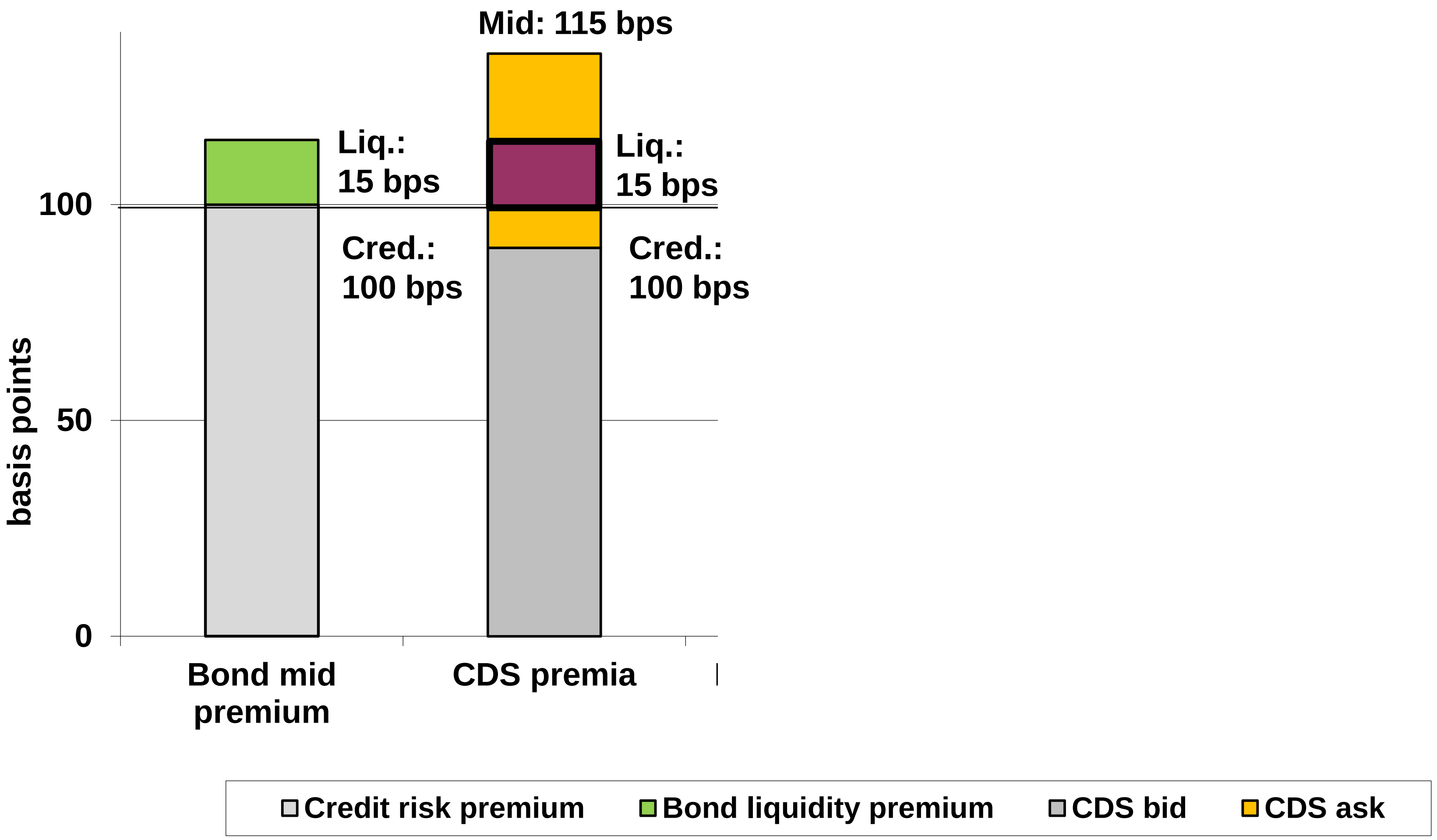
Bid-induced transactions: negative liquidity premia

The liquidity relation



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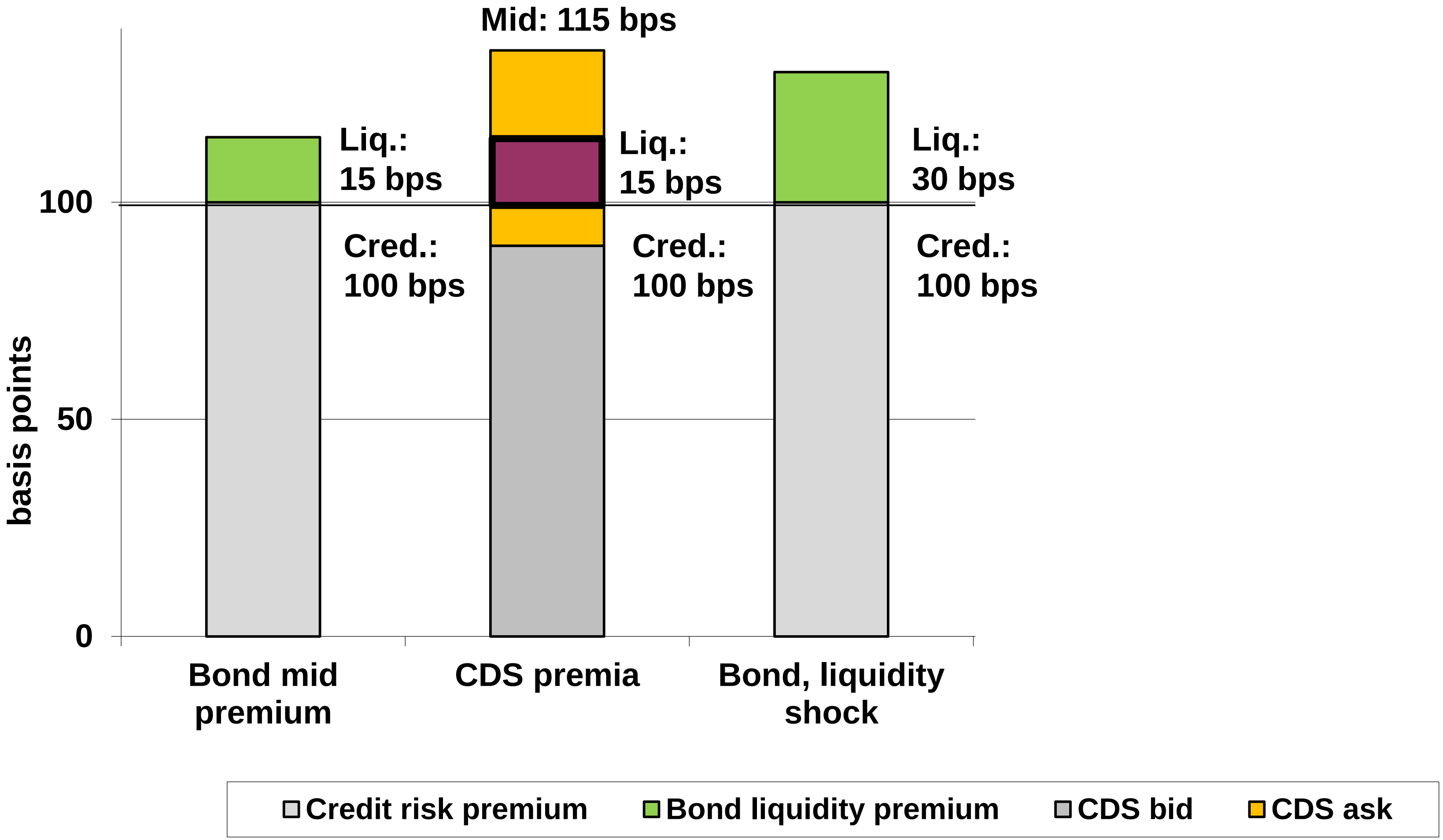

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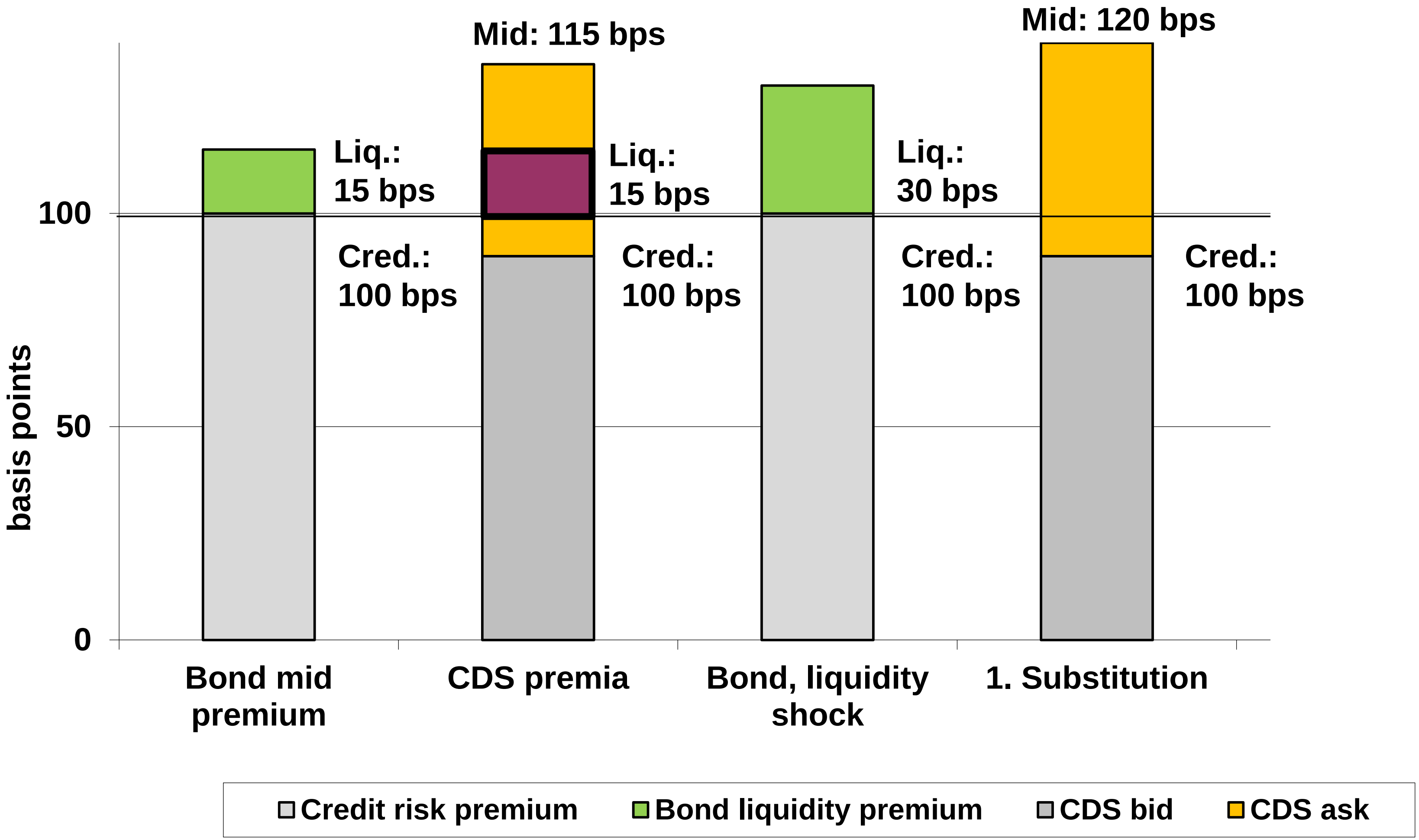

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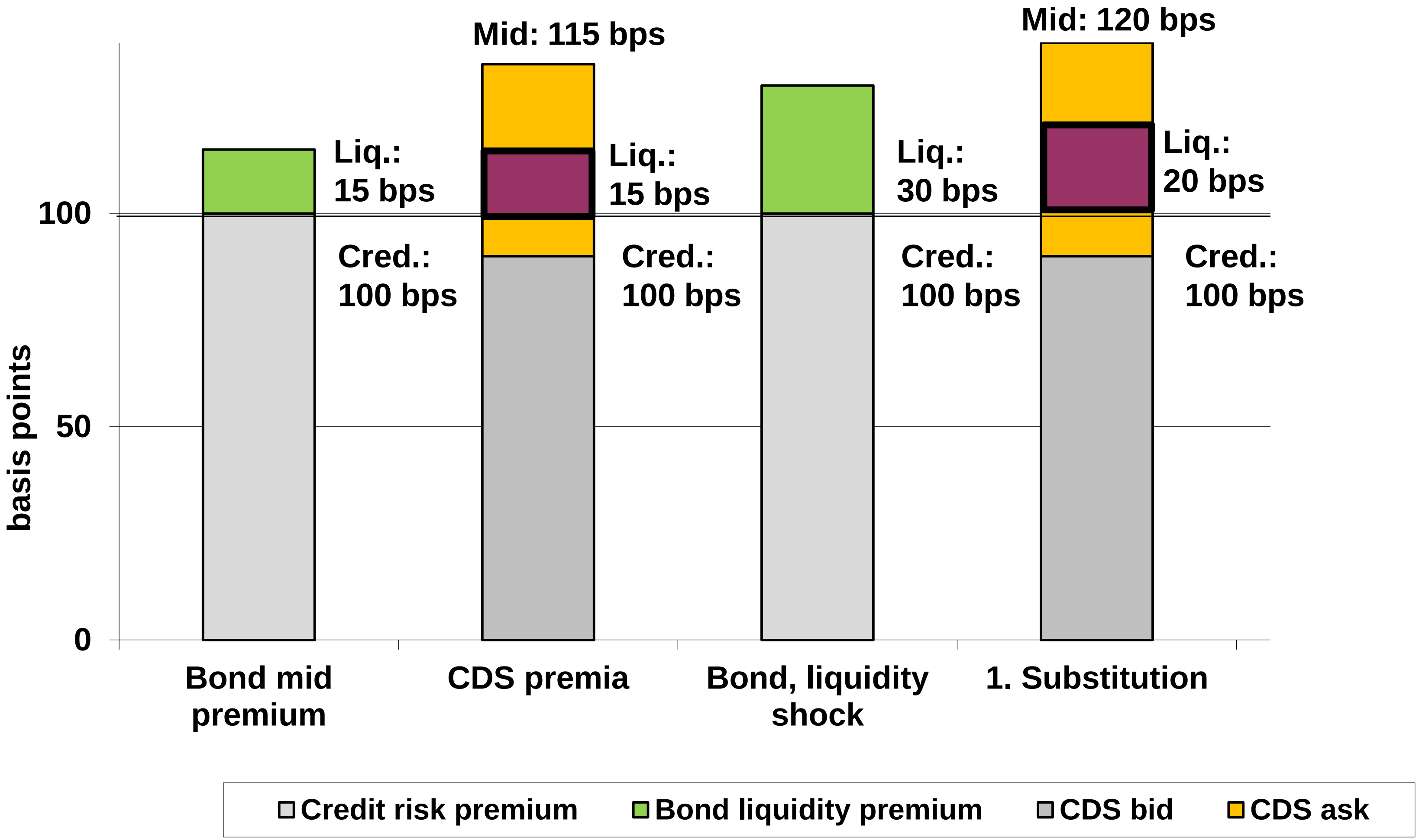

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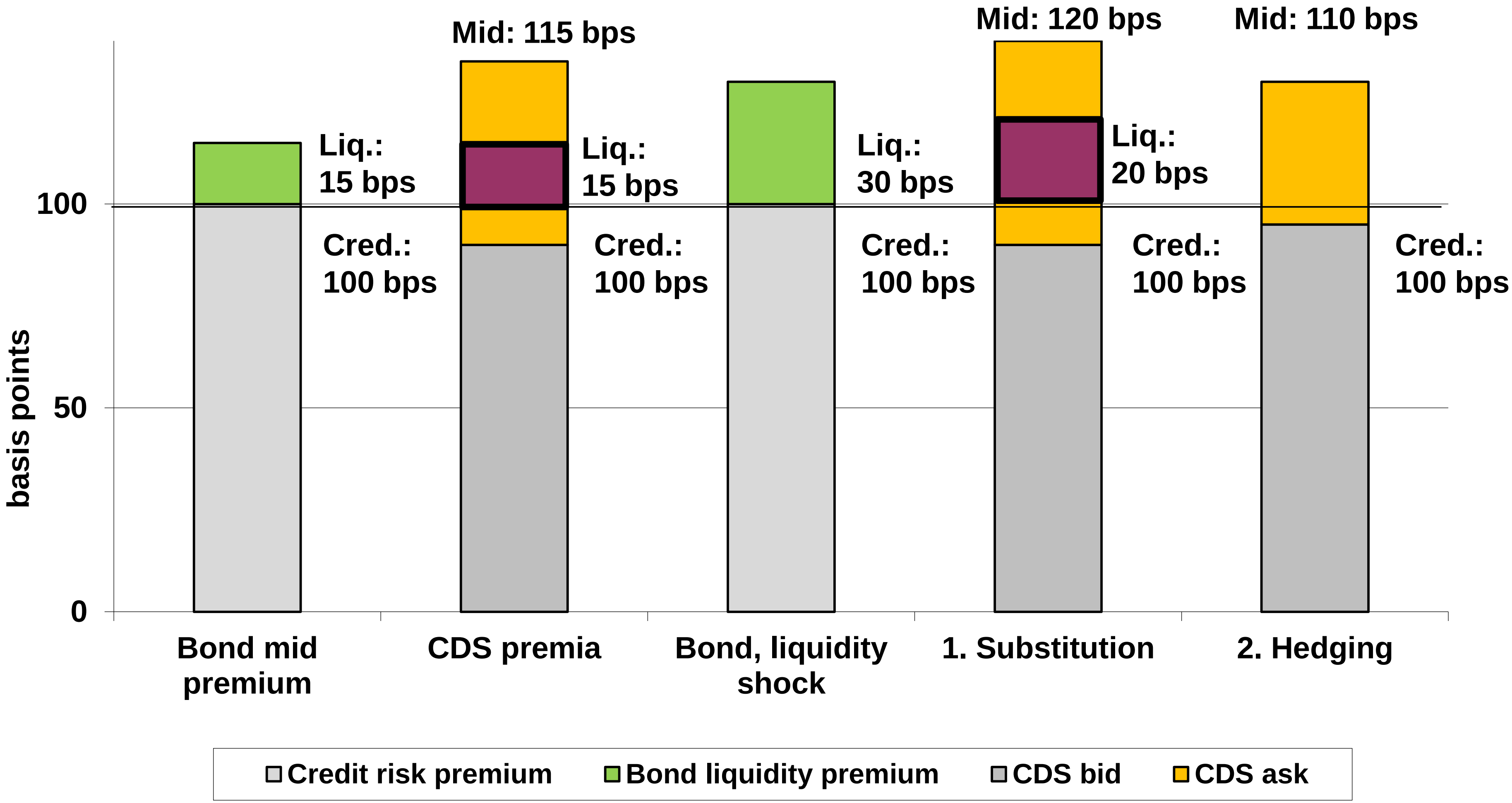

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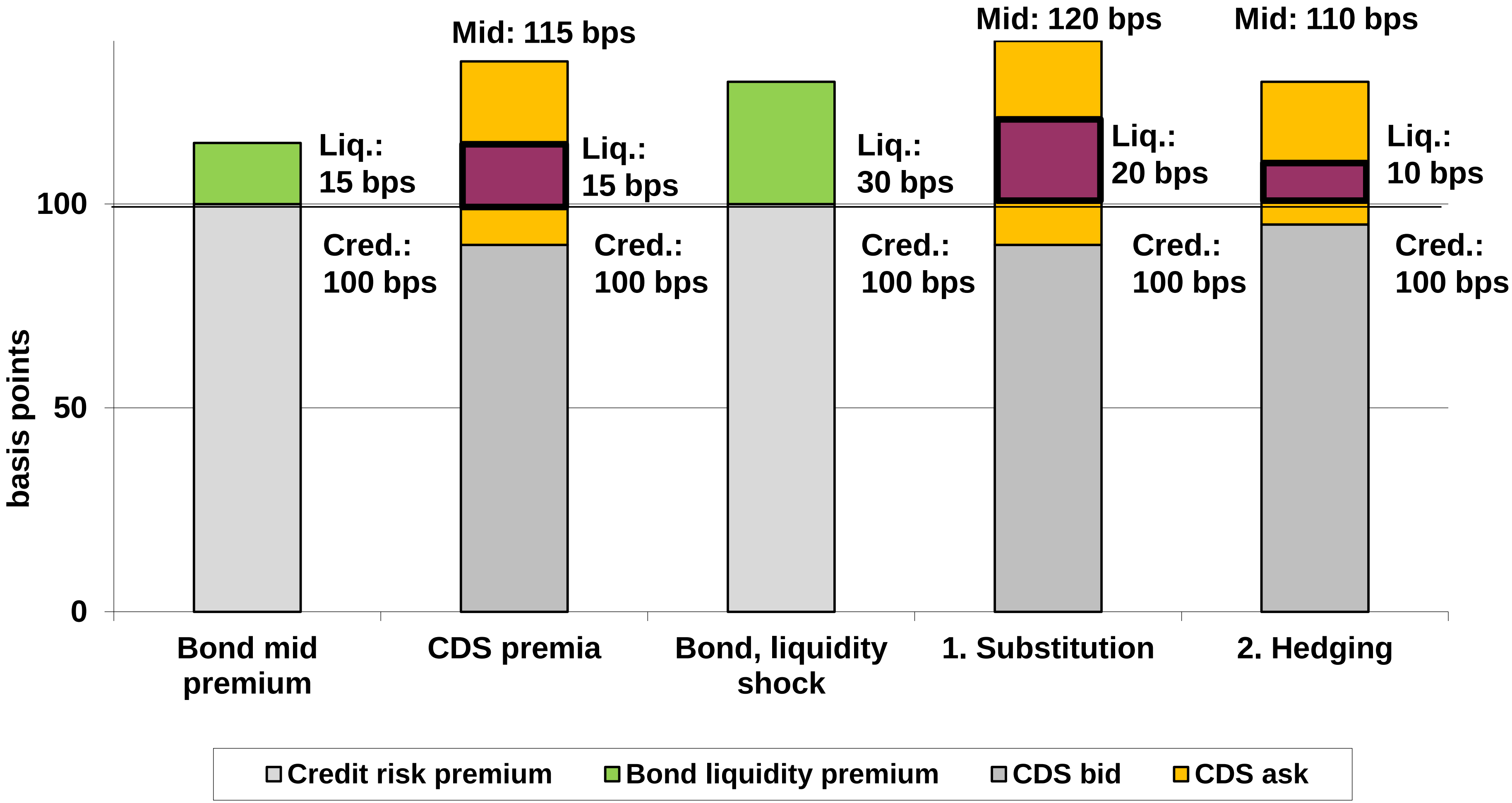
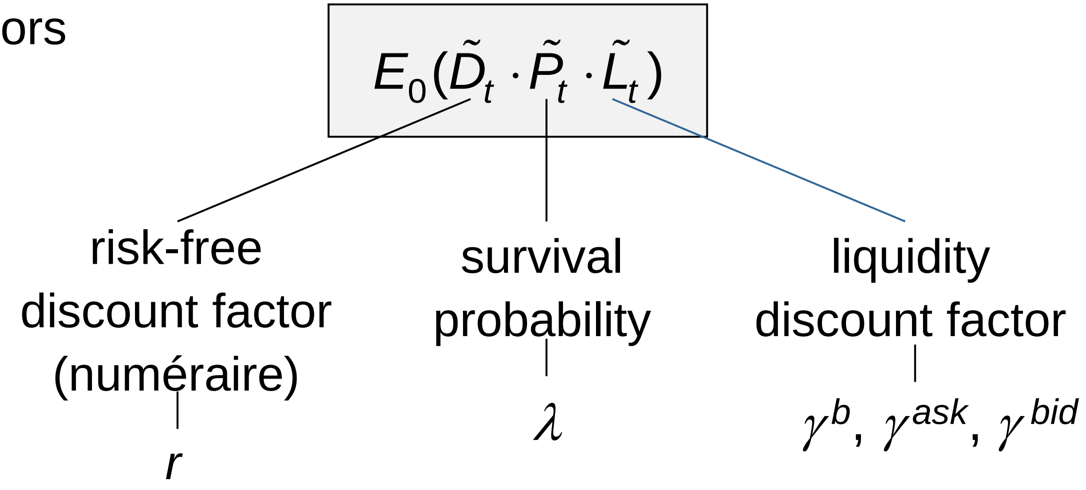


The model – I

- **Reduced-form model**

Discount factors



$$E_0(\tilde{D}_t \cdot \tilde{P}_t \cdot \tilde{L}_t)$$

risk-free discount factor (numéraire)

r

survival probability

λ

liquidity discount factor

$\gamma^{b}, \gamma^{ask}, \gamma^{bid}$

- **Duffie-Singleton setting**

$$E_0(\tilde{D}_t \cdot \tilde{P}_t \cdot \tilde{L}_t) = \underbrace{E_0\left[\exp\left(-\int_0^t r_s + \lambda_s + \gamma_s ds\right)\right]}_{\text{analytical solution}}$$

The model – II

- **Factor structure**

$$\begin{pmatrix} d\lambda \\ d\gamma^{b} \\ d\gamma^{ask} \\ d\gamma^{bid} \end{pmatrix} = \begin{pmatrix} 1 & g_b & g_{ask} & g_{bid} \\ f_b & 1 & \omega_{b,ask} & \omega_{b,bid} \\ f_{ask} & \omega_{b,ask} & 1 & \omega_{ask,bid} \\ f_{bid} & \omega_{b,bid} & \omega_{ask,bid} & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy^{b} \\ dy^{ask} \\ dy^{bid} \end{pmatrix} \begin{array}{l} \leftarrow \text{pure credit risk} \\ \leftarrow \text{pure bond liquidity} \\ \left.\begin{array}{l} \leftarrow \text{ask} \\ \leftarrow \text{bid} \end{array}\right\} \text{pure CDS liquidity} \end{array}$$

$$= \begin{pmatrix} 1 & g_b & g_{ask} & g_{bid} \\ f_b & 1 & \omega_{b,ask} & \omega_{b,bid} \\ f_{ask} & \omega_{b,ask} & 1 & \omega_{ask,bid} \\ f_{bid} & \omega_{b,bid} & \omega_{ask,bid} & 1 \end{pmatrix} \left(\begin{pmatrix} \alpha - \beta x_t \\ \mu^{b} \\ \mu^{ask} \\ \mu^{bid} \end{pmatrix} dt + \begin{pmatrix} \sigma\sqrt{x_t}\, dW_{x,t} \\ \eta^{b} dW_{y^{b},t} \\ \eta^{ask} dW_{y^{ask},t} \\ \eta^{bid} dW_{y^{bid},t} \end{pmatrix} \right)$$

The model – II

- **Factor structure**

$$\underbrace{\begin{pmatrix} d\lambda \\ d\gamma^{b} \\ d\gamma^{ask} \\ d\gamma^{bid} \end{pmatrix}}_{\text{priced intensities}} = \underbrace{\begin{pmatrix} 1 & g_{b} & g_{ask} & g_{bid} \\ f_{b} & 1 & \omega_{b,ask} & \omega_{b,bid} \\ f_{ask} & \omega_{b,ask} & 1 & \omega_{ask,bid} \\ f_{bid} & \omega_{b,bid} & \omega_{ask,bid} & 1 \end{pmatrix}}_{\text{factor matrix H}} \underbrace{\begin{pmatrix} dx \\ dy^{b} \\ dy^{ask} \\ dy^{bid} \end{pmatrix}}_{\text{independent latent factors}} \begin{array}{l} \leftarrow \text{pure credit risk} \\ \leftarrow \text{pure bond liquidity} \\ \left. \begin{array}{l} \leftarrow \text{ask} \\ \leftarrow \text{bid} \end{array} \right\} \text{pure CDS liquidity} \end{array}$$

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The model – II

- **Relation at the factor structure level**

$$\begin{pmatrix} d\lambda \\ d\gamma^{b} \\ d\gamma^{ask} \\ d\gamma^{bid} \end{pmatrix} = \begin{pmatrix} 1 & g_{b} & g_{ask} & g_{bid} \\ f_{b} & 1 & \omega_{b,ask} & \omega_{b,bid} \\ f_{ask} & \omega_{b,ask} & 1 & \omega_{ask,bid} \\ f_{bid} & \omega_{b,bid} & \omega_{ask,bid} & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy^{b} \\ dy^{ask} \\ dy^{bid} \end{pmatrix}$$

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2.b

$\vec{f} = \vec{g} = 0$ $\Rightarrow$ credit risk and liquidity uncorrelated

$\vec{f} \neq 0$ $\Rightarrow$ pure credit risk affects liquidity (Ericsson/Renault 2006)

$\vec{g} \neq 0$ $\Rightarrow$ pure liquidity affects credit risk (He/Xiong 2012)

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2.c

$\vec{\omega} \neq 0$ $\Rightarrow$ liquidity spillover across markets

$\omega_{b,ask} > 0,\ \omega_{b,bid} < 0$ $\Rightarrow$ ask increases, bid decreases $\Rightarrow$ substitution

$\omega_{b,ask} < 0,\ \omega_{b,bid} > 0$ $\Rightarrow$ ask decreases, bid increases $\Rightarrow$ hedging

The model – III

- **Comparable premia**

Construct par bond with given payment dates and maturity

credit risk: $bd = \text{mid yield spread}(x, \vec{y} = 0, \vec{f} = 0, \vec{g} = 0)$

liquidity: $bl = \text{mid yield spread}(x, \vec{y}, \vec{f} = 0, \vec{g} = 0) - bd$

correlation: $bc = \text{mid yield spread}(x, \vec{y}, \vec{f}, \vec{g}) - bd - bl$

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Construct CDS with identical payment dates and maturity

credit risk: $sd = \text{mid quote}(x, \vec{y} = 0, \vec{f} = 0, \vec{g} = 0)$

liquidity: $sl = \text{mid quote}(x, \vec{y}, \vec{f} = 0, \vec{g} = 0) - sd$

correlation: $sc = \text{mid quote}(x, \vec{y}, \vec{f}, \vec{g}) - sd - sl$

The model – III

- **Relation at the premium level**

2.b

$bc = sc = 0$ $\Rightarrow$ credit risk and liquidity uncorrelated

$bc,\, sc \neq 0$ $\Rightarrow$ premium for correlation between credit risk and liquidity

The model – III

- **Relation at the premium level**

2.b

$bc = sc = 0$ ⇒ credit risk and liquidity uncorrelated

$bc, sc \neq 0$ ⇒ premium for correlation between credit risk and liquidity

2.c

$\text{cor}(bl, sl) \neq 0$ ⇒ liquidity spillover across markets

$\text{cor}(bl, sl) > 0$ ⇒ both liquidity premia increase jointly

⇒ substitution

$\text{cor}(bl, sl) < 0$ ⇒ increases in one premium associated with decreases in the other premium

⇒ hedging

Data

- Time period June 2001 to December 2013
- Risk-free interest rate: Nelson-Siegel-Svensson curve derived from German sovereign bonds
- CDS: CMA bid and ask quotes, EUR, 5-year maturity
- Bonds: Bloomberg mid quotes
- Ratings: Thompson-Reuters S&P rating
- Selection procedure:
  - European underlying reference entity (to harmonize contract specifics)
  - S&P rating from AAA to CCC
  - At least 2 plain vanilla corporate bonds outstanding
  - At least 30 consecutive days with quotes for at least 2 bonds, CDS bid and ask
- Yields a sample of 257 firms

Estimation procedure

1. Set up a grid for the latent factor process parameters, choose an initial value $\theta_0 = \left(\alpha_0, \beta_0, \ldots, \mu_0^{bid}, \sigma_0^{bid}\right)$
2. Initialize factor matrix $H_0 = I$

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2. Initialize factor matrix $H_0 = I$
3. Determine estimates of $\left(\hat{\lambda}_t, \hat{\gamma}_t^{b}, \hat{\gamma}_t^{c_{ask}}, \hat{\gamma}_t^{c_{bid}}\right)$ for each observation date t via
$$\min\ S_0 = \sum_{t=1}^{T} \sum_{i=1}^{N_t} (P_{t,i}^{theor.}(\hat{\lambda}_t, \hat{\gamma}_t^{b}, \hat{\gamma}_t^{ask}, \hat{\gamma}_t^{bid}; \mathrm{H}_0, \theta_0) - P_{t,i}^{obs.})^2$$
4. Determine $\left(\hat{x}_t, \hat{\mathrm{y}}_t^{b}, \hat{\mathrm{y}}_t^{ask}, \hat{\mathrm{y}}_t^{bid}\right)_{t=1,\ldots,T} = H_0^{-1}\left(\hat{\lambda}_t, \hat{\gamma}_t^{b}, \hat{\gamma}_t^{ask}, \hat{\gamma}_t^{bid}\right)$
5. Determine M_0 as the SSD between the theoretical and empirical moments of the latent factor changes

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6. Determine $H_1 := \arg\min M_0$ via numerical minimization, return to 3., repeat until max-norm of change in H < 1% twice
7. Determine grid point with smallest S, set up finer local grid around it, return to 1., repeat until smallest S decreases by < 1% twice

Cross-sectional results

- **Results at the factor level**

	f_b	f_{ask}	f_{bid}	g_b	g_{ask}	g_{bid}
Av.	0.16	0.37	-0.07	0.01	0.01	0.00
# sign.	240	238	166	7	9	6

2.b

- Credit risk affects liquidity, not vice versa
- Liquidity relation via credit risk impact

Cross-sectional results

- **Results at the factor level**

	f_b	f_{ask}	f_{bid}	g_b	g_{ask}	g_{bid}	$w_{b,ask}$	$w_{b,bid}$	$w_{ask,bid}$
Av.	0.16	0.37	-0.07	0.01	0.01	0.00	-0.02	0.01	-0.38
# sign.	240	238	166	7	9	6	223	185	231

2.b
- Credit risk affects liquidity, not vice versa
- Liquidity relation via credit risk impact

2.c
- Bond liquidity shock decreases CDS ask premia, increases bid premia
- Consistent with hedging between markets

Cross-sectional results

- **Results at the premium level**

	bd	bl	bc	sd	sl	sc
Av. %	88.8 60	52.8 35	7.2 5	88.8 95	3.9 4	0.7 1

1.
- Liquidity significantly more important for bond than for CDS markets
- Strictly positive bond liquidity premium

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2.b

- Correlation premia on average positive
- Negative correlation premia for AAA/AA firms consistent with flight-to-quality

Time series analysis

- **Results at the premium level**

  - Run vector error correction model for liquidity premia

	Investment grade firms			Speculative grade firms		
	Coint.	ECT bond	ECT CDS	Coint.	ECT bond	ECT CDS
Av. % sign.	8.86 95	-0.04 11	-0.13 94	17.29 98	-0.16 72	-0.26 98

- Liquidity premia countermove contemporaneously
- Bond liquidity premia affect CDS liquidity premia in IG, not vice versa
- Impact of CDS liquidity in speculative grade w/ limited economic significance ⇒ hedging

2.c

Extensions and robustness

- Simulation study for negative default probabilities
- Maximum likelihood instead of SSE
- Calibrate model using only CDS mid or bid or ask quotes
- Extend model to bond bid and ask quotes for transactions subsample
- Swap rate as alternative risk-free interest rate proxy
- Construct a specialness-adjusted reference rate as in Duffie (1996)
- Include counterparty risk in fixed and floating leg of CDS
- Time split financial crisis: bc, sc, sl become relatively more important, pointing at flight-to-quality and higher importance of inventory management for CDS dealers

Thank you very much for your attention!

Correlation structure

$$Corr(d\lambda_t, d\gamma_t^b) = \frac{f_b \sigma^2 x_t + g_b \eta^{b^2} + g_{ask} \omega_{b,ask} \eta^{ask^2} + g_{bid} \omega_{b,bid} \eta^{bid^2}}{\sqrt{\left(\sigma^2 x_t + g_b^2 \eta^{b^2} + g_{ask}^2 \eta^{ask^2} + g_{bid}^2 \eta^{bid^2}\right)\left(f_b^2 \sigma^2 x_t + \eta^{b^2} + \omega_{b,ask}^2 \eta^{ask^2} + \omega_{b,bid}^2 \eta^{bid^2}\right)}}$$

$$Corr(d\gamma_t^{ask}, d\gamma_t^{bid}) = \frac{f_{ask} f_{bid} \sigma^2 x_t + \omega_{b,ask} \omega_{b,bid} \eta^{b^2} + \omega_{ask,bid} \eta^{ask^2} + \omega_{ask,bid} \eta^{bid^2}}{\sqrt{\left(f_{ask}^2 \sigma^2 x_t + \omega_{b,ask}^2 \eta^{b^2} + \eta^{ask^2} + \omega_{ask,bid}^2 \eta^{bid^2}\right)\left(f_{bid}^2 \sigma^2 x_t + \omega_{b,bid}^2 \eta^{b^2} + \omega_{ask,bid}^2 \eta^{ask^2} + \eta^{bid^2}\right)}}$$

Discount factor structure

BOND PRICE

Bond Price:

$$c \cdot \sum_{i=1}^{n} D(t_i) \cdot P(t_i, x_{t_0}; f) \cdot L\left(t_i, y_{t_0}^{b}; g\right) + D(t_n) \cdot P(t_n, x_{t_0}; f) \cdot L(t_n, y_{t_0}^{b}; g)$$
$$+ R \cdot \sum_{j=1}^{N} D(\theta_j) \cdot \Delta P(\theta_j, x_{t_0}; f) \cdot L(\theta_j, y_{t_0}^{b}; g)$$

CDS ASK PREMIUM

Fixed Leg :

$$s^{ask} \cdot \left(\begin{array}{l} \sum_{i=1}^{m} D(T_i) \cdot P(T_{i-1}, x_{T_0}; f) \cdot L\left(T_i, y_{T_0}^{ask}; g\right) \\ + \sum_{j=1}^{M} \delta_j D(\theta_j) \cdot \Delta P(\theta_j, x_{t_0}; f) \end{array} \right)$$

Floating Leg :

$$\sum_{j=1}^{M} D(\theta_j) \cdot \Delta P(\theta_j, x_{t_0}; f) \cdot \left[1 - R \cdot L(\theta_j, y_{t_0}^{b}; g)\right]$$

Analytical solutions for the discount factors – I

- Assume that risk-free discount factor is known / independent of the latent factors

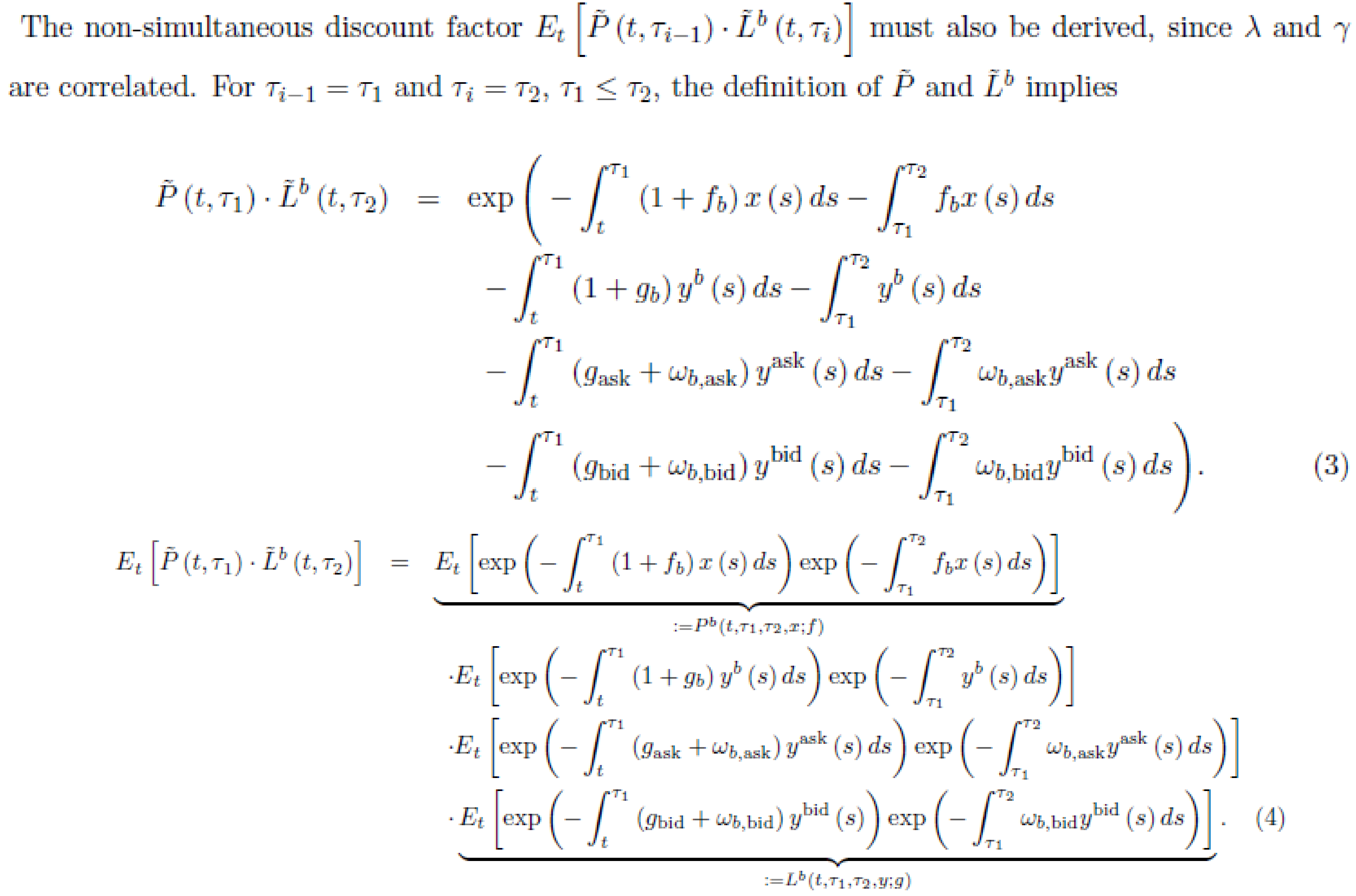
$$
\begin{aligned}
& E_t\left[\tilde{P}(t,\tau)\cdot\tilde{L}^b(t,\tau)\right] = E_t\left[\exp\left(-\int_t^\tau \lambda(s)+\gamma^b(s)\,ds\right)\right] \\
= \quad & E_t\left[\exp\left(-\int_t^\tau (1+f_b)\,x(s)+(1+g_b)\,y^b(s)+(g_{\text{ask}}+\omega_{b,\text{ask}})\,y^{\text{ask}}(s)+(g_{\text{bid}}+\omega_{b,\text{bid}})\,y^{\text{bid}}(s)\,ds\right)\right] \\
= \quad & E_t\left[\exp\left(-\int_t^\tau (1+f_b)\,x(s)\,ds\right)\right]\cdot E_t\left[\exp\left(-\int_t^\tau (1+g_b)\,y^b(s)\,ds\right)\right] \\
& \cdot E_t\left[\exp\left(-\int_t^\tau (g_{\text{ask}}+\omega_{b,\text{ask}})\,y^{\text{ask}}(s)\,ds\right)\right]\cdot E_t\left[\exp\left(-\int_t^\tau (g_{\text{bid}}+\omega_{b,\text{bid}})\,y^{\text{bid}}(s)\,ds\right)\right] \\
=: \quad & \underbrace{P(t,\tau,x;1+f_b)}_{:=P(t,\tau,x;f)}\underbrace{L\left(t,\tau,y^b;1+g_b\right)L\left(t,\tau,y^{\text{ask}};g_{\text{ask}}+\omega_{b,\text{ask}}\right)L\left(t,\tau,y^{\text{bid}};g_{\text{bid}}+\omega_{b,\text{bid}}\right)}_{:=L^b(t,\tau,y;g)}, \qquad (1)
\end{aligned}
$$

Analytical solutions for the discount factors – II

$$\begin{aligned} P\left(t,\tau,x;k\right) &:= a_1\left(t,\tau;k\right)\cdot\exp\left[-a_2\left(t,\tau;k\right)\,k\,x\left(t\right)\right], \\ L\left(t,\tau,y^l;k\right) &:= a_3^l\left(t,\tau;k\right)\cdot\exp\left[-a_4^l\left(t,\tau;k\right)\,k\,y^l\left(t\right)\right], \end{aligned}$$

$$\begin{aligned} a_1\left(t,\tau;k\right) &= \left(\frac{1-\kappa\left(k\right)}{1-\kappa\left(k\right)\exp\left[\phi\left(k\right)\left(\tau-t\right)\right]}\right)^{\frac{2\alpha}{\sigma^2}}\exp\left[\frac{\alpha\left(\beta+\phi\left(k\right)\right)}{\sigma^2}\left(\tau-t\right)\right], \\ a_2\left(t,\tau;k\right) &= \frac{\phi\left(k\right)-\beta}{\sigma^2 k}+\frac{2\phi\left(k\right)}{\sigma^2\,k\left(\kappa\left(k\right)\exp\left[\phi\left(k\right)\left(\tau-t\right)\right]-1\right)}, \\ a_3^l\left(t,\tau;k\right) &= \exp\left[\frac{k^2\eta^{l2}\left(\tau-t\right)^3}{6}-\frac{k\mu^l\left(\tau-t\right)^2}{2}\right], \\ a_4^l\left(t,\tau;k\right) &= \tau-t, \\ \phi\left(k\right) &= \sqrt{2\sigma^2\,k+\beta^2},\quad \kappa\left(k\right)=\frac{\beta+\phi\left(k\right)}{\beta-\phi\left(k\right)}. \end{aligned}$$

Analytical solutions for the discount factors – III



The non-simultaneous discount factor $E_t\left[\tilde{P}\left(t,\tau_{i-1}\right)\cdot\tilde{L}^b\left(t,\tau_i\right)\right]$ must also be derived, since λ and γ are correlated. For $\tau_{i-1}=\tau_1$ and $\tau_i=\tau_2$, $\tau_1\leq\tau_2$, the definition of $\tilde{P}$ and $\tilde{L}^b$ implies

$$\begin{aligned}
\tilde{P}\left(t,\tau_1\right)\cdot\tilde{L}^b\left(t,\tau_2\right) = \exp\Bigg(& -\int_t^{\tau_1}\left(1+f_b\right)x\left(s\right)ds-\int_{\tau_1}^{\tau_2}f_bx\left(s\right)ds \\
& -\int_t^{\tau_1}\left(1+g_b\right)y^b\left(s\right)ds-\int_{\tau_1}^{\tau_2}y^b\left(s\right)ds \\
& -\int_t^{\tau_1}\left(g_{\text{ask}}+\omega_{b,\text{ask}}\right)y^{\text{ask}}\left(s\right)ds-\int_{\tau_1}^{\tau_2}\omega_{b,\text{ask}}y^{\text{ask}}\left(s\right)ds \\
& -\int_t^{\tau_1}\left(g_{\text{bid}}+\omega_{b,\text{bid}}\right)y^{\text{bid}}\left(s\right)ds-\int_{\tau_1}^{\tau_2}\omega_{b,\text{bid}}y^{\text{bid}}\left(s\right)ds\Bigg). \qquad (3)
\end{aligned}$$

$$\begin{aligned}
E_t\left[\tilde{P}\left(t,\tau_1\right)\cdot\tilde{L}^b\left(t,\tau_2\right)\right] = & \underbrace{E_t\left[\exp\left(-\int_t^{\tau_1}\left(1+f_b\right)x\left(s\right)ds\right)\exp\left(-\int_{\tau_1}^{\tau_2}f_bx\left(s\right)ds\right)\right]}_{:=P^b\left(t,\tau_1,\tau_2,x;f\right)} \\
& \underbrace{\begin{aligned}
& \cdot E_t\left[\exp\left(-\int_t^{\tau_1}\left(1+g_b\right)y^b\left(s\right)ds\right)\exp\left(-\int_{\tau_1}^{\tau_2}y^b\left(s\right)ds\right)\right] \\
& \cdot E_t\left[\exp\left(-\int_t^{\tau_1}\left(g_{\text{ask}}+\omega_{b,\text{ask}}\right)y^{\text{ask}}\left(s\right)ds\right)\exp\left(-\int_{\tau_1}^{\tau_2}\omega_{b,\text{ask}}y^{\text{ask}}\left(s\right)ds\right)\right] \\
& \cdot E_t\left[\exp\left(-\int_t^{\tau_1}\left(g_{\text{bid}}+\omega_{b,\text{bid}}\right)y^{\text{bid}}\left(s\right)\right)\exp\left(-\int_{\tau_1}^{\tau_2}\omega_{b,\text{bid}}y^{\text{bid}}\left(s\right)ds\right)\right]
\end{aligned}}_{:=L^b\left(t,\tau_1,\tau_2,y;g\right)}. \qquad (4)
\end{aligned}$$

Analytical solutions for the discount factors – IV

$$P\left(t,\tau_1,\tau_2,x;k_1,k_2\right)=a_1\left(\tau_1,\tau_2;k_1\right)\,b_1\left(t,\tau_1,\tau_2;k_1,k_2\right)\exp\left[-b_2\left(t,\tau_1,\tau_2;k_1,k_2\right)\,x\left(t\right)\right], \tag{5}$$

where ϕ, a_1, and a_2 are defined as above and

$$b_1\left(t,\tau_1,\tau_2;k_1,k_2\right)={\frac{2\phi\left(k_2\right)\exp\left[\frac{\tau_1-t}{2}\left(\phi\left(k_2\right)+\beta\right)\right]}{\sigma^2 a_2\left(\tau_1,\tau_2;k_1\right)k_1\left(\exp\left[\phi\left(k_2\right)\left(\tau_1-t\right)\right]-1\right)+\phi\left(k_2\right)-\beta+\exp\left[\phi\left(k_2\right)\left(\tau_1-t\right)\right]\left(\phi\left(k_2\right)+\beta\right)}}^{\frac{2\alpha}{\sigma^2}},$$

$$b_2\left(t,\tau_1,\tau_2;k_1,k_2\right)=\frac{a_2\left(\tau_1,\tau_2;k_1\right)k_1\left[\phi\left(k_2\right)+\beta+\exp\left[\phi\left(k_2\right)\left(\tau_1-t\right)\right]\left(\phi\left(k_2\right)-\beta\right)\right]+2k_2\left(\exp\left[\phi\left(k_2\right)\left(\tau_1-t\right)\right]-1\right)}{\sigma^2 a_2\left(\tau_1,\tau_2;k_1\right)k_1\left(\exp\left[\phi\left(k_2\right)\left(\tau_1-t\right)\right]-1\right)+\phi\left(k_2\right)-\beta+\exp\left[\phi\left(k_2\right)\left(\tau_1-t\right)\right]\left(\phi\left(k_2\right)+\beta\right)}.$$

Simultaneously, we obtain

$$L\left(t,\tau_1,\tau_2,y^l;k_1,k_2\right)=a_3^l\left(\tau_1,\tau_2;k_2\right)\,b_3^l\left(t,\tau_1,\tau_2;k_1,k_2\right)\,\exp\left[-b_4^l\left(t,\tau_1,\tau_2;k_1,k_2\right)\,y^l\left(t\right)\right], \tag{6}$$

where a_3^l and a_4^l are defined as above and

$$\begin{aligned} b_3^l\left(t,\tau_1,\tau_2;k_1,k_2\right) &= \exp\left[\frac{{\eta^l}^2 k_1^2}{6}\left(\tau_1-t\right)^3+\frac{{\eta^l}^2 k_2 a_4\left(\tau_1,\tau_2;k_2\right)-\mu^l}{2}k_1\left(\tau_1-t\right)^2\right. \\ &\quad \left.+\left(\frac{{\eta^l}^2 k_2 a_4\left(\tau_1,\tau_2;k_2\right)}{2}-\mu^l\right)a_4\left(\tau_1,\tau_2;k_2\right)k_2\left(\tau_1-t\right)\right], \\ b_4^l\left(t,\tau_1,\tau_2;k_1,k_2\right) &= a_4^l\left(\tau_1,\tau_2;k_2\right)k_2+k_1\left(\tau_1-t\right). \end{aligned}$$