Asymptotics of utilities from terminal wealth under partial information

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Problem

$W_T(u)$ wealth at time T, u investment strategy

U - utility function

Find maximal λ such that

$$\sup_u E\left[U(W_T(u))\right] \sim U(e^{\lambda T})$$

In other words:

$$\lambda = \lim_{T \to \infty} \frac{1}{T} \ln \left[U^{-1}\left(E\left[U(W_T(u))\right]\right) \right]$$

We shall assume that on a given market we have risky asset(s) $(S(t))$ the dynamics of which depend on unobserved economic factors

Particular cases:

1. $U(W) = \ln W$, then $\lambda = \lim_{T\to\infty} \frac{1}{T} \left(E \left[\ln(W_T(u)) \right] \right)$ - growth optimal portfolio: Kelly capital growth investment criterion (Maclean, Thor, Ziemba 2011)

2. $U(W) = W^{\alpha}$, $\alpha \in (0,1)$, then

$$\lambda = \lim_{T\to\infty} \frac{1}{\alpha T} \ln \left[\left(E \left[(W_T(u))^{\alpha} \right] \right) \right] = \lim_{T\to\infty} \frac{1}{\alpha T} \ln \left[\left(E \left[exp \left(\alpha \ln(W_T(u)) \right) \right] \right) \right] \quad (1)$$

3. $U(W) = 1 - W^{-\alpha}$, $\alpha > 0$, then

$$\lambda = \lim_{T\to\infty} \frac{-1}{\alpha T} \ln \left[\left(E \left[(W_T(u))^{-\alpha} \right] \right) \right] = \lim_{T\to\infty} \frac{1}{-\alpha T} \ln \left[\left(E \left[exp \left(-\alpha \ln(W_T(u)) \right) \right] \right) \right] \quad (2)$$

risk seeking and risk sensitive controls of terminal growth: dynamic limit growth indices (Bielecki, Cialenco, Pitera 2013)

Discrete time model

L. Stettner, Appl. Math. 2012

$S(n)$, economic factors (x_n) on a Polish space E with transition operator P, which is Feller

$X^n = \sigma\{x_i, i \leq n\}$ and $Y^n = \sigma\{S(i), i \leq n\}$

$P\left\{x_{n+1} \in A | X^n, Y^n\right\} = P(x_n, A)$, P a.e. for $A \in \mathcal{B}(E)$.

$r_n := \frac{S(n)}{S(n-1)}$; $Y^n = \sigma\{r_i, i \leq n\}$

$P\left\{r_{n+1} \in B | X^{n+1}, Y^n\right\} = \int_B q(x_{n+1}, r_n, y)\nu(dy)$, P a.e. for $B \in \mathcal{B}((0,\infty))$, and where $\nu \in \mathcal{P}((0,\infty))$

wealth W_0 the wealth W_n at time n is given by the formula for $n \geq 1$;

$$W_n = W_{n-1}\left(1 - b_{n-1} + b_{n-1} r_n\right).$$

Filtering process

The pair (r_n, x_n) forms a Markov process. Given initial law μ of the Markov process (x_n) define the following sequence of random measures $\pi_0(A) = \mu(A)$,

$\pi_{n+1}(A) = \frac{\int_A q(z,r_n,r_{n+1})P(\pi_n,dz)}{\int_E q(z,r_n,r_{n+1})P(\pi_n,dz)} =: \frac{N(r_n,r_{n+1},\pi_n)(A)}{N(r_n,r_{n+1},\pi_n)(E)}$ for $A \in \mathcal{B}(E)$ with notation $P(\pi_n, A) = \int_E P(x, A)\pi_n(dx)$, defining moreover implicitly random measures $N(r_n, r_{n+1}, \pi_n)(\cdot) \in M(E)$, where $M(E)$ is the set of finite measures on E.

We have $\pi_n(A) = P\{x_n \in A|Y^n\}$ for $A \in \mathcal{B}(E)$, P a.e., and the pair (r_n, π_n) forms a Markov process with transition operator Π.

Let $\Lambda_n(\omega) = \prod_{i=0}^{n-1} q(x_{i+1}(\omega), r_i(\omega), r_{i+1}(\omega))$ and let P^0 be a probability measure such that for the restrictions $P^0_{|n}$ and $P_{|n}$ of P^0 and P we have $P_{|n}(d\omega) = \Lambda_n(\omega)P^0_{|n}(d\omega)$.

Lemma 1. Under P^0, (r_n) is i.i.d. with law ν independent of (x_n) and (x_n) is Markov with transition operator $P(x, dx')$.

Lemma 2. Let for $A \in \mathcal{B}(E)$

$$N_n(r, r_1, \ldots, r_n, \eta)(A) = N(r_{n-1}, r_n, N_{n-1}(r, r_1, \ldots, r_{n-1}, \eta))(A)$$

The transition operator Π of the pair (r_n, π_n) and its iterations are respectively of the form $\Pi F(r, \mu) = E_{r\mu}\{F(r_1, \pi_1)\} = E^0\{SF(r_1, N(r, r_1, \mu))\}$ and $\Pi^n F(r, \mu) = E_{r\mu}\{F(r_n, \pi_n)\} = E^0\{SF(r_n, N_n(r, r_1, \ldots, r_n, \mu))\}$ for a bounded Borel function $F : (0, \infty) \times \mathcal{P}(E) \mapsto R$, with $SF(r, \zeta) := \zeta(E)F(r, \frac{\zeta}{\zeta(E)})$ for $\zeta \in M(E)$.

(A) there is a probability measure $\phi \in \mathcal{P}(E \times (0, \infty))$ such that

$$P\{(x_n, r_n) \in \cdot\} \to \phi(\cdot), \text{ in variation norm as } n \to \infty.$$

Theorem 1 (van Handel 2010) Under (A) there is a unique invariant measure Φ for the pair (r_n, π_n) and $\Pi^n(F)$ converges to $\Phi(F)$, as $n \to \infty$ for each F continuous in the first coordinate and in the total variation norm with respect to the second coordinate.

Asymptotics of the logarithmic utility function

Under (A), assuming that

$(I) \quad L = \sup_{x\in E}\sup_{r\in(0,\infty)}\sup_{b\in[0,1]} |\int_0^\infty \int_E \ln(1-b+by)q(x',r,y)P(x,dx')\nu(dy)| < \infty$

and

$\forall_{\epsilon>0} \quad \forall_{C-compact\ in\ E} \quad \forall_{C_1-compact\ in\ (0,\infty)} \quad \exists_{K-compact\ in\ (0,\infty)\times E}$ $\sup_{x\in C}\sup_{r\in C_1}\sup_{b\in\mathcal{S}}\int_{K^c}|\ln(1-b+by)|q(x',r,y)P(x,dx')\nu(dy) < \epsilon$ we have

$$\lim_{T\to\infty}\sup_{(b_n)}\frac{1}{T}E\{\ln(W_T)\} = \lambda,$$

where $\lambda = \int_{(0,\infty)\times\mathcal{P}(E)} F(\tilde{b}(r,\eta),r,\eta)\Phi(dr,d\eta)$ with $F(b,r,\eta) = \int_E\int_E\int_{(0,\infty)} \ln(1-b+by)q(x',r,y)\nu(dy)P(x,dx')\eta(dx)$ and $\tilde{b}(r,\eta) = argmax_{b\in\mathcal{S}}F(b,r,\eta)$. Furthermore an optimal control is of the form $b_i = \tilde{b}(r_i,\pi_i)$.

Asymptotics of negative power utility functions

$U(W) = 1 - W^{-\alpha}$, with $\alpha > 0$

Maximizing expected utility from terminal wealth in this case we minimize

$$E\left\{\prod_{i=0}^{T-1}(1-b_i+b_i r_{i+1})^{-\alpha}\right\} = E\left\{\prod_{i=0}^{T-1} e^{-\alpha G(r_{i+1},b_i)}\right\},$$

with $G(r,b) = \ln(1-b+br)$. Consequently we are looking for λ such that

$$e^{-\alpha\lambda T} \sim \inf_{(b_i)} E\left\{\prod_{i=0}^{T-1}(1-b_i+b_i r_{i+1})^{-\alpha}\right\}.$$

Discounted risk sensitive control problem

minimizing for given $\beta \in (0,1)$, $\gamma \in (0,1]$ the cost functional

$$J_{r,\mu}^{\beta\gamma}((b_n)) = E\left[\prod_{i=0}^{\infty}(1-b_i+b_i r_{i+1})^{-\alpha\beta^i\gamma}\right] = E\left\{\prod_{i=0}^{\infty} e^{-\alpha\beta^i\gamma G(r_{i+1},b_i)}\right\}.$$

Let $w^\beta(r,\eta,\gamma) = \inf_{(b_n)} J_{r,\eta}^{\beta\gamma}((b_n))$ We have

Proposition 1. Under the assumption (I), function w^β is a solution to the Bellman equation

$$w^\beta(r,\eta,\gamma) = \inf_{b\in\mathcal{S}} \int_0^\infty e^{-\alpha\gamma G(y,b)} S w^\beta(y, N(r,y,\eta), \gamma\beta)\nu(dy),$$

with the operator S defined above and w^β takes values in the interval $(0,1]$ and is uniformly bounded away from 0 (i.e. there is an $a > 0$ such that $w^\beta \geq a$). Moreover the mapping $\mathcal{P}(E) \ni \eta \mapsto w^\beta(r,\eta,\gamma)$ is concave.

Vanishing discount approach concavity properties

(B1) $\quad \sup_{r,r'\in(0,\infty)} \sup_{x\in E} \sup_{y\in(0,\infty)} \frac{q(x,r,y)}{q(x,r',y)} := \bar{q} < \infty$

(B2) $\quad \sup_{x,x'\in E} \sup_{A} \frac{P(x,A)}{P(x',A)} := \bar{p} < \infty$

We have

Proposition 2. Under (B1), (B2) and (I) we have that

$$Sw^\beta(y, N(r,y,\eta), \gamma\beta) \geq \frac{1}{\bar{p}\bar{q}} Sw^\beta(y, N(r',y,\eta'), \gamma\beta)$$

for $r, y \in (0,\infty)$ and $\eta, \eta' \in \mathcal{P}(E)$. Furthermore

$$(\bar{q}(r',r)\bar{p}(\eta',\eta))^{-1} \leq \frac{w^\beta(r,\eta,\gamma)}{w^\beta(r',\eta',\gamma)} \leq \bar{q}(r,r')\bar{p}(\eta,\eta')$$

with $\bar{p}(\eta,\eta') = \sup_A \frac{P(\eta,A)}{P(\eta',A)}$ and $\bar{q}(r,r') = \sup_{x\in E, y\in(0,\infty)} \frac{q(x,r,y)}{q(x,r',y)}$.

Vanishing discount approach cont.

Fix $\bar{\eta} \in P(E)$ and $\bar{r} \in (0, \infty)$ and define

$v^\beta(r,\eta,\gamma) := \frac{w^\beta(r,\eta,\gamma)}{w^\beta(\bar{r},\bar{\eta},\gamma)}$ and $\kappa^\beta(\gamma) := \frac{w^\beta(\bar{r},\bar{\eta},\gamma)}{w^\beta(\bar{r},\bar{\eta},\gamma\beta)}$.

Then we have

$$v^\beta(r,\eta,\gamma)\kappa^\beta(\gamma) = \inf_{b\in\mathcal{S}} \int_0^\infty e^{-\alpha\gamma G(y,b)} S v^\beta(y, N(r,y,\eta), \gamma\beta)\nu(dy).$$

Lemma 3. $e^{-\alpha\gamma L} \leq \kappa^\beta(\gamma) \leq \overline{pq}$

further assumption

(C1) $\bar{p}(\eta,\eta') \to 0$ when $\eta' \Rightarrow \eta$ in the weak topology of $\mathcal{P}(E)$ and $\bar{q}(r,r') \to 0$, when $r' \to r$.

Asymptotics of negative power utility functions - main result

Theorem 2. Under (B1), (B2), (I) and (C1) for $\gamma > 0$ there is $\lambda(\gamma)$ and a continuous bounded function $(r,\eta) \mapsto v(r,\eta,\gamma)$ such that

$$v(r,\eta,\gamma)e^{-\alpha\lambda(\gamma)} = \inf_{b\in\mathcal{S}} \int_0^\infty e^{-\alpha\gamma G(y,b)} Sv(y, N(r,y,\eta),\gamma)\nu(dy)$$

and

$$\lambda(\gamma) = \lim_{T\to\infty} \frac{-1}{\alpha\gamma T} \ln \inf_{(b_i)} E\left\{\prod_{i=0}^{T-1}(1-b_i+b_i r_{i+1})^{-\alpha\gamma}\right\},$$

so that $\lambda(1)$ is the optimal asymptotics of the negative power utility function and the optimal control is of the form $b_i = \tilde{b}(r_i,\pi_i,1)$ for $i = 0,1,2\ldots,$ where $\tilde{b}$ is a continuous selector for which infimum on the right hand side of the Bellman equation with $\gamma = 1$ is attained.

We have two methods to study asymptotics: limit of finite horizon problems or to study control problem over infinite horizon

Continuous time problem without transaction costs

P. Lakner, 1995, 1998, N. Bauerle U. Rieder 2005, H. Pham MC Quenez (2001), H. Pham 2008

$$dS_t = diag S_t(a(x_t)dt + \sigma dB_t)$$

$c(x_t) = \sigma^{-1}a(x_t)$; $\sigma'^{-1}b_t$ portion of capital invested in assets at time t

$$\frac{dP^0}{dP} = Z_T = exp\left(-\int_0^T c(x_t)'dB_t - \frac{1}{2}\int_0^T |c(x_t)|^2 dt\right)$$

under P^0, $B_t^0 = B_t + \int_0^t c(x_s)ds$;

$L_t = \frac{1}{Z_t}$; $L_t^0 = E_0\left[L_t|Y^t\right]$, where $Y^t = \sigma\{S_u, u \leq t\}$,

Asymptotics for logarithmic and power utilities

$\pi_t(A) = P\left\{x_t \in A|Y^t\right\}$ then

$L_t^0 = exp\left(\int_0^t (\pi_u(c))' dB_u^0 - \frac{1}{2}\int_0^t |\pi_u(c)|^2 du\right)$

Proposition 3. If $U(x) = \ln x$ then $v_T(x) = \ln x + \frac{1}{2}E\left\{\int_0^T |\pi_t(c)|dt\right\}$ and

$\liminf_{T\to\infty} \frac{1}{T}\left(\ln x + \frac{1}{2}E\left\{\int_0^T |\pi_t(c)|^2 dt\right\}\right) = \frac{1}{2}\int_{\mathcal{P}(E)} |\nu(c)|^2 \Phi(d\nu)$

Proposition 4. If $U(x) = x^\alpha$, and $H_0^T = E^0\left[(L_T^0)^\beta\right]$ with $\beta = \frac{1}{1-\alpha}$ we have $v_T(x) = x^\alpha (H_0^T)^{1-\alpha}$ and

$\liminf_{T\to\infty} \frac{1}{\alpha T} \ln(x^\alpha (H_0^T)^{1-\alpha}) = \liminf_{T\to\infty} \frac{1-\alpha}{\alpha T} \ln H_0^T$

Björk Davis Landen 2007

Models with transaction costs - impulse control approach

T. Duncan, B. Pasik Duncan, L.S. (2011), L.S. (2011) (complete observation case)

ζ_t^{-i}, ζ_t^i capital invested in the i-th asset before and after transaction at time t

$W_t^- = \sum_{i=1}^d \zeta_t^{-i}$, $W_t = \sum_{i=1}^d \zeta_t^i$ wealth processes

transaction costs:

$\sum_{i=1}^d \kappa_i^1(\zeta_t^i - \zeta_t^{-i})^+ + \kappa_i^2(\zeta_t^i - \zeta_t^{-i})^-$

$\kappa(v) = \sum_{i=1}^d \kappa_i^1(v)^+ + \kappa_i^2(v)^-$

$b_t = (b_t^1, \ldots, b_t^d)'$ portions of wealth invested in assets at time t, $b_t \in S$

Lemma 3. There is a unique continuous map $e : S \times S \mapsto (0,1]$ such that for b^- and $b \in S$ we have

$$\kappa(be(b^-, b) - b^-) + e(b^-, b) = 1$$

and after change of portfolio from b^- to b the wealth W^- is diminished to $W = e(b^-, b)W^-$.

$S_t^i = S_0^i e^{F_t^i}$, where F_t depends on an unobserved process x_t

for b and $d \in [0, \infty)^d \setminus \{(0, 0, \ldots, 0)\}$ let $b \cdot d = \sum_{i=1}^d b^i d^i$,

$b \diamond d = (b^1 d^1, b^2 d^2, \ldots, b^d d^d)$ and $g(d) = (\frac{d^1}{\sum d^i}, \frac{d^2}{\sum d^i}, \ldots, \frac{d^d}{\sup d^i})$

Then if the are not transactions in the time interval $[0, t)$ we have

$W_t = W_0 b_0 \cdot e^{F_t^i}$, and $b_t = g(b_0 \diamond e^{F_t^i})$

Impulse strategies

decision lag or execution delay h

impulse strategy $V = (\tau_n, \tilde{b}_n)$, where $\tilde{b}_n$ presents portions of wealth invested at time τ_n in assets

we have $W_t = W_{\tau_n} \sum_{i=1}^{d} b^i_{\tau_n} \cdot e^{F^i_t - F^i_{\tau_n}}$

$b_t = g(b_{\tau_n} \diamond e^{F^i_t - F^i_{\tau_n}})$ for $\tau_n < t < \tau_{n+1}$ and

$W_{\tau_n} = e(b^-_{\tau_n}, \tilde{b}_n) W^-_{\tau_n}$

additional assumptions: $X = (x_s)$ is a finite state ergodic continuous time Markov process on $E = \{1, 2, \ldots, R\}$ with generator $Q = (q_{ij})$;

$\pi_t(k) = P\left\{x_t = k | Y^t\right\}$

whenever $F_t^i = \int_0^t (a(x_s) - \frac{1}{2}\sigma\sigma')ds + \int_0^t \sigma dB_s$

$d\pi_t(i) = \sum_j q_{ij}\pi_t(j)dt + \pi_t(i)(\sigma^{-1}a(i) - \pi_t(\sigma^{-1}a))'dN_t$

where $N_t^i = B_t - \int_0^t (\sigma^{-1}a(x_t) - \pi_s(\sigma^{-1}a))ds$ is the innovation process.

Problems and results for logarithmic utility

$U(x) = \ln x$

Find a constant λ and a continuous bounded function $w : S \times \mathcal{P}(E) \to R$ such that

$$w(b,\mu) = \sup_{\tau} E_{b,\mu}\left[\ln(b \cdot e^{F_\tau}) - \lambda\tau + Mw(b_\tau, \pi_\tau)\right]$$

where $Mw(b,\mu) = sup_{b \in S'}\left[\ln(e(b,b')) + w(b',\mu)\right]$, and λ is an optimal growth

S' a closed subset of S which does not contains the boundary of S to which we change our portfolio; we make an obligatory transaction when b_t leaves the set S'

the optimal strategy $\hat{\tau} = \inf\{s \geq 0 : w(b_s, \pi_s) = Mw(b_s, \pi_s)\}$; and transaction b' for which sup in $Mw(b_\tau, \pi_\tau)$ is attained; $\tau_{n+1} = \tau_n + h + \tau \circ \theta_{\tau_n + h}$

Problems and results for power utility

$U(x) = x^\alpha$ - find a constant λ and a continuous bounded function $w : S \times \mathcal{P}(E) \times (0,1) \to R$ such that

$$w(b,\mu,\alpha) = \sup_\tau \ln E_{b,\mu}\left[exp\left(\alpha\left[\ln(b \cdot e^{F_\tau})\right] + M_h w(b_\tau, \pi_\tau, \alpha) - (\tau + h)\lambda(\alpha)\right)\right]$$

where $M_h w(b,\mu) = \sup_{b \in S'}\left[e(b,b') + E_{b',\mu}\left[(\ln(b' \cdot e^{F_h}) + w(b_h, \pi)h)\right]\right]$

$\frac{\lambda(\alpha)}{\alpha}$ optimal rate λ.

Further assumptions: we assume a delay h in execution of the transaction, transactions are to the set S' and we make an obligatory transaction when b_t leaves the set S'

Remarks about the proofs: continuity of the optimal stopping problem (by penalty method of time discretization), vanishing discount approach using compactness of suitably defined operators M and M_h.

Discounted problem for logarithmic utility function

$$J^{\beta}_{b,\mu}(V) = E_{b,\mu}\left[\sum_{i=1}^{\infty} e^{-\beta\tau_i}\left[\ln(\tilde{b}_n \cdot e^{F_{\tau_i}-F_{\tau_{i-1}}}) + \ln e(b^-_{\tau_i}, \tilde{b}_n))\right]\right]$$

$$v^{\beta}(b,\mu) = \sup_V J^{\beta}_{b,\mu}(V)$$

Bellman equation

$$v^{\beta}(b,\mu) = \sup_{\tau} E_{b,\mu}\left[e^{-\beta\tau}\left[\ln(b \cdot e^{F_\tau}) + Mv^{\beta}(b_\tau, \pi_\tau)\right]\right]$$

with $Mv^{\beta}(b,\mu) = \sup_{b' \in S'}\left[lne(b,b') + v^{\beta}(b',\pi)\right]$.

$w^{\beta}(b,\mu) = v^{\beta}(b,\mu) - v^{\beta}(\hat{b},\hat{\mu})$, $\lim_{\beta\to 0} w^{\beta}(b,\mu) = w(b,\mu)$ and $\lim_{\beta\to 0} \beta w^{\beta}(\hat{b},\hat{\mu}) = \lambda$ for a suitably chosen subsequence.

Discounted problem for power utility function

$$J^{\beta}_{b,\mu,\alpha}(V) = \ln E_{b,\mu}\left[exp\left[\alpha \sum_{i=1}^{\infty} e^{-\beta\tau_i}\left[\ln(\tilde{b}_n \cdot e^{F_{\tau_i} - F_{\tau_{i-1}}}) + \ln e(b^-_{\tau_i}, \tilde{b}_n)\right)\right]\right]\right]$$

$$v^{\beta}(b,\mu,\alpha) = \sup_V J^{\beta}_{b,\mu,\alpha}(V)$$

Bellman equation

$$v^{\beta}(b,\mu,\alpha) = \sup_{\tau} \ln E_{b,\mu}\left[exp\left[\alpha e^{-\beta\tau}\left[(b \cdot e^{F_\tau} + M^h v^{\beta}(b_\tau, \pi_\tau, \alpha e^{-\beta\tau})\right]\right]\right]$$

with $M^h v^{\beta}(b,\pi,\alpha) = \sup_{b' \in S'} \ln E_{b,\mu}\left[exp\left[\alpha e^{-\beta h} \ln(b \cdot e^{F_h}) + \alpha e^{-\beta h} \ln(e(b_h, b')) + v^{\beta}(b', \pi_h, \alpha e^{-\beta h}\right]\right]$

$v^{\beta}(b,\pi,\gamma) - v^{\beta}(\tilde{b},\tilde{\pi},\gamma) \to w(b,\pi,\gamma)$ for a suitably chosen subsequence

Thank you !