Robust strategies, pathwise Itō calculus, and generalized Takagi functions

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In mainstream finance, the price evolution of a risky asset is usually modeled as a stochastic process defined on some probability space.

Problem

– only one single trajectory of the asset price process is observable
– there are no repeated "experiments"
– the price evolution typically lacks stationarity

It follows that the law of the stochastic process cannot be measured accurately by means of statistical observation. We are facing model ambiguity.

Practically important consequence: model risk

Occam's razor: do without a probability space

1. Continuous-time finance without probability

Let X_t, $0 \le t \le T$, be the price evolution of a risky asset. We assume for simplicity that X is a continuous function and that there is a riskless asset with prices $B_t = 1$.

Trading strategy (ξ, η):

- ξ_t shares of the risky asset
- η_t shares of the riskless asset

at time t.

Portfolio value at time t:

$$V_t = \xi_t X_t + \eta_t$$

Key notion for continuous-time finance: self-financing strategy

If trading is only possible at times $0 = t_0 < t_1 < \cdots < t_N = T$, a strategy (ξ, η) is self-financing if and only if

$$(1) \qquad V_{t_i} = V_0 + \sum_{k=1}^{i} \xi_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}), \qquad i = 1, \dots, N$$

Now let $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions (i.e., $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \cdots$ and $\mathrm{mesh}(\mathbb{T}_n) \to 0$). Then (ξ, η) can be called self-financing if we may pass to the limit in (1). That is,

$$V_t = V_0 + \int_0^t \xi_s \, dX_s, \qquad 0 \le t \le T,$$

where the integral should be understood as the limit of the corresponding Riemann sums:

$$\int_0^t \xi_s \, dX_s = \lim_{n\uparrow\infty} \sum_{s\in\mathbb{T}_n,\, s\le t} \xi_s (X_{s'} - X_s)$$

(Here, s' denotes the successor of s in $\mathbb{T}_n$).

A special strategy

Here we give a version of an argument from Föllmer (2001)

Proposition 1. *For* $K \in \mathbb{R}$ *let*

$$\xi_t = 2(X_t - K) \qquad 0 \le t \le T.$$

Then $\int_0^t \xi_t \, dX_t$ *exists for all* t *as the limit of Riemann sums if and only if the* quadratic variation of X,

$$\langle X \rangle_t := \lim_{N \uparrow \infty} \sum_{s \in \mathbb{T}_N, \, s \le t} (X_{s'} - X_s)^2,$$

exists for all t. *In this case*

$$\int_0^t \xi_s \, dX_s = (X_t - K)^2 - (X_0 - K)^2 - \langle X \rangle_t$$

For $K = X_0$

Proposition 1. *Let*

$$\xi_t = 2(X_t - X_0), \qquad 0 \le t \le T.$$

Then $\int_0^t \xi_t \, dX_t$ *exists for all* t *as the limit of Riemann sums if and only if the* quadratic variation of X,

$$\langle X \rangle_t := \lim_{N \uparrow \infty} \sum_{s \in \mathbb{T}_N, \, s \le t} (X_{s'} - X_s)^2,$$

exists for all t*. In this case*

$$\int_0^t \xi_s \, dX_s = (X_t - X_0)^2 - \langle X \rangle_t$$

We always have $\langle X \rangle_t = 0$ if X is of bounded variation or Hölder continuous for some exponent $\alpha > 1/2$ (e.g., fractional Brownian motion with $H > 1/2$)

Otherwise, the quadratic variation $\langle X \rangle$ depends strongly on the choice of $(\mathbb{T}_n)$.

Indeed, for instance it is well known that for any continuous function X there exists a refining sequence of partitions along which $\langle X \rangle_t = 0$ (e.g., Freedman (1983))

If $\langle X \rangle_t$ exists and is continuous in t, Itō's formula holds in the following strictly pathwise sense (Föllmer 1981):

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X \rangle_s$$

where

$$\int_0^t f'(X_s)\, dX_s = \lim_{n\uparrow\infty} \sum_{s\in\mathbb{T}_n,\, s\leq t} f'(X_s)(X_{s'} - X_s)$$

is sometimes called the Föllmer integral and $\int_0^t f''(X_s)\, d\langle X \rangle_s$ is a standard Riemann Stieltjes integral.

This formula was extended by Dupire (2009) and Cont & Fournié (2010) to a functional context.

Incomplete list of financial applications of pathwise Itō calculus

- Strictly pathwise approach to Black–Scholes formula (Bick & Willinger 1994)
- Robustness of hedging strategies and pricing formulas for exotic options (A.S. & Stadje 2007, Cont & Riga 2015)
- Model-free replication of variance swaps (e.g., Davis et al. (2010))
- CPPI strategies (A.S. 2014)
- Functional and pathwise extension of the Fernholz–Karatzas stochastic portfolio theory (A.S., Speiser & Voloshchenko 2015)

The key to many of these results is the following associativity property of the Föllmer integral:

$$\int_0^t \eta_s \, d\Big(\int_0^s \xi_r \, dX_r\Big) = \int_0^t \eta_s \xi_s \, dX_s$$

(A.S. 2014, A.S. & Voloshchenko 2015)

2. In search of a class of test integrators

Let's fix the sequence of dyadic partitions,

$$\mathbb{T}_n := \{k2^{-n} \mid k = 0, \dots, 2^n\}, \qquad n = 1, 2, \dots$$

Goal: Find a rich class of functions $x \in C[0,1]$ that admit a nontrivial continuous quadratic variation along $(\mathbb{T}_n)$.

Of course this is true for the sample paths of Brownian motion or other continuous semimartingales—as long as these sample paths do not belong to a certain nullset A.

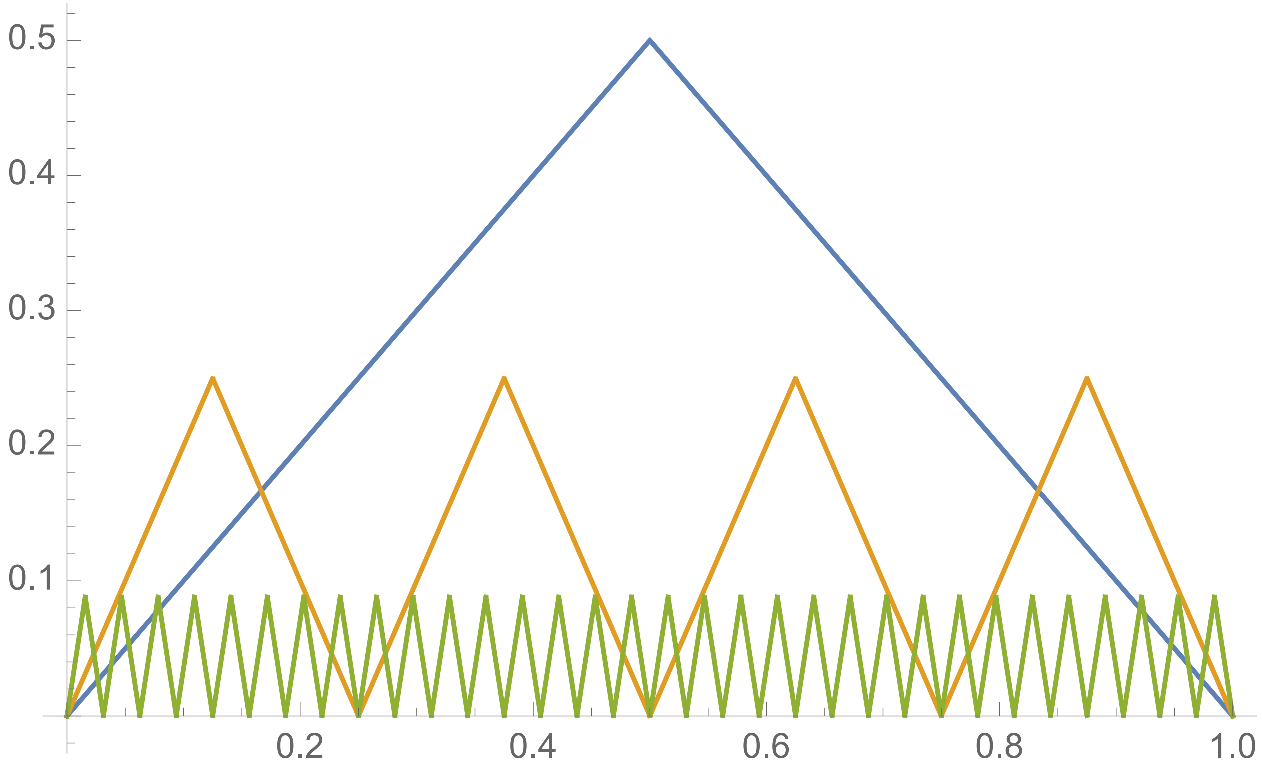
But A is not explicit, and so it is not possible to tell whether a specific realization x of Brownian motion does indeed admit the quadratic variation $\langle x \rangle_t = t$ along $(\mathbb{T}_n)_{n\in\mathbb{N}}$.

Moreover, this selection principle for functions x lets a probabilistic model enter through the backdoor...

A result of N. Gantert

Recall that the *Faber–Schauder functions* are defined as

$$e_{0,0}(t) := (\min\{t, 1-t\})^+ \qquad e_{m,k}(t) := 2^{-m/2} e_{0,0}(2^m t - k)$$



Functions $e_{n,k}$ for $n = 0$, $n = 2$, and $n = 5$

Every function $x \in C[0,1]$ with $x(0) = x(1) = 0$ can be represented as

$$x = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m - 1} \theta_{m,k} e_{m,k}$$

where

$$\theta_{m,k} = 2^{m/2} \left(2x\left(\frac{2k+1}{2^{m+1}}\right) - x\left(\frac{k}{2^m}\right) - x\left(\frac{k+1}{2^m}\right) \right).$$

Gantert (1991, 1994) showed that

$$\langle x \rangle_t^n := \sum_{s \in \mathbb{T}_n,\, s \le t} (x(s') - x(s))^2$$

can be computed for $t = 1$ as

$$\langle x \rangle_1^n = \frac{1}{2^n} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m - 1} \theta_{m,k}^2$$

By letting

$$\mathscr{X} := \left\{ x \in C[0,1] \,\middle|\, x = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k} \text{ for coefficients } \theta_{m,k} \in \{-1,+1\} \right\}$$

(which is easily shown to be possible) we hence get a class of functions with $\langle x \rangle_1 = 1$ for all $x \in \mathscr{X}$.

As a matter of fact:

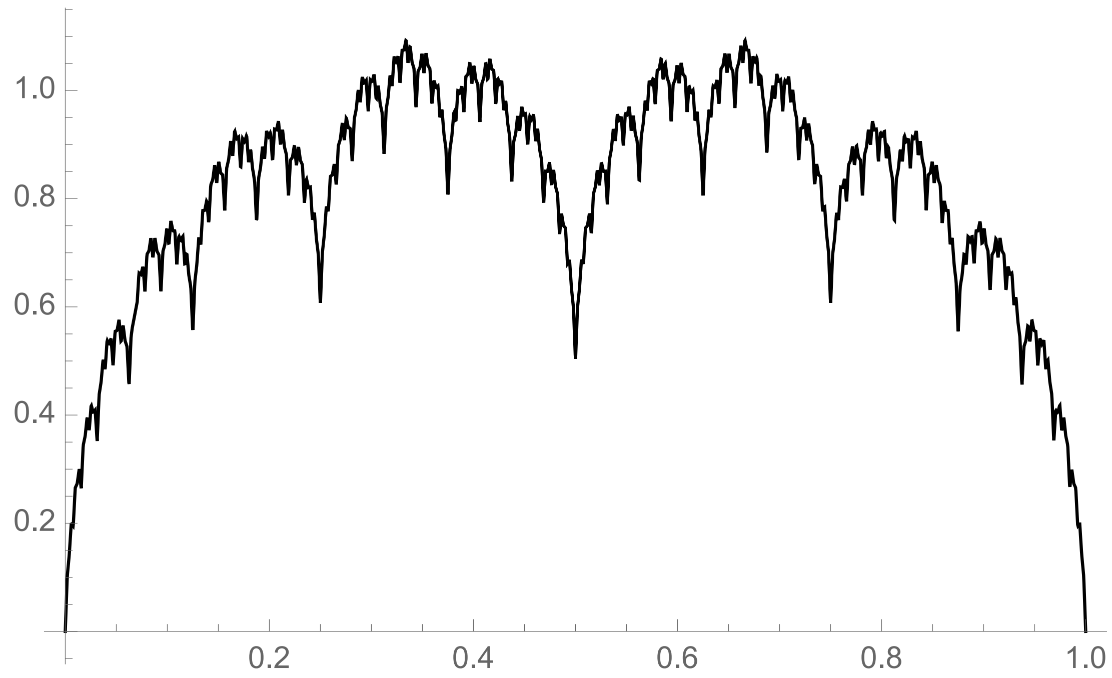
Proposition 2. *Every* $x \in \mathscr{X}$ *has the quadratic variation* $\langle x \rangle_t = t$ *along* $(\mathbb{T}_n)$.

Link to the Takagi function and its generalizations

The specific function

$$\widehat{x} := \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} e_{m,k}$$

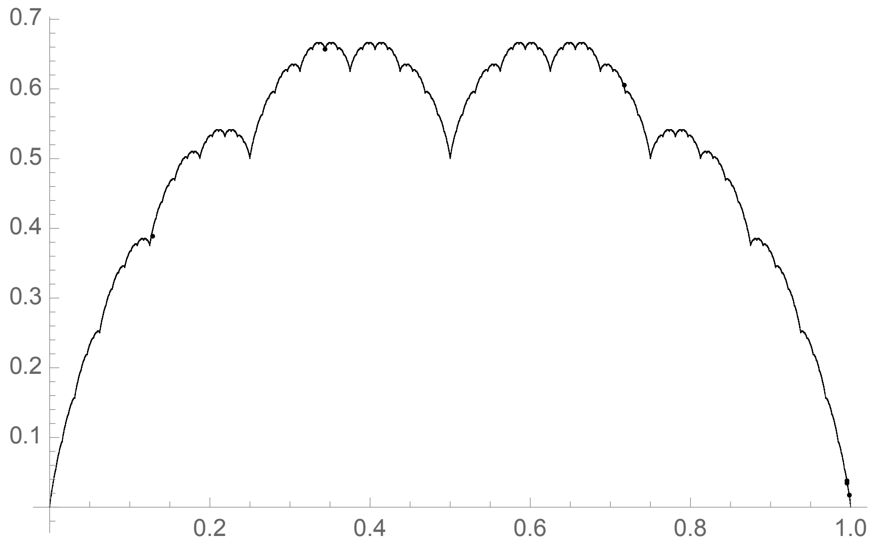
has some interesting properties.



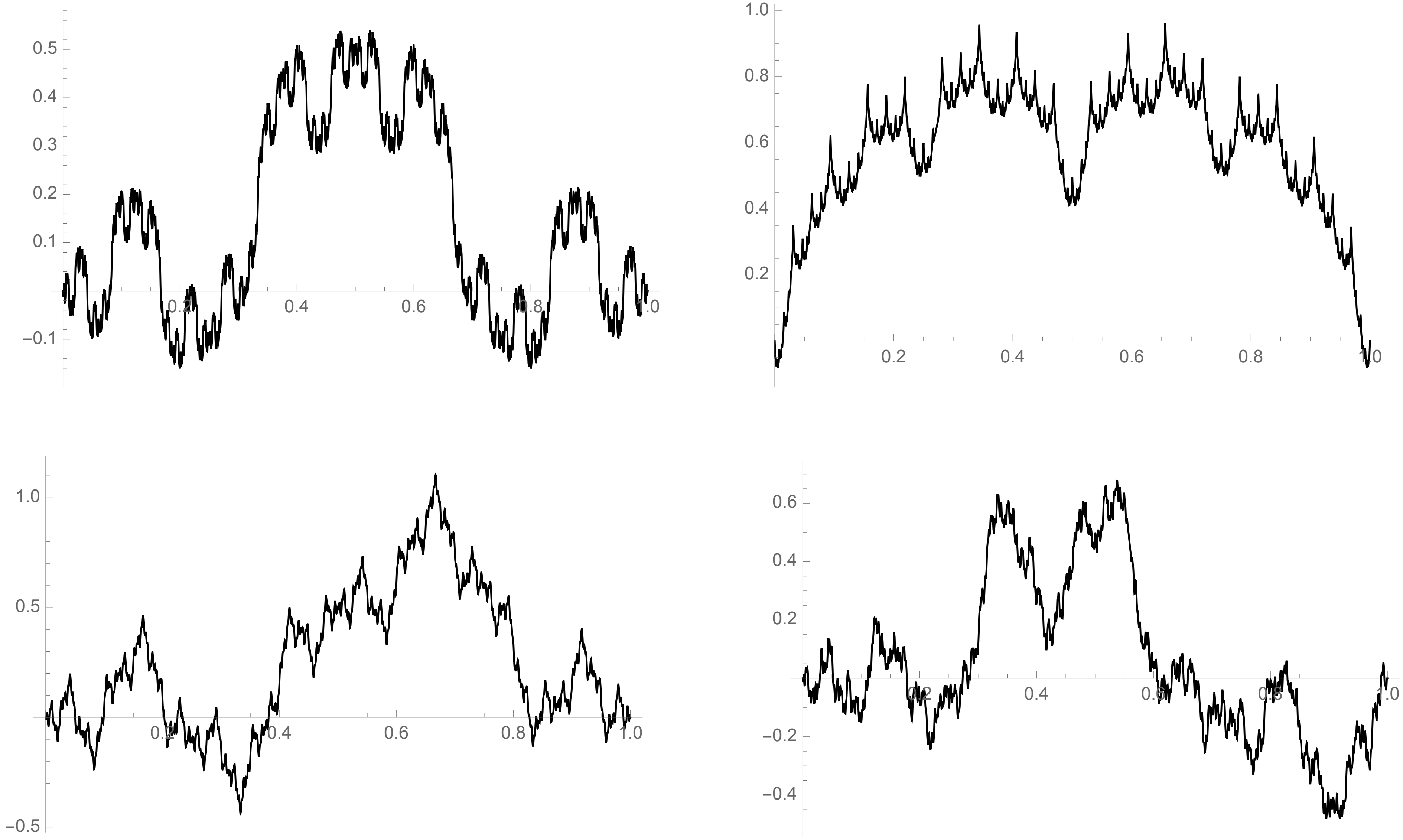
The function $\widehat{x}$ is closely related to the celebrated Takagi function,

$$\tau = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} 2^{-m/2} e_{m,k}$$

which was first found by Takagi (1903) and rediscovered many times (e.g., by van der Waerden (1930), Hildebrandt (1933), Tambs–Lyche (1942), and de Rham (1957))

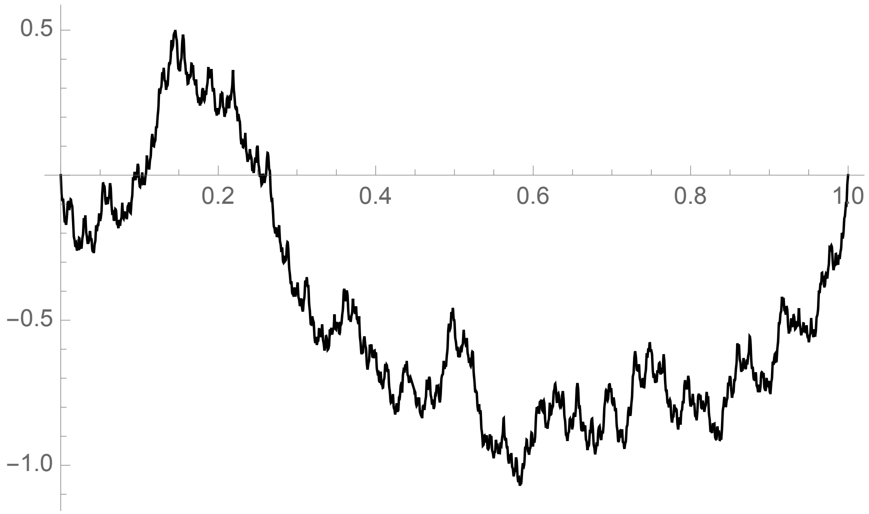


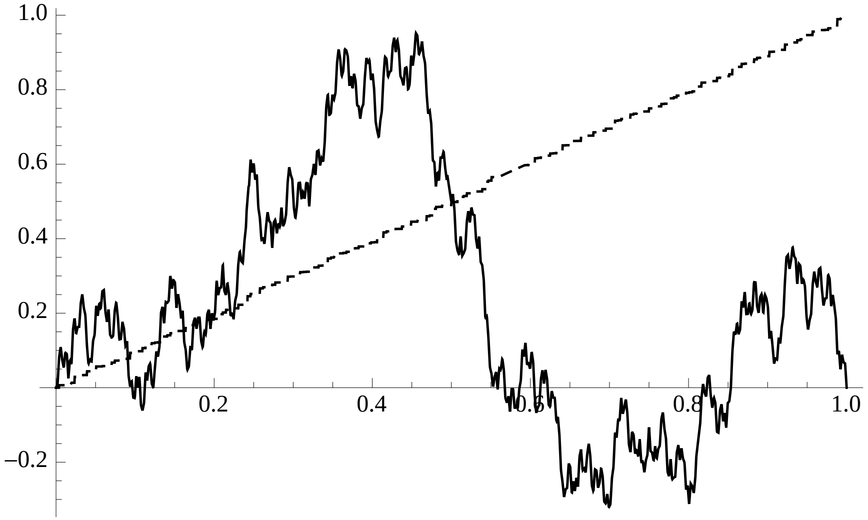
Our class $\mathscr{X}$ has a nonempty intersection with the "Takagi class" introduced by Hata & Yamaguti (1984) and is a subset of the class of generalized Takagi functions studied by Allaart (2009).



Functions in $\mathscr{X}$ for various (deterministic) choices of $\theta_{m,k} \in \{-1,1\}$

Similarities with sample paths of a Brownian bridge





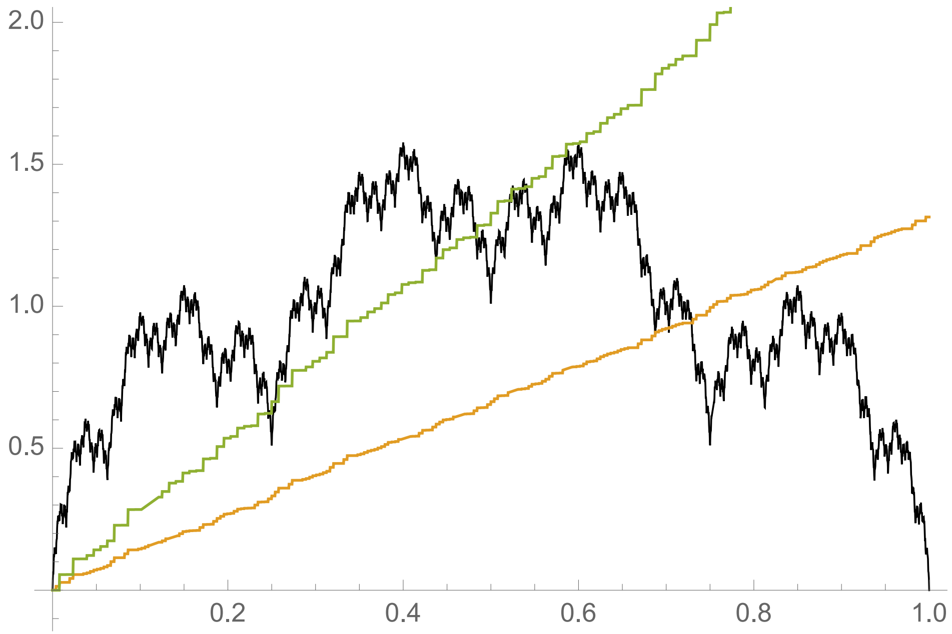
Plots of $x \in \mathscr{X}$ when the $\theta_{m,k}$ form a $\{-1,+1\}$-valued i.i.d. sequence

- Lévy–Ciesielski construction of the Brownian bridge
- Quadratic variation
- Nowhere differentiability (de Rham 1957, Billingsley 1982, Allaart 2009)
- Hausdorff dimension of the graph of $\widehat{x}$ is $\frac{3}{2}$ (Ledrappier 1992)

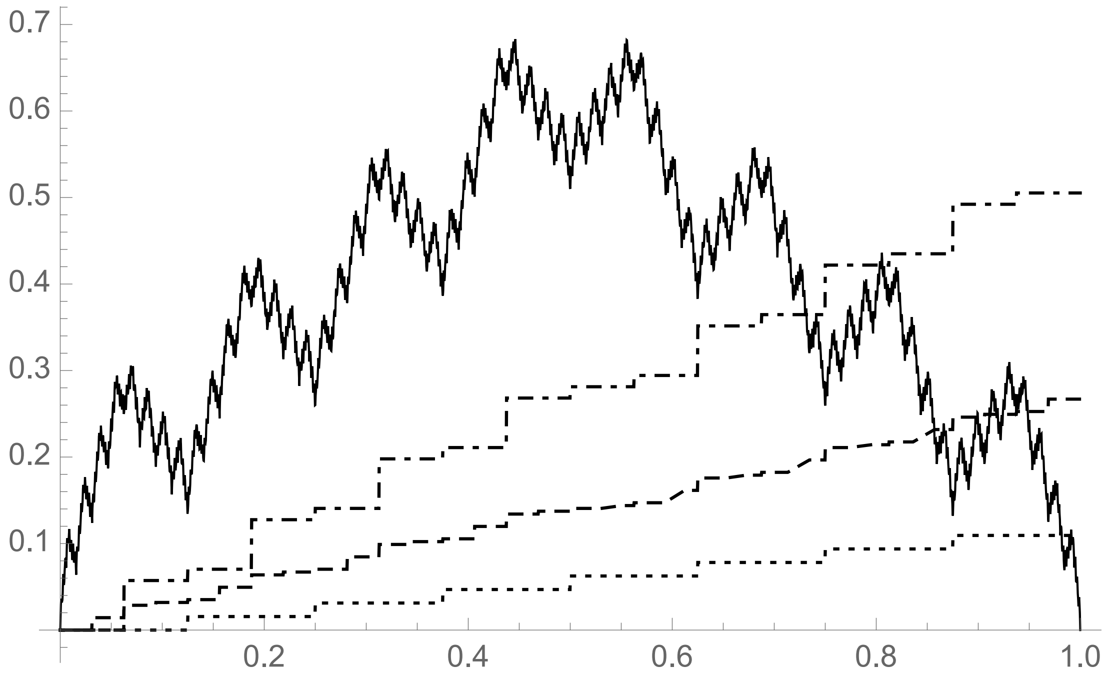
The class of functions with quadratic variation is not a vector space

Proposition 3. *Consider the function $y \in \mathscr{X}$ defined through $\theta_{m,k} = (-1)^m$. Then*

$$\lim_{n\uparrow\infty} \langle \widehat{x} + y \rangle_t^{2n} = \frac{4}{3}t \qquad \text{and} \qquad \lim_{n\uparrow\infty} \langle \widehat{x} + y \rangle_t^{2n+1} = \frac{8}{3}t$$



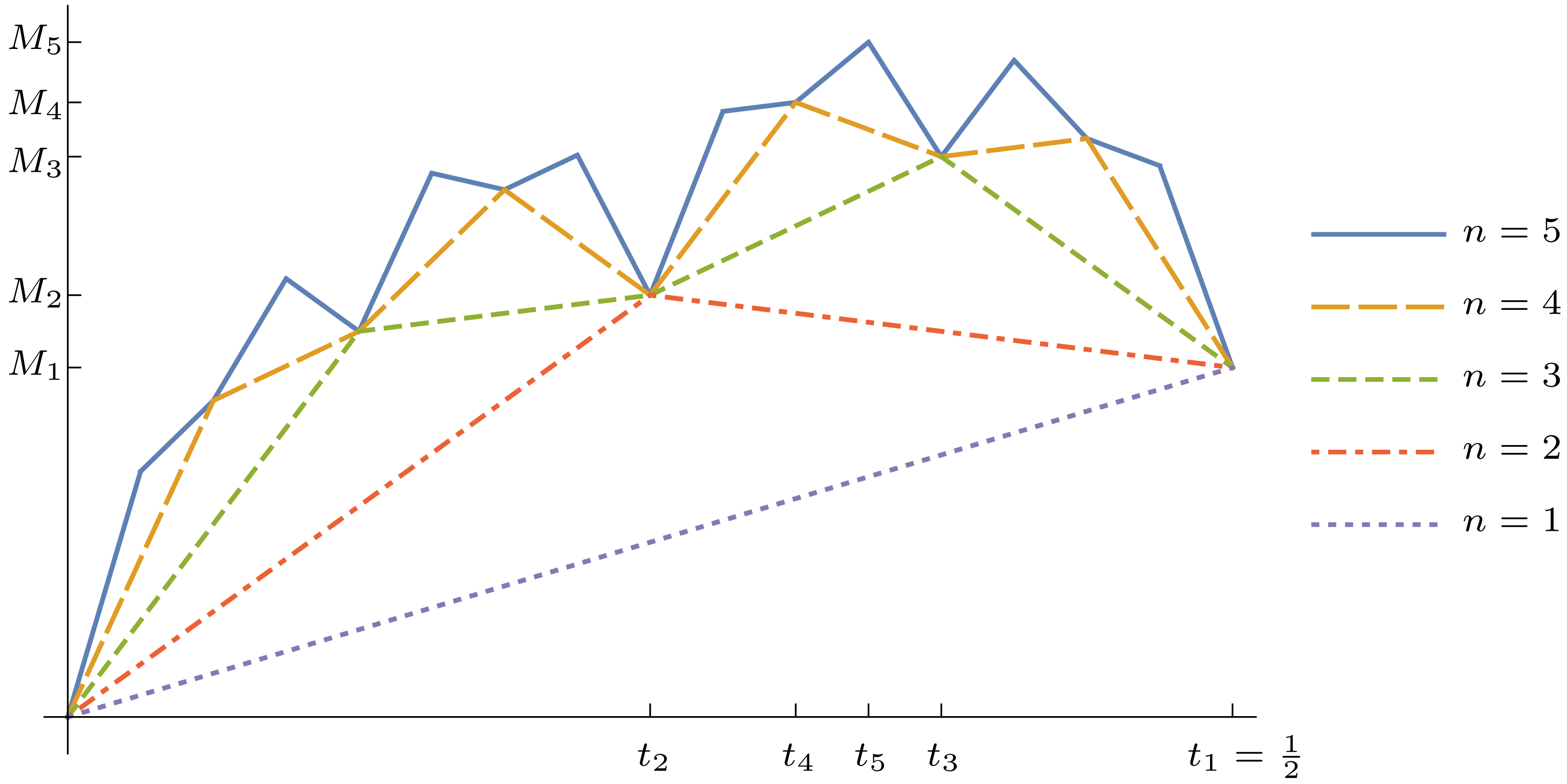
The function $\widehat{x} + y$ with $\langle \widehat{x} + y \rangle^7$ and $\langle \widehat{x} + y \rangle^8$



A function $z \notin \mathscr{X}$ with exactly three distinct accumulation points for $\langle z \rangle_t^n$

The maximum of $\widehat{x}$

Kahane (1959) showed that the maximum of the Takagi function is $\frac{3}{2}$. For $\widehat{x}$, we need different arguments.



Functions $\widehat{x}^n(t) := \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} e_{m,k}(t)$ and their maxima on $[0, \frac{1}{2}]$

The preceding plot suggests the recursions

$$t_{n+1} = \frac{t_n + t_{n-1}}{2} \qquad \text{and} \qquad M_{n+1} = \frac{M_n + M_{n-1}}{2} + 2^{-\frac{n+2}{2}}$$

These are solved by

$$t_n = \frac{1}{3}(1-(-1)^n 2^{-n}) \qquad \text{and} \qquad M_n = \frac{1}{3}\Big(2+\sqrt{2}+(-1)^{n+1}2^{-n}(\sqrt{2}-1)\Big)-2^{-n/2}$$

By sending $n \uparrow \infty$, we obtain:

Theorem 1. *The uniform maximum of functions in* $\mathscr{X}$ *is attained by* $\widehat{x}$ *and given by*

$$\max_{x\in\mathscr{X}} \max_{t\in[0,1]} |x(t)| = \max_{t\in[0,1]} \widehat{x}(t) = \frac{1}{3}(2+\sqrt{2}).$$

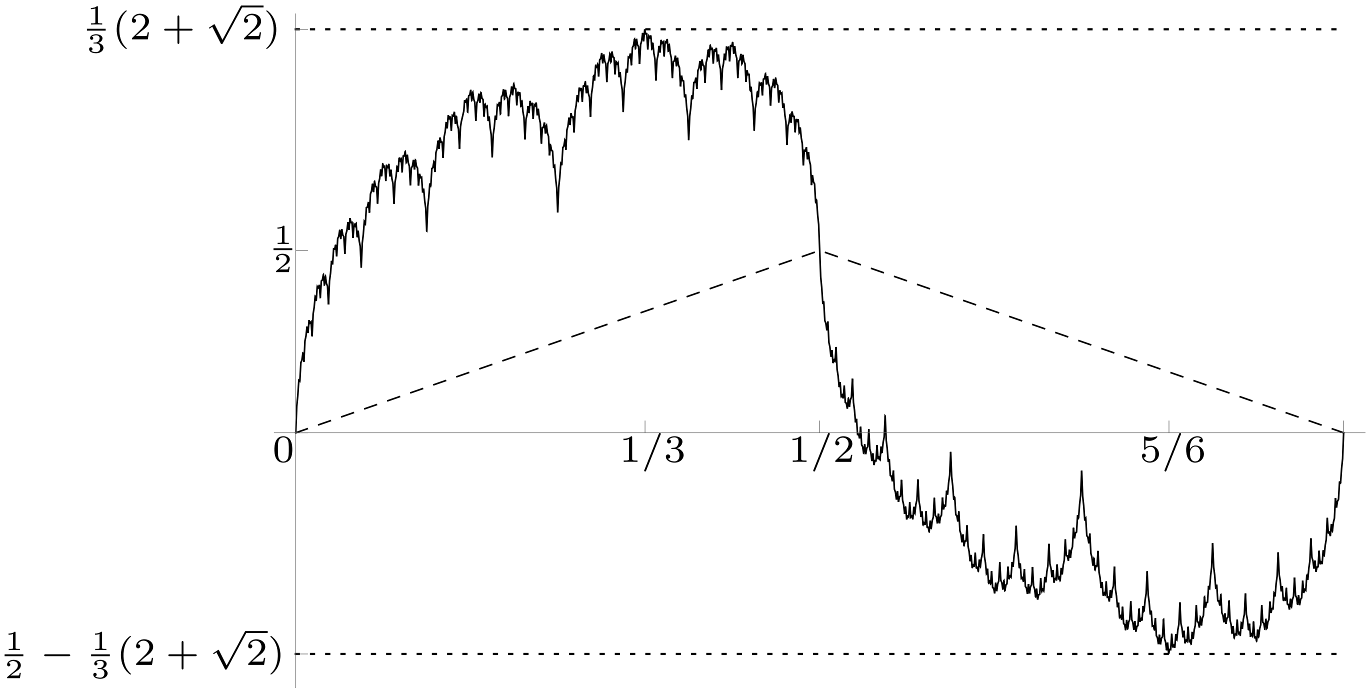
Maximal points are $t = \frac{1}{3}$ *and* $t = \frac{2}{3}$.

Corollary 1. *The maximal uniform oscillation of functions in $\mathscr{X}$ is*

$$\max_{x\in\mathscr{X}} \max_{s,t\in[0,1]} |x(t)-x(s)| = \frac{1}{6}(5+4\sqrt{2})$$

where the respective maxima are attained at $s = 1/3$, $t = 5/6$, *and*

$$x^* := e_{0,0} + \sum_{m=1}^{\infty}\left(\sum_{k=0}^{2^{m-1}-1} e_{m,k} - \sum_{\ell=2^{m-1}}^{2^m-1} e_{m,\ell}\right)$$



Uniform moduli of continuity

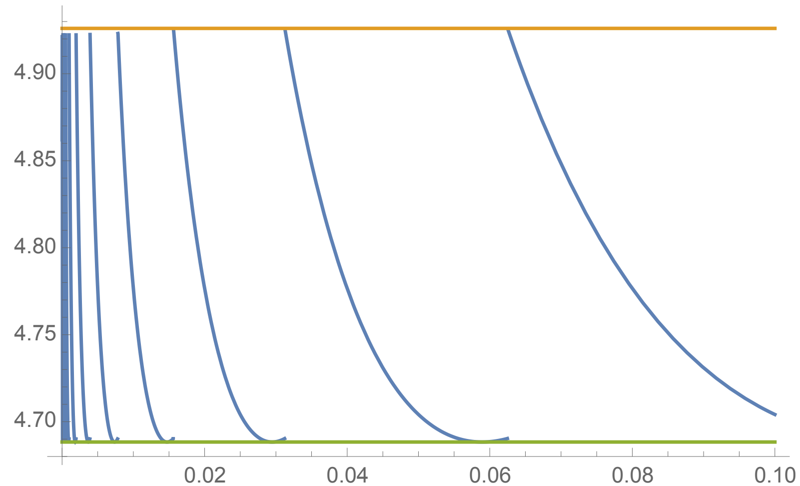
Kahane (1959), Kôno (1987), Hata & Yamaguti (1984), and Allaart (2009) studied moduli of continuity for (generalized) Takagi functions. However, their arguments are not applicable to the functions in $\mathscr{X}$.

Let

$$\omega(h) := \Big(1 + \frac{1}{\sqrt{2}}\Big) h 2^{\lfloor -\log_2 h \rfloor/2} + \frac{1}{3}(\sqrt{8}+2)2^{-\lfloor -\log_2 h \rfloor/2}$$

Then $\omega(h) = O(\sqrt{h})$ as $h \downarrow 0$. More precisely,

$$\liminf_{h\downarrow 0} \frac{\omega(h)}{\sqrt{h}} = 2\sqrt{\frac{4}{3} + \sqrt{2}} \qquad \limsup_{h\downarrow 0} \frac{\omega(h)}{\sqrt{h}} = \frac{1}{6}(11 + 7\sqrt{2})$$



Theorem 2 (Moduli of continuity).

(a) *The function* $\widehat{x}$ *has* ω *as its modulus of continuity. More precisely,*

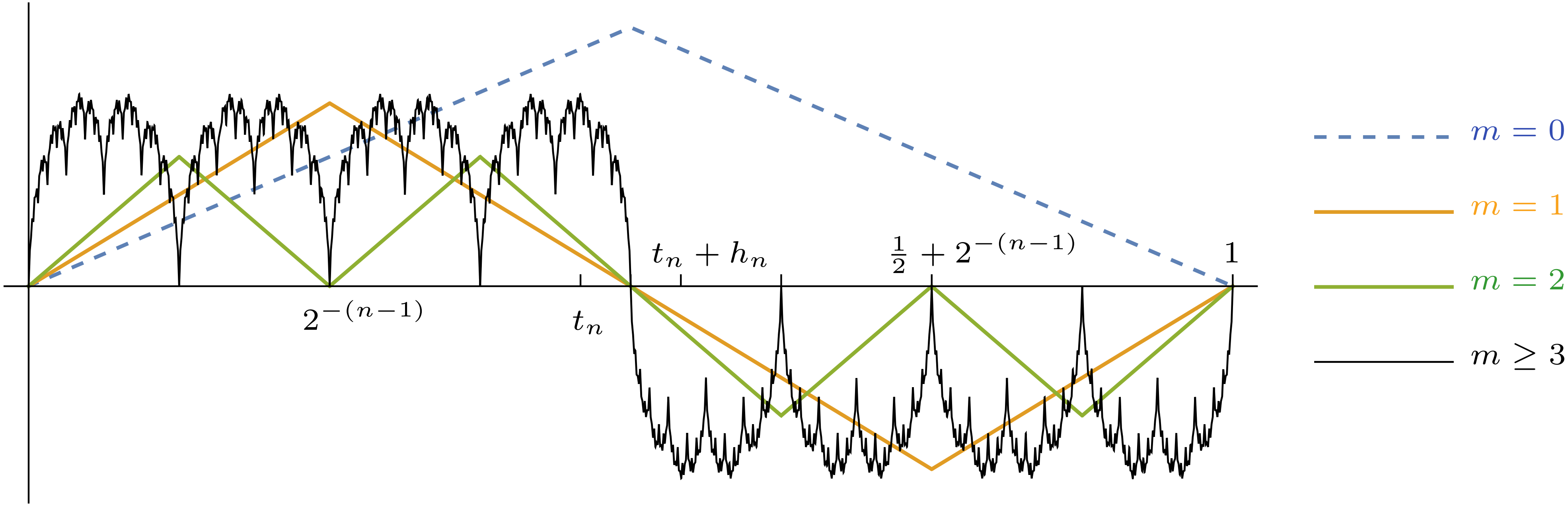
$$\limsup_{h\downarrow 0} \max_{0\le t\le 1-h} \frac{|\widehat{x}(t+h)-\widehat{x}(t)|}{\omega(h)} = 1$$

(b) *An exact uniform modulus of continuity for functions in* $\mathscr{X}$ *is given by* $\sqrt{2}\omega$*. That is,*

$$\limsup_{h\downarrow 0} \sup_{x\in\mathscr{X}} \max_{0\le t\le 1-h} \frac{|x(t+h)-x(t)|}{\omega(h)} = \sqrt{2}$$

Moreover, the above supremum over functions $x \in \mathscr{X}$ *is attained by the function* x^* *in the sense that*

$$\limsup_{h\downarrow 0} \max_{0\le t\le 1-h} \frac{|x^*(t+h)-x^*(t)|}{\omega(h)} = \sqrt{2}$$



The Faber–Schauder development of x^* is plotted individually for generations $m \leq n-1$ (with $n = 3$ here).
The aggregated development over all generations $m \geq n$ corresponds to a sequence of rescaled functions $\widehat{x}$.

$$\omega(h) = \left(1 + \frac{1}{\sqrt{2}}\right)h2^{\lfloor -\log_2 h \rfloor/2} + \frac{1}{3}(\sqrt{8} + 2)2^{-\lfloor -\log_2 h \rfloor/2}$$

linear part   self-similar part

Consequences

- Functions in $\mathscr{X}$ are uniformly Hölder continuous with exponent $\frac{1}{2}$
- Functions in $\mathscr{X}$ have a finite 2-variation and hence can serve as integrators in rough path theory
- $\mathscr{X}$ is a compact subset of $C[0,1]$

Thank you

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