Derivative pricing for a multi curve extension of the Gaussian, exponentially quadratic short rate model

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Introduction

We study a short rate model for the term structure of interest rates, which is of the exponentially quadratic type and which takes into account the fact that, after the crisis, the classical relationship between bonds and Libor rates have broken down.

- → We study also the pricing of interest rate derivatives in such a model (*due to time limitations, only linear derivatives (FRAs)*).
- → It is based on a recent paper by Z.Grbac,L.Meneghello, W.J.Runggaldier

Discounting curve

In line with post-crisis market practice, we consider one curve for discounting ($p(t, T)$) and various curves for generating future cash flows (Libor rates), where the latter depend on the tenor structure.

$\rightarrow$ Here, for simplicity, only one tenor.

- Discounting is performed via the OIS curve $T \rightarrow p^{OIS}(t, T)$ that can be stripped from OIS rates. We identify it with the classical risk-free bonds $p(t, T)$.

Discounting curve

Considering the corresponding (infinitesimal) forward and short rates $f(t,T) := -\frac{\partial}{\partial T}p(t,T)$ and $r_t := f(t,t)$, let $B_t := \exp\left[\int_0^t r_u du\right]$ be the money market account.

- It leads to the standard martingale measure Q under which prices, expressed in units of B_t, are (local) martingales.
- Given B_t and the OIS bonds $p(t,T)$, one can define the various forward measures, typically used for the pricing of derivatives, and one has for given $T > t$

$$\frac{dQ^T}{dQ}_{|\mathcal{F}_t} = \frac{p(t,T)}{B_t p(0,T)}$$

Preliminaries

In view of obtaining arbitrage-free dynamic models for the bonds and the Libors, needed in derivative pricing, we start by recalling some basic notions.

FRA and FRA rates

Recall first the classical discrete compounding forward rate $F(t; T, T+\Delta)$, evaluated at $t \leq T$ for the interval $[T, T+\Delta]$.

→ By absence of arbitrage arguments it can be related to the OIS bond prices $p(t, T)$ via

$$F(t; T, T+\Delta) = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right)$$

FRA and FRA rates

Recall next a basic derivative contract, namely a FRA (*forward rate agreement*), established at a certain time t for a future time interval $[T, T+\Delta]$ where, at $T+\Delta$, a fixed rate R is exchanged against a floating rate $F(T; T, T+\Delta)$, usually determined in T.

→ Using the forward measure, its price in t is (*for a unitary nominal*)

$$P^{FRA}(t; T, T+\Delta) = p(t, T+\Delta)\,\Delta\, E^{T+\Delta}\left\{F(T; T, T+\Delta) - R \mid \mathcal{F}_t\right\}$$

→ The fixed rate, making the contract fair in t, is then

$$R_t = E^{T+\Delta}\left\{F(T; T, T+\Delta) \mid \mathcal{F}_t\right\}$$

FRA and FRA rates

If $F(T;T,T+\Delta)$ is the discrete compounding spot rate in T for the interval $[T,T+\Delta]$, then for the fair rate R_t in t we have that

$$\begin{aligned} R_t &= E^{T+\Delta}\{F(T;T,T+\Delta) \mid \mathcal{F}_t\} \\ &= \frac{1}{\Delta}\left(E^{T+\Delta}\left\{\frac{p(T,T)}{p(T,T+\Delta)} \mid \mathcal{F}_t\right\} - 1\right) \\ &= \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right) = F(t;T,T+\Delta) \end{aligned}$$

$\rightarrow$ It coincides with the discrete compounding forward rate.

Libor rates

A basic rate in the interest rate (derivative) market is the Libor (Euribor) rate which is determined by a panel that takes into account various risk factors, in particular counterpart and liquidity risk. It is a discrete compounding forward rate $L(t; T, T+\Delta)$ that, before the crisis, was identified with $F(t; T, T+\Delta)$

- After the crisis different Libor rates had to be considered for different tenors Δ.

$\rightarrow$ It has led to the multi curve phenomenon.

Libor rates

If in a FRA the floating rate received in $T+\Delta$ is the spot Libor $L(T;T,T+\Delta)$ as determined in T by the Libor panel, then the fair fixed rate in t is

$$L(t;T,T+\Delta) = E^{T+\Delta}\left\{L(T;T,T+\Delta) \mid \mathcal{F}_t\right\}$$

which is a $Q^{T+\Delta}-$martingale by construction and is called the forward Libor rate.

$\rightarrow$ Since, in general, $L(T;T,T+\Delta) \neq F(T;T,T+\Delta)$, also

$$L(t;T,T+\Delta) \neq F(t;T,T+\Delta) = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right)$$

Libor rates

The Libors are the main rates underlying the interest rate derivatives and, to price the latter, one needs arbitrage-free dynamics for the Libors.

- Since after the crisis the Libor rates became disconnected from the OIS bond prices, their dynamics need to be modeled separately from the OIS bonds.

$\rightarrow$ How should the (arbitrage-free) dynamics of $L(t; T, T + \Delta)$ be modeled?

Libor rate

In the classical one-curve setting there are various approaches, among them (*in a top-down sequencing*): LMM, HJM setup, short rate models. Each of them may be extended to the multi curve setting.

$\rightarrow$ Here we consider an extension of the short rate approach.

Libor rate

In view of obtaining a short rate model and following some of the recent literature, one may postulate the classical relationship between forward rates and OIS bonds also for the Libors, namely

$$L(t; T, T+\Delta) = \frac{1}{\Delta}\left(\frac{\bar{p}(t,T)}{\bar{p}(t,T+\Delta)} - 1\right)$$

where, however, $\bar{p}(t,T)$ are fictitious bond prices that can also be seen as average bonds issued by a representative bank from the Libor group and therefore affected by the same risk factors as the Libors.

$\rightarrow$ We thus reduce ourselves to obtaining arbitrage-free dynamics for $\bar{p}(t,T)$ (analogously to the Libor we need one $\bar{p}(t,T)$ for each tenor Δ; here we consider just one.)

Libor rate

- Recall now that, according to standard martingale modeling, one has

$$p(t,T) = E^Q\left\{\exp\left[-\int_t^T r_u du\right] \mid \mathcal{F}_t\right\}$$

  - $p(t,T)/B_t$ are $Q-$martingales.

Fictitious bonds

Inspired by a credit risk analogy and by a common practice of deriving post-crisis quantities by adding a spread over the corresponding pre-crisis quantities, we may postulate

$$\bar{p}(t,T) = E^Q \left\{ \exp\left[-\int_t^T (r_u + s_u)du \right] \mid \mathcal{F}_t \right\}$$

where s_t is the short rate spread (*for the given tenor*), assumed to be affected by the same factors as the Libor rate.

→ The definition implies that $\bar{p}(T,T) = 1$.

Fictitious bonds

The $\bar{p}(t,T)$ are not traded assets and so, differently from $p(t,T)$, the discounted values $\bar{p}(t,T)/B_t$ are not necessarily $Q-$martingales.

$\rightarrow$ However, provided one replaces B_t by $\bar{B}_t := \exp\left[\int_0^t (r_u + s_u)\, du\right]$ one has that, by the above definition of $\bar{p}(t,T)$, the $\bar{p}(t,T)/\bar{B}_t$ are $Q-$ (local) martingales .

Short rate and spread

- We need now arbitrage-free dynamics for r_t and s_t,namely dynamics under the martingale measure Q that, in practice, has to be calibrated to the market.

Classical short rate model are the square-root, exponentially affine models (*CIR dynamics for* r_t*, exponentially affine expression for* $p(t, T)$). Dual to this class are the less well known Gaussian exponentially quadratic models resulting from expressing the short rate as a second order polynomial of Gaussian factors.

→ The main advantage, especially for derivative pricing, is that computations reduce to expectations involving Gaussian factors.

Gaussian factor model

We shall consider a factor model for both, r_t and s_t. We need a minimum of three factors if we want also to model correlation between r_t and s_t. Let then, under Q,

$$d\Psi_t^i = -b^i \Psi_t^i dt + \sigma^i dw_t^i \ , \quad i = 1, 2, 3$$

(*for simplicity of notation, we consider just* $0-$*mean reversion*) and put

$$\begin{cases} r_t &= \Psi_t^1 + \left(\Psi_t^2\right)^2 \\ s_t &= \kappa \Psi_t^1 + \left(\Psi_t^3\right)^2 \end{cases}$$

Gaussian factor model

The common (*systematic*) factor Ψ_t^1 allows for instantaneous correlation between r_t and s_t with intensity κ. The linearity in Ψ_t^1 allows for more flexibility (*the "adjustment factor" below*) and, although it allows for negative values also for the spreads, the corresponding probability is small if one considers a positive mean reversion.

$\rightarrow$ For the short rate one may also consider a "deterministic shift extension" $r_t = \phi_t + \Psi_t^1 + \left(\Psi_t^2\right)^2$

Exponentially quadratic structure

By adapting results from exponentially quadratic term structure models (see e.g. Gombani,R. (2001)) one obtains the following exponentially quadratic term structure

$$p(t,T) = E^Q\left\{e^{-\int_t^T r_u\,du} \mid \mathcal{F}_t\right\} = E^Q\left\{e^{-\int_t^T (\Psi_u^1 + (\Psi_u^2)^2) du} \mid \mathcal{F}_t\right\}$$
$$= \exp\left[-A(t,T) - \sum_{i=1}^3 B^i(t,T)\Psi_t^i - \sum_{i,j=1}^3 C^{ij}(t,T)\Psi_t^i\Psi_t^j\right]$$

Analogously, putting $R_t := r_t + s_t$,

$$\bar{p}(t,T) = E^Q\left\{e^{-\int_t^T R_u\,du} \mid \mathcal{F}_t\right\}$$
$$= E^Q\left\{e^{-\int_t^T ((1+\kappa)\Psi_u^1 + (\Psi_u^2)^2 + (\Psi_u^2)^2) du} \mid \mathcal{F}_t\right\}$$
$$= \exp\left[\bar{A}(t,T) - \sum_{i=1}^3 \bar{B}^i(t,T)\Psi_t^i - \sum_{i,j=1}^3 \bar{C}^{ij}(t,T)\Psi_t^i\Psi_t^j\right]$$

Exponentially quadratic structure

As in standard term structure models, the coefficients can be determined on the basis of no-arbitrage constraints (or equivalently via the HJM drift condition).

- Since $p(t, T)$ is a traded quantity, no-arbitrage implies that $p(t, T)/B_t$ has to be a $Q-$martingale. This property follows however also from the definition of $p(t, T)$ as an expectation under Q and it is more precisely this property that implies the well-known HJM drift condition.
- $\bar{p}(t, T)$ on the other hand is not a traded quantity so that it does not have to satisfy no-arbitrage conditions. However, by its definition, $\bar{p}(t, T)/\bar{B}_t$ is a $Q-$martingale, just as is $p(t, T)/B_t$. Consequently here too we can derive a HJM-type drift condition to determine the coefficients in the exponential-quadratic representation of $\bar{p}(t, T)$.

Exponentially quadratic structure

Using a result for exponentially quadratic term structures in Gombani, Runggaldier (2001), the HJM drift conditions lead to

$$p(t,T) = \exp\left[-A(t,T) - B^1(t,T)\Psi_t^1 - C^{22}(t,T)(\Psi_t^2)^2\right]$$

and

$$\begin{aligned}\bar{p}(t,T) &= \exp\Big[-\bar{A}(t,T) - (\kappa+1)B^1(t,T)\Psi_t^1 \\ &\qquad -C^{22}(t,T)(\Psi_t^2)^2 - \bar{C}^{33}(t,T)(\Psi_t^3)^2\Big] \\ &= p(t,T)\exp\left[-\tilde{A}(t,T) - \kappa B^1(t,T)\Psi_t^1 - \bar{C}^{33}(t,T)(\Psi_t^3)^2\right]\end{aligned}$$

where $\tilde{A}(t,T) = \bar{A}(t,T) - A(t,T)$.

Exponentially quadratic structure

The coefficients $C^{22}(t,T)$, $\bar{C}^{33}(t,T)$ satisfy Riccati equations and allow for an explicit expression. Given their values, the other coefficients $B^1(t,T)$, $A(t,T)$ and $\bar{A}(t,T)$ satisfy linear 1st order ODEs.

- → With the above expression for $\bar{p}(t,T)$, we obtained a factor to pass from $p(t,T)$ to $\bar{p}(t,T)$ (*will be used to obtain an "adjustment factor" below*).
- → In the general case of multiple tenors, we obtain one factor for each tenor.

Forward measure

- The underlying factor model was defined under a standard martingale measure Q.
- For derivative prices, forward measures turn out to be convenient and here we introduce a generic $(T + \Delta)-$forward measure $Q^{T+\Delta}$.

Forward measure

Recall the density process to change the measure from Q to $Q^{T+\Delta}$ is

$$\mathcal{L}_t := \frac{d\,Q^{T+\Delta}}{d\,Q}_{|\mathcal{F}_t} = \frac{p(t, T+\Delta)}{p(0, T+\Delta)} \frac{1}{B(t)}$$

from which

$$d\mathcal{L}_t = \mathcal{L}_t \left(-B^1(t, T+\Delta)\sigma^1 dw_t^1 - 2C^{22}(t, T+\Delta)\Psi_t^2 \sigma^2 dw_t^2 \right)$$

implying that

$$\begin{cases} dw_t^{1,T+\Delta} & = & dw_t^1 + \sigma^1 B^1(t, T+\Delta)dt \\ dw_t^{2,T+\Delta} & = & dw_t^2 + 2C^{22}(t, T+\Delta)\Psi_t^2 \sigma^2 \\ dw_t^{3,T+\Delta} & = & dw_t^3 \end{cases}$$

Forward measure

Recalling $d\Psi_t^i = -b^i \Psi_t^i dt + \sigma^i \, dw_t^i, \; i = 1, 2, 3,$ under $Q^{T+\Delta}$ we then have

$$\begin{aligned} d\Psi_t^1 &= -\left[b^1 \Psi_t^1 + (\sigma^1)^2 B^1(t, T+\Delta) \right] dt + \sigma^1 dw_t^{1,T+\Delta} \\ d\Psi_t^2 &= -\left[b^2 \Psi_t^2 + \sigma^1 + 2(\sigma^2)^2 C^{22}(t, T+\Delta) \Psi_t^2 \right] dt + \sigma^2 dw_t^{2,T+\Delta} \\ d\Psi_t^3 &= -b^3 \Psi_t^3 dt + \sigma^3 dw_t^{3,T+\Delta} \end{aligned}$$

→ Under $Q^{T+\Delta}$ the processes $\Psi_t^i, \; i = 1, 2, 3$ have a time-dependent drift and are described by a Gaussian distribution that, given $B^1(\cdot)$ and $C^{22}(\cdot)$, has mean and variance

$$E^{T+\Delta}\{\Psi_t^i\} = \bar{\alpha}_t^i = \bar{\alpha}_t^i(b^i, \sigma^i) \quad , \quad Var^{T+\Delta}\{\Psi_t^i\} = \bar{\beta}_t^i = \bar{\beta}_t^i(b^i, \sigma^i)$$

and that can be explicitly computed.

FRAs, FRA rates and adjustment factors

Recall that the price in $t < T$ of a *(text-book)* FRA for the interval $[T, T+\Delta]$ with fixed rate R, notional N, and with the Libor rate as floating rate, is given by

$$\begin{aligned} &P^{FRA}(t; T, T+\Delta, R, N) \\ &= N\Delta p(t, T+\Delta) E^{T+\Delta}\{L(T; T, T+\Delta) - R \mid \mathcal{F}_t\} \\ &= N p(t, T+\Delta) E^{T+\Delta}\left\{\frac{1}{\bar{p}(T, T+\Delta)} - (1+\Delta R)\mathcal{F}_t\right\} \end{aligned}$$

→ The fair value of the FRA rate (here representing the forward Libor) is then

$$\bar{R}_t = \frac{1}{\Delta}\left(\bar{\nu}_{t,T} - 1\right) \quad \text{where} \quad \bar{\nu}_{t,T} := E^{T+\Delta}\left\{\frac{1}{\bar{p}(T, T+\Delta)} \mid \mathcal{F}_t\right\}$$

FRAs, FRA rates and adjustment factors

- In the single-curve case we have instead

$$R_t^{\Delta} = \frac{1}{\Delta}\left(\nu_{t,T} - 1\right)$$

where

$$\nu_{t,T} := E^{T+\Delta}\left\{\frac{1}{p(T, T+\Delta)} \mid \mathcal{F}_t\right\} = \frac{p(t,T)}{p(t,T+\Delta)}$$

since $\frac{p(t,T)}{p(t,T+\Delta)}$ is a $Q^{T+\Delta}-$martingale. The expression can be explicitly computed on the basis of bond price data without requiring an interest rate model.

FRAs, FRA rates and adjustment factors

Concerning $\bar{\nu}_{t,T}$, on the basis of the expression for $\bar{p}(t,T)$ and the density process to pass from Q to $Q^{T+\Delta}$, one obtains the following expression

$$\bar{\nu}_{t,T} = \frac{1}{p(t,T+\Delta)} \exp[\tilde{A}(T,T+\Delta)] E^Q\left\{ e^{\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2} \middle| \mathcal{F}_t \right\}$$

$$\cdot E^Q\left\{ e^{-\int_t^T \Psi_u^1 du} e^{\kappa B^1(T,T+\Delta)\Psi_T^1} \middle| \mathcal{F}_t \right\} E^Q\left\{ e^{-\int_t^T (\Psi_u^2)^2 du} \middle| \mathcal{F}_t \right\}$$

→ We can thus explicitly compute FRA prices and FRA rates. It leads also to an interesting relationship between pre-crisis and post-crisis values as shown in the following proposition.

FRAs, FRA rates and adjustment factors

Proposition. We have

$$\bar{\nu}_{t,T} = \nu_{t,T} \cdot Ad_t^{T,\Delta} \cdot Res_t^{T,\Delta}$$

where

$$Ad_t^{T,\Delta} := E^Q\left\{\frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} \mid \mathcal{F}_t\right\} = E^Q\Big\{\exp\Big[\tilde{A}(T,T+\Delta) + \kappa B^1(T,T+\Delta)\Psi_T^1 + \bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2\Big] \mid \mathcal{F}_t\Big\}$$

and

$$Res_t^{T,\Delta} = \exp\left[-\kappa\frac{(\sigma^1)^2}{2(b^1)^3}\left(1+e^{-b^1\Delta}\right)\left(1-e^{-b^1(T-t)}\right)^2\right]$$

→ *Convenient for calibration.*

Thank you for your attention

Optional derivatives/Caps

- As example of an optional derivative, we start from a Cap. We may limit ourselves to the pricing of just a single Caplet for the generic interval $[T, T+\Delta]$ (*single tenor*) and for a fixed rate R.
- In view of Cap pricing, recall that, under $Q^{T+\Delta}$, the factors Ψ^i_T are independent Gaussian with mean $\bar{\alpha}^i_T$ and variance $\bar{\beta}^i_T$, i.e.

$$f_{(\Psi^1_T, \Psi^2_T, \Psi^3_T)}(x, y, z) = \prod_{i=1}^{3} f_{\Psi^i_T}(\cdot) = \prod_{i=1}^{3} \mathcal{N}(\cdot, \bar{\alpha}^i_T, \bar{\beta}^i_T)$$

We shall use the shorthand $f_i(\cdot)$ for $f_{\Psi^i_T}(\cdot)$.

- We shall also use the shorthand $\bar{A}, B^1, C^{22}, \bar{C}^{33}$ for the corresponding functions, evaluated at $(T, T+\Delta)$ and put $\tilde{R} := 1 + \Delta R$.

Caps

The price, in $t = 0$, of a Caplet can now be written as

$$P^{Cpl}(0; T+\Delta, R) = \Delta\, p(0, T+\Delta) E^{Q^{T+\Delta}} \left\{ \left(L(T; T, T+\Delta) - R \right)^+ \right\}$$

$$= p(0, T+\Delta) E^{Q^{T+\Delta}} \left\{ \left(\frac{1}{\bar{p}(T, T+\Delta)} - \tilde{R} \right)^+ \right\}$$

$$= p(0, T+\Delta) E^{Q^{T+\Delta}} \left\{ \left(e^{\bar{A} + (\kappa+1) B^1 \Psi^1_T + C^{22} (\Psi^2_T)^2 + \bar{C}^{33} (\Psi^3_T)^2} - \tilde{R} \right)^+ \right\}$$

$$= p(0, T+\Delta) \int_{\mathbb{R}^3} \left(e^{\bar{A} + (\kappa+1) B^1 x + C^{22} y^2 + \bar{C}^{33} z^2} - \tilde{R} \right)^+ f_{(\Psi^1_T, \Psi^2_T, \Psi^3_T)}(x, y, z) d(x, y, z)$$

Caps

- Inspired by Jamshidian, consider the function

$$g(x, y, z) := \exp\left[\bar{A} + (\kappa + 1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2\right]$$

- *Noticing that* $\bar{C}^{33}(T, T + \Delta) > 0$*, for fixed* x, y*, the function* $g(x, y, z)$ *is continuous and increasing for* $z \geq 0$ *and decreasing for* $z < 0$ *with* $\lim_{z \to \pm\infty} g(x, y, z) = +\infty$*.*
  - → Define then $M := \{(x, y) \in \mathbb{R}^2 \mid g(x, y, 0) \leq \tilde{R}\}$ letting M^c be its complement.

Caps

- For $(x, y) \in M$, define the values $\bar{z}^1 = \bar{z}^1(x, y)$, $\bar{z}^2 = \bar{z}^2(x, y)$ as the solutions of $g(x, y, z) = \tilde{R}$ with $\bar{z}^1 \leq 0 \leq \bar{z}^2$.
- For $z \leq \bar{z}^1 \leq 0$ and $z \geq \bar{z}^2 \geq 0$ we have $g(x, y, z) \geq g(x, y, \bar{z}^k) = \tilde{R}$, while for $z \in (\bar{z}^1, \bar{z}^2)$ we have $g(x, y, z) < \tilde{R}$.
- In M^c we have $g(x, y, z) \geq g(x, y, 0) > \tilde{R}$ and thus no solution of the equation $g(x, y, z) = \tilde{R}$.

Our main result is now

Caps

Under the assumption $b^3 \geq \sigma^3/\sqrt{2}$ one has

$$P^{Cpl}(0; T+\Delta, R) = p(0, T+\Delta)\Big[\int_M e^{\bar{A}+(\kappa+1)B^1 x + C^{22}(y)^2}$$

$$\cdot\Big[\gamma(\bar{\alpha}^i_T, \bar{\beta}^i_T, \bar{C}^{33})\left(\Phi(d^1(x,y)) + \Phi(d^2(x,y))\right)$$
$$-e^{\bar{C}^{33}(\bar{z}^2)^2}\Phi(d^3(x,y)) + e^{\bar{C}^{33}(\bar{z}^1)^2}\Phi(d^4(x,y))\Big] f_1(x) f_2(y)\, dx dy$$

$$+\gamma(\bar{\alpha}^i_T, \bar{\beta}^i_T, \bar{C}^{33}) \int_{M^c} e^{\bar{A}+(\kappa+1)B^1 x + C^{22}(y)^2} f_1(x) f_2(y)\, dx dy$$
$$-\tilde{R}\, Q^{T+\Delta}\left\{(\Psi^1_T, \Psi^2_T) \in M^c\right\}\Big]$$

→ $\Phi(\cdot)$ is the cumulative std. Gaussian distribution function.

→ $d^i(x,y)$ depend on (x,y) via $z^2(x,y)$ for $i = 1, 2$ and via $z^1(x,y)$ for $i = 3, 4$; they depend also on $\bar{\alpha}^3_T, \bar{\beta}^3_T, \bar{C}^{33}$.

Swaptions

- For Swaption pricing one can proceed analogously to the Caps; in fact, the Swaption price can be expressed as a sum, over the intervals of the tenor structure, of expressions that have the same structure as that of the Caplets.
- The main difference is that, instead of comparing $g(x, y, z)$ with $\tilde{R}$, one has to consider a suitable function $h(x, y)$ and define

$$M := \{(x, y) \in \mathbb{R}^2 \mid g(x, y, 0) \leq h(x, y)\}$$

with M^c its complement.

→ One then defines, for $(x, y) \in M$, the values $\bar{z}^1 = \bar{z}^1(x, y)$, $\bar{z}^2 = \bar{z}^2(x, y)$ as the solutions of $g(x, y, z) = h(x, y)$.