(Almost) Model-Free Recovery

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Motivation and Intuition

- No-arbitrage (NA) reasonable model for financial markets
- Breeden and Litzenberger (1978): relation between derivatives of option prices and forward-neutral density under NA
- No such model-free result for conditional physical probability measure
- Evidence only from
  - Parametric models (for example affine or polynomial)
  - Assumptions on classes of processes
    - Diffusion processes
    - (Stationary) Markov processes
- *Trading Strategies:* What information do signs of expected profits carry under risk aversion?

In a Nutshell

- Bound expected market returns with observed option prices
  $\Rightarrow$ *Upper and lower bounds on conditional physical moments*
- Moment bounds parameterize family of feasible physical probability measures through *truncated moment problem*
- Projection of pricing kernel on market return in $\mathcal{L}^2$ space weighted with physical probability measure developed in terms of moments
- Minimum variance such projection identifies set of moments $\Rightarrow$ *"Recovers" feasible set of physical probability measures*
- *(Almost) Model-free:* no assumption on underlying stochastic process
  $\Rightarrow$ No misspecification error
- *No time series estimation:* No parameter uncertainty

Literature

- (Semi-(Non))Parametric projections
  - Jackwerth (2000)
  - Gagliardini, Gourieroux, and Renault (2011)
- Non-parametric conditional pricing kernel projections using asympt.
  - Chapman (1997)
  - Aït-Sahalia and Lo (1998)
  - Aït-Sahalia and Duarte (2003)
- Empirical Likelihood with pricing constraints
  - Julliard and Gosh (2012)
  - Almeida and Garcia (2015)
- (Non) *Recovery* of conditional physical measure
  - Carr and Yu (2012)
  - Borovička, Hansen, and Scheinkman (2014)
  - Ross (2015)

Notation

- We assume that markets are arbitrage-free in the sense of Acciaio et al. (2013)
- Spot market means the S&P 500 index supported on compact state space $\mathcal{D}$
- Forward price of S&P 500 at time t for delivery at T is $F_{t,T}$
- European SPX call and put options with prices $C_{t,T}(K)$ and $P_{t,T}(K)$ at strikes $K > 0$
- Zero-coupon bond for delivery of one unit currency at time T is $p_{t,T}$
- Power divergence function in terms of market gross returns $R := \frac{F_{T,T}}{F_{t,T}}$

$$D_p(R) := \frac{R^p - pR + p - 1}{p^2 - p},$$

$$D_1(R) := R\log(R) - R + 1, \text{ and } D_0(R) := R - \log(R) - 1,$$

Notation II

- True and unobserved forward pricing kernel $\mathcal{M}_{\mathbb{P}} := \frac{d\mathbb{Q}_T}{d\mathbb{P}}$ satisfies for any traded $g(F_{T,T})$

$$F_{t,T}(g(F_{T,T})) = \mathbb{E}_t^{\mathbb{P}}\left[\mathcal{M}_{\mathbb{P}} \cdot g(F_{T,T})\right] = \mathbb{E}_t^{\mathbb{Q}_T}\left[g(F_{T,T})\right], \tag{1}$$

  in particular $F_{t,T} = F_{t,T}(F_{T,T}) = \mathbb{E}_t^{\mathbb{Q}_T}\left[F_{T,T}\right]$

- Expectation of kernel conditional on (time T-measurable) $R = \frac{F_{T,T}}{F_{t,T}}$

$$\mathcal{M}_{\mathbb{P}}(R) := \mathbb{E}^{\mathbb{P}}\left[\mathcal{M} \mid R\right],$$

- $\mathcal{M}_{\mathbb{P}}(R)$ is our object of interest
- Along with the conditional moments

$$\mu_{t,n}^{\mathbb{P}} := \mathbb{E}_t^{\mathbb{P}}\left[R^n\right]$$

Options and Nonlinear Payoffs

- To accommodate nonlinear payoffs we define a linearization operator $\mathcal{J}_t$ for f twice continuously differentiable

$$\mathcal{J}_t f(R) := \int_{a_t}^{1} f''(K)(K-R)^+ dK + \int_{1}^{b_t} f''(K)(R-K)^+ dK$$
$$= \begin{cases} f(a_t) + f'(a_t)(R-a_t) & R < a_t \\ f(R) & a_t \leq R \leq b_t \\ f(b_t) + f'(b_t)(R-b_t) & R > b_t. \end{cases} \tag{2}$$

- We get forward prices of nonlinear corridor payoffs as

$$\mathbb{E}_t^{\mathbb{Q}_T}[\mathcal{J}_t f(R)] = \frac{1}{p_{t,T}} \left(\int_{a_t}^{1} f''(K) P_{t,T}(K) dK + \int_{1}^{b_t} f''(K) C_{t,T}(K) dK \right) \tag{3}$$

Upper Bounds on Physical Polynomial Moments of the Market

Assumption (Negative Divergence Premium (NDP))

We define the negative n-power divergence premium NDP(p,n) assumption of orders p and n as

$$-Cov_t^{\mathbb{P}}[\mathcal{M}, D_p(R^n)] = -Cov_t^{\mathbb{P}}[\mathcal{M}(R), D_p(R^n)] \leq 0. \tag{4}$$

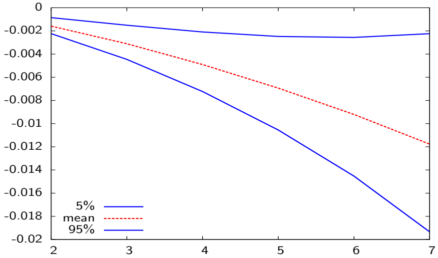
Proposition (Upper Bounds on Conditional $\mathbb{P}$ Moments)

Suppose that $NDP(p, n)$ holds, then

$$S(\mathcal{J}_t D_p(R^n)) := \mathbb{E}_t^{\mathbb{Q}_T}[\mathcal{J}_t D_p(R^n)] \geq \mathcal{J}_t D_p(\mathbb{E}_t^{\mathbb{P}}[R^n]) \tag{5}$$

Does the NDP Hold?

- It holds in any model that exhibits first-order risk (uncertainty) aversion
- There is a trading strategy swapping implied for realized divergence
- Below picture shows average trading profits (for $p = 1/2$ in picture below)



Lower Bounds on Physical Polynomial Moments of the Market

Assumption (Negative Covariance Condition (NCC))

For $p, q \in \mathbb{R}$ we define the negative covariance condition $NCC(p, q)$ as the inequality

$$-Cov_t^{\mathbb{P}}\left[\mathcal{M}R^q, R^p\right] = \mathbb{E}_t^{\mathbb{Q}_T}\left[R^q\right]\mathbb{E}_t^{\mathbb{P}}\left[R^p\right] - \mathbb{E}_t^{\mathbb{Q}_T}\left[R^{p+q}\right] \geq 0. \tag{6}$$

Proposition (Lower Bounds on Conditional $\mathbb{P}$ Moments)

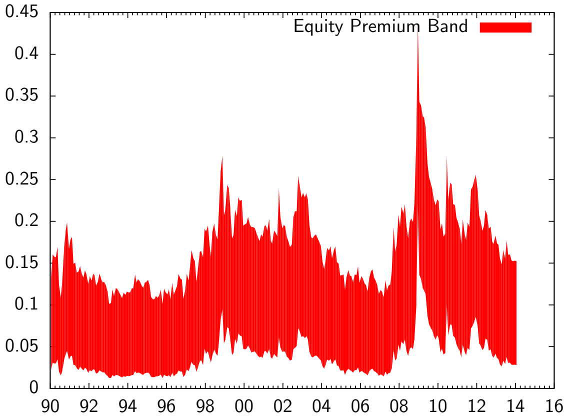
Suppose $NCC(1, q)$ holds for $q \in (0, 1]$, then for $p \geq 1$

$$L(1, q)^p := \left(\frac{\mathbb{E}_t^{\mathbb{Q}_T}\left[\mathcal{J}_t R^{1+q}\right]}{\mathbb{E}_t^{\mathbb{Q}_T}\left[\mathcal{J}_t R^q\right]}\right)^p \leq \mathbb{E}_t^{\mathbb{P}}\left[R^p\right] \tag{7}$$

Does the NCC Hold?

- Like the NDP, the NCC holds in benchmark economies with risk-averse agents (expected utility $\gamma \geq 1$)
- The smaller q in $L(1, q)$ the more conservative is the bound
- Nests the Martin (2015) bound as most *un-conservative case*
- Martin (2015) performs a battery of empirical tests on the validity of the $NCC(1, 1)$ and finds that *it holds*
- If the $NCC(1, 1)$ holds then
  - $NCC(1, q)$, $0 < q < 1$ holds $\Rightarrow L(1, q) < L(1, 1)$
  - $L(1, q)^p \leq \mathbb{E}_t^{\mathbb{P}}[R^p]$ any $p > 1$
- $NCC(1, 0)$ means the equity premium is positive!
- Next slide shows the equity premium band implied from $NCC(1, 1)$ and $NDP(2, 1)$

Locking in the Annual Equity Premium



Moments

- There are upper and lower bounds for conditional moments on R

$$\mu_{t,i}^{\mathbb{P}\text{lower}} \leq \mu_i^{\mathbb{M}} \leq \mu_{t,i}^{\mathbb{P}\text{upper}},$$

- Which moments within the bounds?
- Must maintain
  - Cauchy-Schwartz inequality
  - Moment monotonicity
  - ...
- Moments $\mu_i^{\mathbb{M}}$ must be supported from *distribution*
- For compact state space $\mathcal{D}$: Hausdorff Truncated Moment Problem
- Denote set of distributions supporting $\mu_i^{\mathbb{M}}$ by $\mathbb{H}$
- $\mathbb{H}$ is very big. Which moments to take?

Conditional Expectation in Weighted Hilbert Space

Assumption (Finite Pricing Kernel Variance)

For given probability measure $\mathbb{M}$

$$\mathbb{E}^{\mathbb{M}}\left[\mathcal{M}^2_{\mathbb{M}}(R)\right] < \infty$$

- Then $\mathcal{M}^2_{\mathbb{M}}(R) \in \mathcal{L}^2_{\mathbb{M}}$ with inner product

$$\langle f, g\rangle_{\mathcal{L}^2_{\mathbb{M}}} := \int_{\mathcal{D}} f(R)g(R)d\mathbb{M}(R), \text{ for } f, g \in \mathcal{L}^2_{\mathbb{M}} \tag{8}$$

- Denote by $\{H_0, H_1, \ldots\}$ orthonormal polynomial basis of $\mathcal{L}^2_{\mathbb{M}}$

Polynomial Expansions and Conditional Expectations

- We can write

$$\mathcal{M}_{\mathbb{M}}(R) = \mathbb{E}^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}} \mid R\right] \overset{\mathcal{L}^2_{\mathbb{M}}}{=} 1 + \sum_{i=1}^{\infty} c_i H_i(R),$$

  with

$$c_i := \langle \mathcal{M}_{\mathbb{M}}(R), H_i(R) \rangle_{\mathcal{L}^2_{\mathbb{M}}} \tag{9}$$

- Coefficients c and polynomials H nonlinear functions of $\mu^{\mathbb{Q}_T}_{t,n}$ and $\mu^{\mathbb{M}}_{t,n}$
- Functional form of $c_i H_i$ known in *closed-form*
- Given a *generic* measure $\mathbb{M}$

Truncated Projection

- Truncated projection

$$\mathcal{M}_{\mathbb{M}}^{(J)}(R) := 1 + \sum_{i=1}^{J} c_i H_i(R)$$

- $\mathcal{M}_{\mathbb{M}}^{(\infty)}(R) = \mathbb{E}^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}} \mid R\right]$
- For any $J \geq 0$

$$\mathbb{E}_t^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}}^{(J)}(R)\right] = 1$$

- For any $n \leq J$

$$\mathbb{E}_t^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}}^{(J)}(R) \cdot R^n\right] = \mu_{t,n}^{\mathbb{Q}_T}$$

- If $\mathcal{M}_{\mathbb{M}}^{(J)}(R) > 0$ $\mathbb{M}$-almost surely, it is a valid pricing kernel

Example: Linear Projection

- Consider a linear projection (truncating the sum at order 1)

$$\mathcal{M}_{\mathbb{M}}^{(1)}(R) = \frac{\mu_{t,1}^{\mathbb{M}}\mu_{t,1}^{\mathbb{Q}_T} - \mu_{t,2}^{\mathbb{M}}}{(\mu_{t,1}^{\mathbb{M}})^2 - \mu_{t,2}^{\mathbb{M}}} + \frac{\mu_{t,1}^{\mathbb{M}} - \mu_{t,1}^{\mathbb{Q}_T}}{(\mu_{t,1}^{\mathbb{M}})^2 - \mu_{t,2}^{\mathbb{M}}} R$$

- Its second moment under $\mathbb{M}$ is

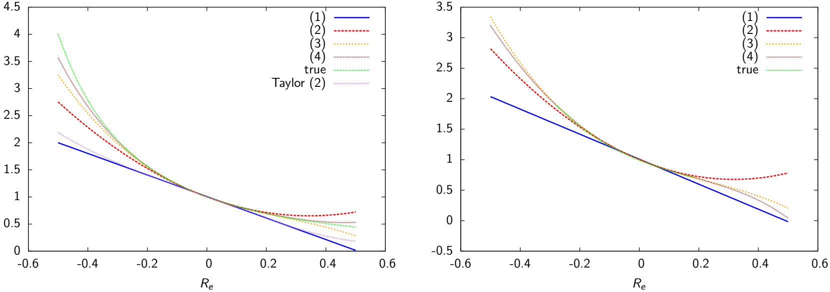
$$\mathbb{E}_t^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}}^{(1)}(R)^2\right] = -\frac{-2\mu_{t,1}^{\mathbb{M}}\mu_{t,1}^{\mathbb{Q}_T} + \mu_{t,2}^{\mathbb{M}} + (\mu_{t,1}^{\mathbb{Q}_T})^2}{(\mu_{t,1}^{\mathbb{M}})^2 - \mu_{t,2}^{\mathbb{M}}}.$$

- Remember that $\mu_{t,1}^{\mathbb{M}} = 1$, hence

$$\mathbb{E}_t^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}}^{(1)}(R)^2\right] = \frac{\mu_{t,2}^{\mathbb{M}} - 2\mu_{t,1}^{\mathbb{M}} + 1}{\mu_{t,2}^{\mathbb{M}} - (\mu_{t,1}^{\mathbb{M}})^2} > 1$$

Black-Scholes and Heston Kernels

- $R_e := R - 1$
- Projection in Black-Scholes model is closed-form
- True projection in Heston model is simulation-based



(a) Black-Scholes Model (b) Heston Model

Identification of Moments

- Define the set

$$\overline{\mathbb{M}}_t(J) := \left\{ \mathbb{M} \mid i = 1, \ldots, 2J; \mu_{t,i}^{\mathbb{P}\text{lower}} \leq \mu_i^{\mathbb{M}} \leq \mu_{t,i}^{\mathbb{P}\text{upper}}, \mathbb{M} \in \mathbb{H} \right\}.$$

- Solve

$$\min_{\mu_1^{\mathbb{M}}, \ldots, \mu_{2J}^{\mathbb{M}}} \mathbb{E}_t^{\mathbb{M}} \left[\mathcal{M}_{\mathbb{M}}^{(J)}(R)^2 \right] \text{ subject to } i = 1, \ldots, J$$

$$-Cov_t^{\mathbb{M}} \left[\mathcal{M}_{\mathbb{M}}^{(J)}(R), D_{1/2}(R^{2^i}) \right] \leq 0, \text{ and } \mathbb{M} \in \overline{\mathbb{M}}_t(J)$$

- *Intuition:* Making the conditional variance as small as possible makes the Hansen-Jagannathan bound as tight as possible
- Condition on return-variance of trading strategies in the economy

Properties of Second Moments of Projections

- For any candidate measure $\mathbb{M}$

$$\mathbb{E}_t^{\mathbb{M}}\left[\mathcal{M}_{\mathbb{M}}^{(J)}(R)^2\right] > 1$$

- The second moment of the $J-$order projection with the optimal moments

$$\mathbb{E}^{\mathbb{M}^\star}\left[\mathcal{M}_{\mathbb{M}^\star}^{(J)}(R)^2\right]$$

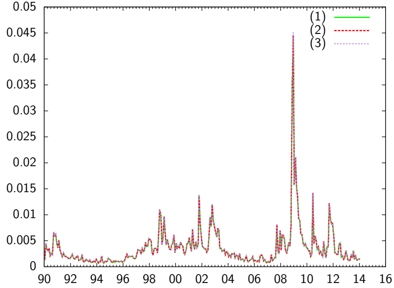
  gives the *option-implied* conditional Hansen-Jagannathan bound
- For any higher-order projection we have a generalization of the Hansen-Jagannathan result
- We can *trade* on the bound with semi-static strategies

Data and Empirical Strategy

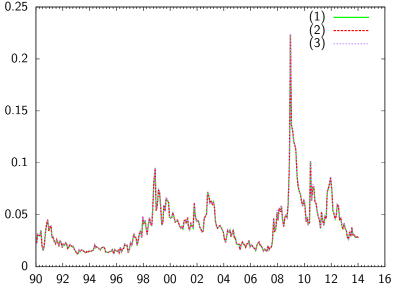
- European options on SPX
- Data set from January 1990 to January 2014.
- Options on the SPX subject to standard no-arbitrage filters
  - Negative bid ask spreads
  - Implied vol < 0.001 or greater than 9
  - Convexity
  - $\dots$
- Forwards implied from options
- Solve Optimization Program every third Friday of the month in gross returns R
- Produce results in $R_e := R - 1$
- Sometimes program has no solution

First Moment Risk Premia $\mathbb{E}_t^{\mathbb{P}}[R_e] - \mathbb{E}_t^{\mathbb{Q}_T}[R_e]$

(J) refers to moment implied by projection $\mathcal{M}^{(J)}_{\mathbb{M}^\star}(R)$



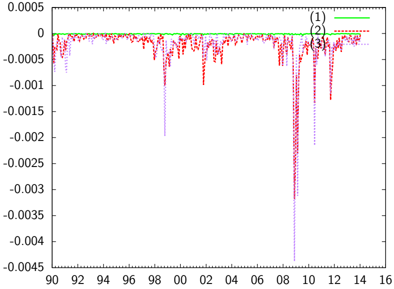
(a) 1 month



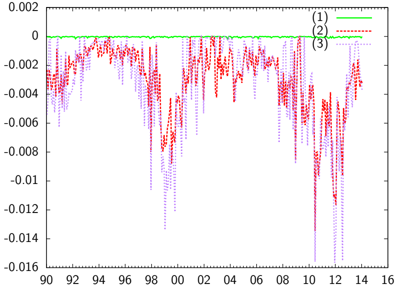
(b) 1 year

Second Moment Risk Premia $\mathbb{E}_t^{\mathbb{P}}\left[R_e^2\right] - \mathbb{E}_t^{\mathbb{Q}_T}\left[R_e^2\right]$

(J) refers to moment implied by projection $\mathcal{M}_{\mathbb{M}^\star}^{(J)}(R)$



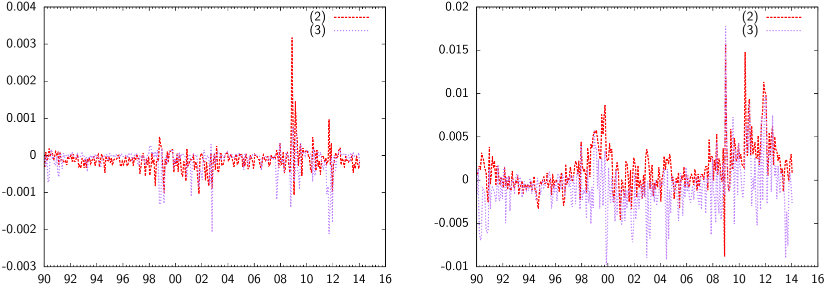
(a) 1 month



(b) 1 year

Third Moment Risk Premia $\mathbb{E}_t^{\mathbb{P}}\left[R_e^3\right] - \mathbb{E}_t^{\mathbb{Q}_T}\left[R_e^3\right]$

(J) refers to moment implied by projection $\mathcal{M}^{(J)}_{\mathbb{M}^\star}(R)$



(a) 1 month

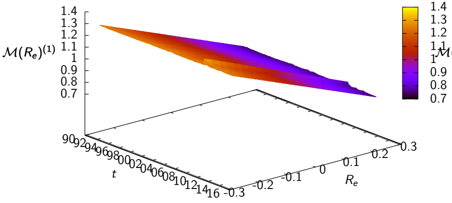
(b) 1 year

Do Conditional Moments Predict Realizations?

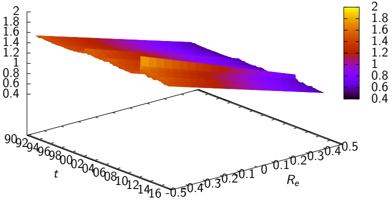
- Numbers are out-of-sample R^2
- Column headers refer to the power of the moment

1 m	1	2	3	4	5	6
J=1	0.019	-0.33				
J=2	0.019	-0.24	-0.42	-97.50		
J=3	0.019	-0.27	-0.86	-72.65	-13517.75	-3948843.88
3 m	1	2	3	4	5	6
J=1	0.013	-0.42				
J=2	0.013	-0.32	-0.12	-4.41		
J=3	0.013	-0.33	-0.24	-5.56	-246.67	-21622.46
6 m	1	2	3	4	5	6
J=1	0.029	-0.082				
J=2	0.029	-0.019	-0.081	-2.69		
J=3	0.029	-0.011	-0.17	-2.99	-141.39	-13257.74
12 m	1	2	3	4	5	6
J=1	0.0034	0.076				
J=2	0.0033	0.12	0.0067	-1.05		
J=3	0.0041	0.13	-0.02	-1.88	-105.28	-9829.60

Linear Projection

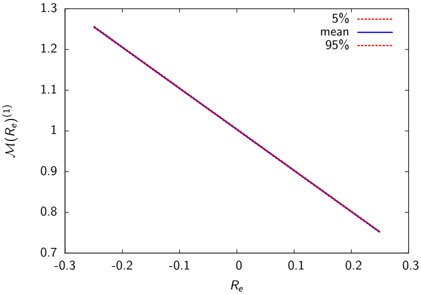


(a) 1 month

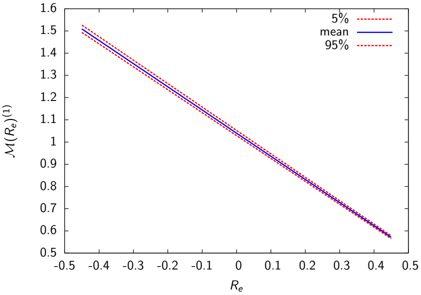


(b) 12 months

Unconditional Linear Projection

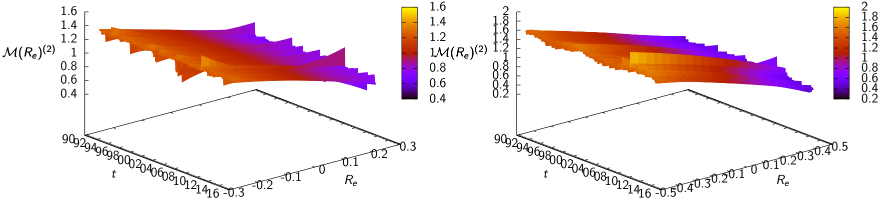


(a) 1 month



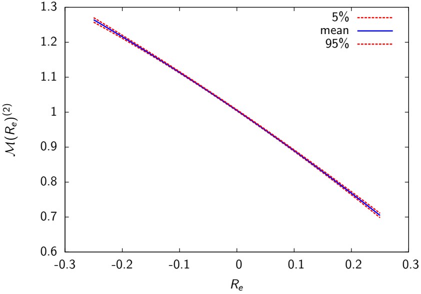
(b) 12 months

Quadratic Projection

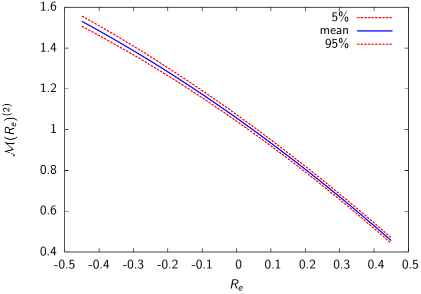


(a) 1 month (b) 12 months

Unconditional Quadratic Projection

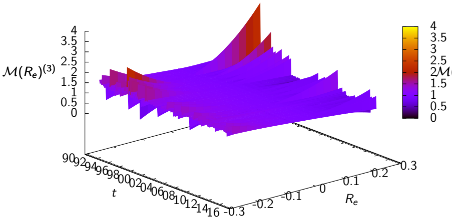


(a) 1 month

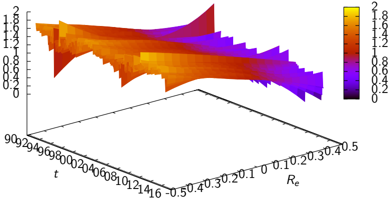


(b) 12 months

Cubic Projection

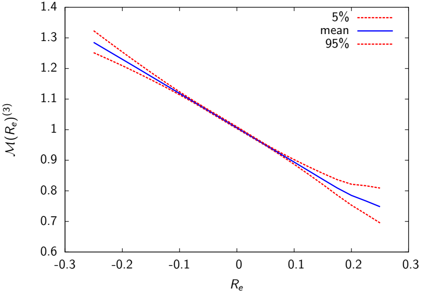


(a) 1 month

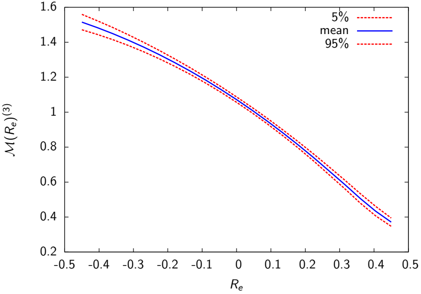


(b) 12 months

Unconditional Cubic Projection

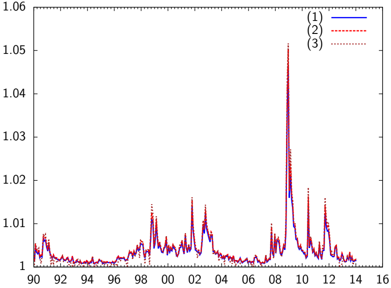


(a) 1 month

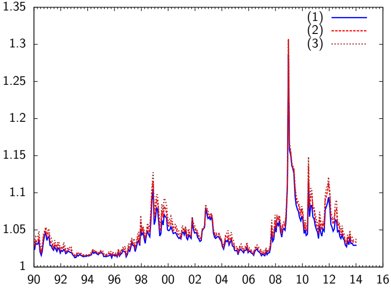


(b) 12 months

The Time-Varying Hansen-Jagannathan Bound



(a) 1 month



(b) 12 months

Trading on the Hansen-Jagannathan Bound

- Trade is on the "bad" asset pricing bound
- Risk Premium is minus the conditional variance of the kernel
- Sharpe ratios from trading order (J) projection

Order	1	2	3
1 month	-0.13	-0.12	-0.067
3 months	-0.22	-0.20	-0.064
6 months	-0.31	-0.31	-0.29
12 months	-0.38	-0.38	-0.38

Interlude: Expected Utility Theory

- Representative agent with Von-Neumann/Morgenstern utility
- Expected Utility Theory tells us

$$\mathcal{M}^{EU} = \frac{U'(W_{t+1})}{U'(W_t)} \tag{10}$$

- Expanding in W_t we have $(R_e := R - 1)$

$$\mathcal{M}^{EU} = 1 + W_t \frac{U''(W_t)}{U'(W_t)} R_e^{EU} + W_t^2 \frac{U'''(W_t)}{U'(W_t)} \left(R_e^{EU}\right)^2 + O\left(R_e^{EU}\right)^3 \tag{11}$$

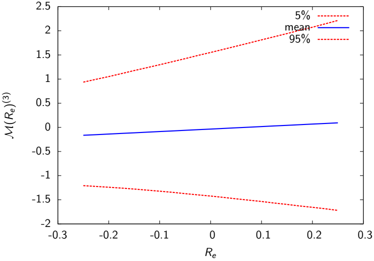
- Alternating sign of coefficients on R
  - $\frac{U''(W_t)}{U'(W_t)} < 0$: *risk aversion*
  - $\frac{U'''(W_t)}{U'(W_t)} > 0$: *prudence*
  - $\frac{U'''(W_t)}{U'(W_t)} < 0$: *temperance*

Projected Utility

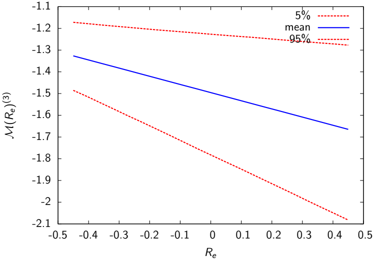
- Collect coefficients $a_{t,i}^{(J)}$ on $i - th$ power of $R_e := R - 1$

$$\mathcal{M}_{\mathbb{M}^\star}^{(J)}(R_e) = a_{t,0}^{(J)} + a_{t,1}^{(J)} R_e + a_{t,2}^{(J)} R_e^2 + \cdots + a_{t,J}^{(J)} R_e^J$$

- Figure shows second derivative of projection

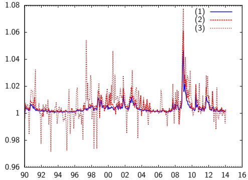


(a) 1 month

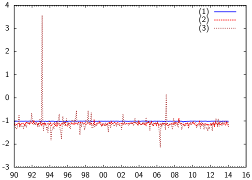


(b) 12 months

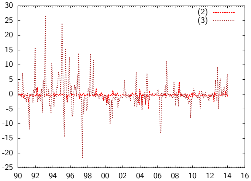
Coefficients Over Time (1 month Maturity)



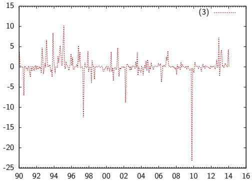
(a) a_0



(b) a_1

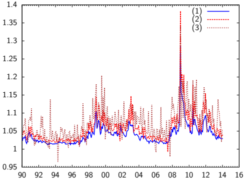


(c) a_2

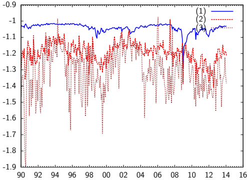


(d) a_3

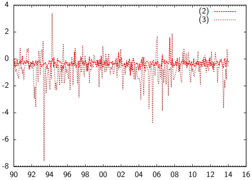
Coefficients Over Time (12 month Maturity)



(a) a_0



(b) a_1



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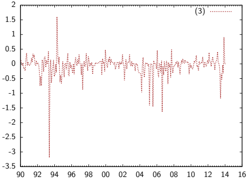

(d) a_3

Conclusion

- Asset pricing bounds on polynomial market returns under mild economic assumptions
- Model-free pricing kernel projection
- Set of conditional physical moments
- Physical moments parameterize physical probability measure
- New evidence on
  - predictability
  - Hansen-Jagannathan type asset pricing bounds
  - option trading
  - aggregate risk aversion