Term Structure Modelling beyond the Intensity Paradigm

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Thorsten Schmidt
joint work with Frank Gehmlich
Abteilung für Mathematische Stochastik

www.stochastik.uni-freiburg.de
thorsten.schmidt@stochastik.uni-freiburg.de
Lausanne, September 2015

Motivation

- Term structure models (risk-free, credit risky, electricity, foreign exchange) are typically build from a family of fundamental instruments which offer a (random) payment H_T at maturity $T \geq t$.
- Under a suitable no-arbitrage criterion (NAFL) there exists an equivalent (local) martingale measure with respect to a certain numéraire, say S^0.
- With sufficient integrability,

$$P(t,T) = E_Q\left[\frac{S_t^0}{S_T^0} H_T | \mathcal{F}_t\right].$$

- If prices ar positive and absolutely continuous, then

$$P(t, T) = e^{-\int_t^T f(t,u)du} \tag{1}$$

(Heath-Jarrow-Morton).

- It is quite typical in financial markets that fundamental decisions (ECB-interest rates, regulation in electricity markets, planned outtakes of power plants, dividend payments, ect.) occur at predictable times (not totally inaccessible).
- This may effect either the numéraire itself or the predictable projection of H_T, such that (1) may lead to arbitrage possibilities.

Credit Risk

- In credit risk (say with $S^0 \equiv 1$)

$$P(t,T) = E_Q[\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_t],$$

  with default time τ, i.e. $H_T = \mathbb{1}_{\{\tau > T\}}$.
- Inspired by the Poisson process one typically assumes that

$$H_t^p = \int_0^{t \wedge \tau} \lambda(s) ds$$

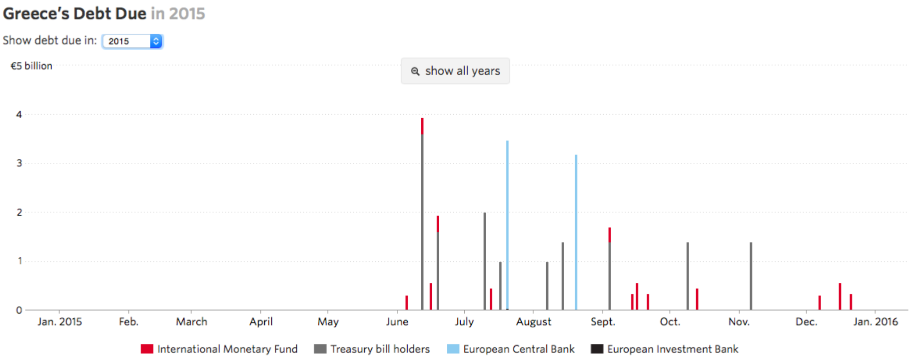
  which implies that τ is totally inaccessible and hence an absolutely continuous term structure.
- Then $P(\tau = t) = 0$ holds for all $t \geq 0$ which contradicts empirical evidence.
- Typical structural models show a totally different behaviour.

Merton (1974)

Simple liability structure: default happens at maturity T of the issued bond if the firm value is not sufficient to cover the liabilities, i.e.

$$P(\tau = T) > 0.$$

- We have a similar situation in many structural models, where default has a positive probability for occurring at a pre-specified times, such as coupon dates.
- Those times may be random, but are always predictable.



Source: The Wall Street Journal,
http://graphics.wsj.com/greece-debt-timeline/

- The default of Greece on 1st of July is a prime example of such a case.

- Bélanger et al. (2004) take this as motivation and study a first-passage time model for the default time which contains the Merton-model.

- As an example we could consider a generalized intensity-based model where

$$H_t^p = \int_0^{t \wedge \tau} \lambda(s) dA(s)$$

with an increasing, determinstic process A. If $\lambda > 0$, we have a positive default probability a times t with $\Delta A(t) \neq 0$.

Credit risk

- Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q^*)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, $Q^* \sim P$.
- The default time τ is an $\mathbb{F}$-stopping time,
- Denote the **default indicator process** H by

$$H_t = \mathbb{1}_{\{t \geq \tau\}}, \qquad t \geq 0$$

- H is a submartingale, such that by the Doob-Meyer decomposition

$$M_t = H_t - H_t^p \tag{2}$$

is a true martingale with an increasing, predictable process H^p.

Simplifying assumption

Assume that

$$H_t^p = \int_0^t h_s ds + \int_0^t \int_{\mathbb{R}} x\Gamma(ds, dx), \quad t \geq 0 \tag{3}$$

with $\Gamma(dt, dx) = \sum_{s>0} \mathbb{1}_{\{\Delta H_s^p \neq 0\}} \delta_{(s, \Delta H_s^p)}(dt, dx)$ being a predictable integer-valued random measure.

Intuition: when $\Delta H_t^p > 0$, there is a positive probability that the company defaults at time t. We call such times **possible default dates**. The set of possible default dates is a thin set which we denote by $\{U_1, U_2, \dots\}$.

We call a random time U **announced** by S if S is an $\mathbb{F}$-stopping time with $S < U$ almost surely and U is $\mathcal{F}_S$-measurable.

We make the following assumptions.

(A1) The process h is progressively measurable and locally integrable,

$$\int_0^{T^*} |h_s| ds < \infty, \quad Q^*\text{-a.s.}$$

(A2) Each possible default date U_i is announced by S_i, $1 \le i \le N$. Moreover, there are positive random variables $\Gamma_1, \Gamma_2, \dots$ such that each Γ_i is $\mathcal{F}_{S_i}$-measurable. The random measure $\Gamma(ds, dx)$ is given by

$$\Gamma([0,t], dx) = \sum_{i=1}^{N} \mathbb{1}_{\{U_i \le t\}} \delta_{\Gamma_i}(dx).$$

Note that, under **(A2)**, the random measure Γ is predictable.

- At time t, all times announced up to t should be taken into account:

$$\mu_t(du) := \sum_{S_i \leq t} \delta_{U_i}(du).$$

- We consider defaultable bond prices given by

$$\begin{aligned} P(t,T) &= \mathbb{1}_{\{\tau > t\}} \exp\left(- \int_t^T f(t,u)du - \int_t^T g(t,u)\mu_t(du) \right) \\ &= \mathbb{1}_{\{\tau > t\}} \exp\left(- \int_t^T f(t,u)du - \sum_{i:S_i \leq t} \mathbb{1}_{\{T_i \in (t,T])\}} g(t,U_i) \right) \end{aligned} \quad (4)$$

for $0 \leq t \leq T \leq T^*$

- Assume that the processes f and g satisfy

$$f(t,T) = f(0,T) + \int_0^t a(u,T)du + \int_0^t b(u,T) \cdot dW_u \tag{5}$$

$$g(t,T) = g(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \beta(u,T) \cdot dW_u \tag{6}$$

with an n-dimensional Q^*-Brownian motion W.

(B1) the initial forward curve is measurable, and integrable on $[0, T^*]$:

$$\int_0^{T^*} |f(0,u)| + |g(0,u)| du < \infty, \qquad Q^*\text{-a.s.},$$

(B2) the **drift parameters** $a(\omega, s, t)$ and $\alpha(\omega, s, t)$ are $\mathbb{R}$-valued $\mathcal{O} \otimes \mathcal{B}$-measurable and

$$\int_0^{T^*} \int_0^{T^*} |a(s,t)| ds\, dt < \infty, \quad Q^*\text{-a.s.},$$

$$\sup_{s,t \le T^*} |\alpha(s,t)| < \infty, \quad Q^*\text{-a.s.},$$

(B3) the **volatility parameter** $b(\omega, s, t)$ is $\mathbb{R}^n$-valued, $\mathcal{O} \otimes \mathcal{B}$-measurable, and

$$\sup_{s,t \le T^*} \| b(s,t) \| < \infty, \quad Q^*\text{-a.s.},$$

while $\beta(\omega, s, t)$ is $\mathbb{R}^n$-valued, $\mathcal{O} \otimes \mathcal{B}$-measurable, and square integrable on $[0, T^*]$:

$$\int_0^{T^*} \int_0^{T^*} \| \beta(s,t) \|^2 ds\, dt < \infty, \quad Q^*\text{-a.s.}$$

(B4) we assume that $\mu(dt, du) = \sum_{i=1}^n \delta_{(S_i, U_i)}(dt, du)$ has an absolutely continuous compensator $\nu_t(du)dt$ and

$$\int_0^{T^*} \int_0^{T^*} |e^{-g(t,u)} - 1| \, \nu(t, du) dt < \infty, \quad Q^*\text{-a.s.}$$

Moreover $Q^*(\tau = S_i) = 0$ for all $i \ge 1$.

$$\bar{a}(t,T) = \int_t^T a(t,u)du, \quad \bar{b}(t,T) = \int_t^T b(t,u)du,$$
$$\bar{\alpha}(t,T) = \int_t^T \alpha(t,u)\mu_t(du), \quad \bar{\beta}(t,T) = \int_t^T \beta(t,u)\mu_t(du).$$

Theorem

Assume that **(A1)**-**(A2)** *and* **(B1)**-**(B4)** *hold. Then Q^* is an ELMM if and only if the following two conditions hold:*

$$\int_0^t f(s,s)ds + \sum_{U_i \le t} g(U_i,U_i) = \int_0^t (r_s + h_s)ds - \sum_{U_i \le t} \log(1-\Gamma_i), \tag{7}$$

$$\bar{a}(t,T) + \bar{\alpha}(t,T) = \frac{1}{2} \parallel \bar{b}(t,T) + \bar{\beta}(t,T) \parallel^2 + \int_t^T \left(e^{-g(t,u)} - 1\right) \nu(t,du), \tag{8}$$

$0 \le t \le T \le T^*$, $dQ^* \otimes dt$*-almost surely on* $\{t < \tau\}$.

Example

Consider $\lambda > 0$, $0 < u_1 < \cdots < u_N$, and positive random variables $\lambda'_1, \ldots, \lambda'_N$. Set

$$\Lambda_t = \lambda t + \sum_{u_i \le t} \lambda'_i.$$

Let E be a standard exponential random variable, independent from Λ, and set

$$\tau = \inf\{t \ge 0 : \Lambda_t \ge E\}.$$

Then $\Delta H^p_{u_i} > 0$ because u_i is a possible default date:

$$Q^*(\tau = u_i) = Q^*(\lambda'_i \ge E) = \mathbb{E}^*[1 - \exp(-\lambda'_i)].$$

If Λ is deterministic and $r = 0$, we obtain

$$P(t,T) = Q^*(\tau > T | \tau > t) = \mathbb{1}_{\{\tau > t\}} \exp\Big(-\lambda(T-t) - \sum_{u_i \in (t,T]} \lambda'_i \Big)$$

Note that

$$H^p_t = \lambda(t \wedge \tau) + \sum_{i: u_i \le (t \wedge \tau)} (1 - e^{-\lambda'_i})$$

The setup simplifies if the defaultable dates are deterministic:

- Consider a set $\mathcal{U} = \{u_1, u_2, \dots\} \subset \mathbb{R}_{>0}$ such that any time outside $\mathcal{U}$ is totally inaccessible for the default time τ, i.e. $P(\tau = t) = 0$ for all $t \notin \mathcal{U}$.
- defaultable bond prices given by (!)

$$P^M(t,T) = \mathbb{1}_{\{\tau > t\}} \exp\left(-\int_t^T f(t,u)\nu(du) \right), \quad 0 \le t \le T \le T^*; \quad (9)$$

with

$$\nu(du) = du + \sum_{i \ge 1} \delta_{u_i}(du).$$

(A2′) Assume $P(\tau = t) = 0$ for all $t \notin \mathcal{U}$ and that there are random variables $0 \le \Gamma_1, \Gamma_2, \dots$ such that Γ_i is $\mathcal{F}_{u_i-}$-measurable. The predictable random measure $\Gamma = \sum_{u_i \in \mathcal{U}} \delta_{(u_i, \Gamma_i)}$ is finite, that is $\Gamma([0,T^*], \mathbb{R}) < \infty$, Q^*-a.s.

A model satisfying **(A2′)** will be called **generalized Merton model**.

Set

$$\bar{a}(t,T) = \int_t^T a(t,u)\nu(du), \quad \bar{b}(t,T) = \int_t^T b(t,u)\nu(du). \tag{10}$$

Theorem

Assume that **(A1)**,**(A2′)** *and* **(B1′)**-**(B3′)** *hold. Then* Q^* *is an ELMM if and only if the following two conditions hold:*

$$\int_0^t f(s,s)\nu(ds) = \int_0^t r_s ds + \int_0^t h(s)ds - \sum_{i:u_i \le t} \log(1-\Gamma_i), \tag{11}$$

$$\bar{a}(t,T) = \frac{1}{2} \parallel \bar{b}(t,T) \parallel^2, \tag{12}$$

for $0 \le t \le T \le T^*$ $dQ^* \otimes dt$*-almost surely on* $\{t < \tau\}$.

It turns out that a generalized Merton model has the default compensator

$$H_t^p = \int_0^{t\wedge\tau} h(s)ds + \sum_{u_i \leq (t\wedge\tau)} \Gamma_i = \int_0^{t\wedge\tau} h'(s)dA(s),$$

thus - it can be viewed also as a generalized intensity-based model.

- We assume $\mathcal{U} = \{u_1, \dots, u_n\}$ and $r_t = 0$.
- The idea is to consider an affine process X and

$$H_t^p = \int_0^t \Big(\alpha_0(s) + \langle \beta_0(s), X_s \rangle\Big) ds + \sum_{i=1}^n \mathbb{1}_{\{t \geq u_i\}} \Big(\alpha_i + \langle \beta_i, X_{u_i} \rangle\Big). \quad (13)$$

- Consider a state space in canonical form $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ for integers $m, n \geq 0$ with $m + n = d$ and a d-dimensional Brownian motion W. Let μ and σ be defined on D by

$$\mu(x) = \mu_0 + \sum_{i=1}^{d} x_i \mu_i, \tag{14}$$

$$\frac{1}{2}\sigma(x)^\top \sigma(x) = \sigma_0 + \sum_{i=1}^{d} x_i \sigma_i, \tag{15}$$

where $\mu_0, \mu_i \in \mathbb{R}^d$, $\sigma_0, \sigma_i \in \mathbb{R}^{d \times d}$, for all $i \in \{1, \dots, d\}$.

- Then the unique strong solution of

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \tag{16}$$

is an **affine** process X on the state space D.

Definition

We call a bond-price model **affine** if there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ such that

$$P^M(t, T) = \mathbb{1}_{\{\tau > t\}} e^{-\phi(t,T) - \langle \psi(t,T), X_t \rangle}, \quad 0 \leq t \leq T. \tag{17}$$

- We consider a generalized Merton model with $\mathcal{U} = \{u_1, \ldots, u_n\}$.
- For the càd-functions ϕ, ψ we denote by ∂_t the partial derivative from the right-hand side.

Proposition

Assume that $\alpha_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\beta_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ are measurable and bounded functions such that $\alpha_0(s) + \langle \beta_0(s), x \rangle \geq 0$ for $s \geq 0$, $x \in D$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $\beta_1, \dots, \beta_n \in \mathbb{R}^d$ satisfy $\alpha_i + \langle \beta_i; x \rangle \geq 0$ for all $1 \leq i \leq n$ and $x \in X$. Moreover,

$$
\begin{aligned}
\phi(T,T) &= 0 \\
-\partial_t \phi(t,T) &= \alpha_0(t) + \langle \mu_0, \psi(t,T) \rangle - \langle \psi(t,T), \sigma_0 \psi(t,T) \rangle, \quad t \neq u_i \\
\Delta \phi(u_i, T) &= \alpha_i
\end{aligned}
$$

$$
\begin{aligned}
\psi(T,T) &= 0 \\
-\partial_t \psi_k(t,T) &= \beta_{0,k}(t) + \langle \mu_k, \psi(t,T) \rangle - \langle \psi(t,T), \sigma_k \psi(t,T) \rangle, \quad t \neq u_i \\
\Delta \psi_k(u_i, T) &= \beta_{k,i}.
\end{aligned}
$$

Then, the affine model given by (13) and (17) satisfies NAFL.

Example

In the one-dimensional case we consider X given as solution of

$$dX_t = (\mu_0 + \mu_1 X_t)dt + \sigma\sqrt{X_t}dW_t, \qquad t \geq 0.$$

We assume that $\mathcal{U} = \{1\}$ and choose $\nu(du) = \delta_{\{1\}}(du)$. Moreover, let $\alpha_0 = 0$, $\beta_0 = 1$ as well as $\alpha_1 = 0$ and $\beta_1 \geq 0$, such that

$$H_t^p = \int_0^t X_s ds + \mathbb{1}_{\{t \geq 1\}} \beta_1 X_1.$$

Hence the probability of default at 1 given the information until time $1-$ is $\beta_1 X_1$.

An arbitrage-free model can be obtained as follows:

$$L_1(t) = 2(e^{\theta t} - 1), \quad L_2(t) = \theta(e^{\theta t} + 1) + \mu_1(e^{\theta t} - 1),$$
$$L_3(t) = \theta(e^{\theta t} + 1) - \mu_1(e^{\theta t} - 1), \quad L_4(t) = \sigma^2(e^{\theta t} - 1).$$

Let

$$A_0(s) = \frac{2\mu_0}{\sigma^2} \log\left(\frac{2\theta e^{\frac{(\sigma-\mu_1)t}{2}}}{L_3(t)}\right), \quad B_0(s) = -\frac{L_1(t)}{L_3(t)}$$

such that with $A(t,T) = A_0(T-t)$ and $B(t,T) = B_0(T-t)$ for $0 \le t \le T < 1$, the conditions of Proposition 5 hold. Similarly, for $1 \le t \le T$ and $T \ge 1$, choosing $A(t,T) = A_0(T-t)$ and $B(t,T) = B_0(T-t)$ implies again the validity of the Riccati equations.

On the other hand, for $0 \le t < 1$ and $T \ge 1$ we set

$$u(T) = B(1-, T) = B(1, T) - \psi_1 = B_0(T - t) - \psi_1,$$

and let

$$A(t, T) = \frac{2\mu_0}{\sigma^2} \log \left(\frac{2\theta e^{\frac{(\sigma-\mu_1)(1-t)}{2}}}{L_3(1-t) - L_4(1-t)u(T)} \right)$$

$$B(t, T) = -\frac{L_1(1-t) - L_2(1-t)u(T)}{L_3(1-t) - L_4(1-t)u(T)}.$$

In this case $\Delta A(1, T) = 0$ and $\Delta B(1, T) = \psi_1$. ◇

Jointly with Martin Keller-Ressel we study the following class of affine processes (talk on Wednesday):

Definition

A semimartingale X is called **affine** if it is adapted and there exist $\mathbb{C}$ and $\mathbb{C}^d$-valued càdlàg functions $\phi(s,t,u)$ and $\psi(s,t,u)$, respectively, such that

$$E\left[e^{\langle u, X_t\rangle} | \mathcal{F}_s\right] = \exp\left(\phi(s,t,u) + \langle \psi(s,t,u), X_s\rangle\right) \tag{18}$$

holds for all $u \in i\mathbb{R}^d$ and $0 \le s \le t$.

This definition does not require **stochastic continuity** for an affine process which is an assumption in the treatments Filipović (2005), Keller-Ressel et al. (2011).

Summary

- Considering semimartingales which are not stochastically continuous requires a modification of the HJM-approach to term structure models.
- Drift conditions can be obtained leading to arbitrage-free models.
- Affine models which are not stochastically continuous have still a high degree of tractability.

Bélanger, A., Shreve, S. E. and Wong, D. (2004), 'A general framework for pricing credit risk', *Mathematical Finance* **14**(3), 317–350.

Filipović, D. (2005), 'Time-inhomogeneous affine processes', *Stochastic Processes and their Applications* **115**(4), 639 – 659.

Keller-Ressel, M., Schachermayer, W. and Teichmann, J. (2011), 'Affine processes are regular', *Probability Theory and Related Fields* **151**(3-4), 591–611.