Exponential ergodicity of the jump-diffusion CIR process

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on joint articles with P. Jin and C. Trabelsi

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$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

- Exp. ergodicity results for JCIR processes (will appear on Proceeding of the conference at CAS in Oslo (2014), Springer Verlag 2015)
- Exp. ergodicity results and Harris recurrence as well as density of BAJD (submitted to international journal)
- (joint also with V. Mandrekar) Harris recurrence and calibration results for CIR process (appeared on Comm. Stoch. An. Vol 7 (2013)).

Using Ergodicity for calibration

Idea: Let π be the invariant measure of an ergodic process X_s.

$$\frac{1}{t}\int_0^t f(X_s)ds \to \int f(x)\pi(dx)\,; \quad \text{time average} \to \ \text{average w.r.t.}\pi$$

Remark:

- the process depends on some constant a,θ,σ, as well as the invariant measure.
- Relation between process and invariant measure is known by stochastic analysis.
- Time avarage can be observed by hystorical data
- The constants can be found by time average, hence for the invariant measure and hence for the process

Jump-diffusion CIR

Consider

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

where $a, \sigma > 0$, $\theta \geq 0$, W_t is a 1-dimensional Brownian motion and J_t is a pure-jump Lévy process with its Lévy measure ν concentrating on $(0, \infty)$ and satisfying

$$\int_{(0,\infty)} (\xi \wedge 1)\nu(d\xi) < \infty.$$

Existence and uniqueness of strong solution follows from results of Fu and Li (2010)

Jump-diffusion CIR

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t$$

- X_t is mean-reverting
- only positive jumps are allowed
- X_t stays non-negative if $X_0 \geq 0$
- X_t is an affine process on $\mathbb{R}_+$
- It follows that it is a Feller process (see Duffie, Filipovic, Schachermeyer An. Appl. Prob 2003, or Keller -Ressel, Schachermeyer, Teichmann PTRF 2011)

JCIR as an affine process on $\mathbb{R}_+$

Characteristic function of X_t (given $X_0 = x$):

$$E_x\left[e^{uX_t}\right] = e^{\phi(t,u)+x\psi(t,u)}, \quad u \in \mathcal{U} := \{u \in \mathbb{C} : \Re u \le 0\},$$

where $\phi(t,u)$ and $\psi(t,u)$ solve

$$\begin{cases} \partial_t \phi(t,u) = F\big(\psi(t,u)\big), & \phi(0,u) = 0, \\ \partial_t \psi(t,u) = R\big(\psi(t,u)\big), & \psi(0,u) = u, \end{cases}$$

and

$$F(u) = a\theta u + \int_{(0,\infty)} (e^{u\xi} - 1)\nu(d\xi),$$

$$R(u) = \frac{\sigma^2 u^2}{2} - au.$$

By solving the ODEs one can get

$$\psi(t,u)=\frac{ue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}$$

and

$$\phi(t,u)=-\frac{2a\theta}{\sigma^2}\log\left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)+\int_0^t\int_{(0,\infty)}\left(e^{\xi\psi(s,u)}-1\right)\nu(d\xi)ds$$

Characteristic function of JCIR

$$
E_x[e^{uX_t}] = \left(1 - \frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right) \cdot \exp\left(\int_0^t \int_0^\infty \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right).
$$

Set

$$
I := \left(1 - \frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right)
$$

$$
II := \exp\left(\int_0^t \int_0^\infty \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right).
$$

Special case 1: ν=0, no jumps!

CIR

$$dY_t = a(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t, \quad Y_0 = x \geq 0$$

$$E_x[e^{uY_t}] = \left(1 - \frac{\sigma^2}{2a}u(1 - e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1 - \frac{\sigma^2}{2a}u(1 - e^{-at})}\right)$$

$$= I$$

- Y_t has a density function $f(t, x, y)$

Special case 2: θ=0 and $x = 0$

JCIR (θ=0 and $x = 0$):

$$dZ_t = -aZ_t dt + \sigma\sqrt{Z_t} dW_t + dJ_t, \quad Z_0 = 0 \geq 0$$

$$E_x[e^{uZ_t}] = \exp\left(\int_0^t \int_{(0,\infty)} \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right) = II$$

where

$$\psi(t,u) = \frac{ue^{-at}}{1 - \frac{\sigma^2}{2a}u(1 - e^{-at})}$$

- II resembles the characteristic function of a compound poisson distribution

Remember that

$$II = E_x[e^{uZ_t}] = \exp\left(\int_0^t \int_{(0,\infty)} \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right)$$

where

$$\psi(t,u) = \frac{ue^{-at}}{1 - \frac{\sigma^2}{2a}u(1-e^{-at})}$$

Theorem 1

Suppose that $\int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty$. Then II is the characteristic function of a compound poisson distribution. In particular, $\mathbb{P}_x(Z_t = 0) > 0$.

Theorem 2 (Lower bound for the transition density of JCIR)

Suppose that $\int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty$. Then for all $A \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{P}(X_t \in A) \geq C(t) \int_A f(t,x,y)dy,$$

where $C(t) > 0$ and $f(t,x,y)$ is the transition density of the CIR process without jumps.

Corollary: if $\int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty$, then the JCIR process X_t is irreducible

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

Theorem 3 (Existence of a Lyapunov function for JCIR)

Suppose that $\int_{(1,\infty)} \xi \, \nu(d\xi) < \infty$. Then the function $V(x) = x, \ x \geq 0$ is a Lyapunov funciton for the JCIR process X_t, i.e. for all $t > 0, \ x \geq 0$,

$$E_x[V(X_t)] \leq e^{-at}V(x) + M,$$

where $0 < M < \infty$ is a constant.

Theorem 4

Let π be the unique invariant probability measure for the JCIR. Suppose that

$$\int_{(1,\infty)} \xi\ \nu(d\xi) < \infty \quad \text{and} \quad \int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty.$$

Then the JCIR process X_t is exponential ergodic, namely there exist constants $0 < \beta < 1$ and $0 < B < \infty$ such that

$$\|P_t(x,\cdot) - \pi\|_{TV} \leq B\big(x+1\big)\beta^t, \quad t \geq 0, \quad x \in \mathbb{R}_+.$$

where $\|\cdot\|_{TV}$ denotes the total-variation norm for signed measures on $\mathbf{R}_+$, namely

$$\|\mu\|_{TV} = \sup_{A \in \mathcal{B}(\mathbf{R}_+)} \{|\mu(A)|\}.$$

Proof: For any $\delta > 0$ we consider the δ-skeleton chain $Y_n^\delta := X_{n\delta},\ n \in \mathbb{Z}_+$. Then $(Y_n^\delta)_{n\in\mathbb{Z}_+}$ is a Markov chain with transition kernel $p(\delta, x, y)$ on the state space $\mathbb{R}_+$ with same invariant measure π.

- The CIR process Y_t is irreducible, aperiodic.
- It follows as Corollary that he JCIR process X_t is irreducible, aperiodic. (This is shown using Theorem 2)
- JCIR is a Feller Process
- There exists a Foster -Lyapunov function.

The BAJD -process was introduced by Duffie and Garleanu (2001) and analyzed also by Filipov (2001) and Keller -Ressel and Steiner (2008). It is the unique strong solution of

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0, \tag{1}$$

where a, θ, σ are positive constants, W_t is a 1-dimensional Brownian motion and J_t is a pure-jump Lévy process with the Lévy measure

$$\nu(dy) = \begin{cases} cde^{-dy}dy, & y \geq 0, \\ 0, & y < 0, \end{cases}$$

for some constants $c \in \mathbb{R}_+$ and $d > 0$.

By solving the system one can get $\phi(t,u)$, $\psi(t,u)$ and finally that the characteristic function of BAJD process X_t is

$$E_x[e^{uX_t}] =$$

$$\begin{cases} \left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \left(\frac{d-\frac{\sigma^2 du}{2a}+\left(\frac{\sigma^2 d}{2a}-1\right)ue^{-at}}{d-u}\right)^{\frac{c}{a-\frac{\sigma^2 d}{2}}} \\ \quad \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right), & \text{if } d \neq d_0, \\ \left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \exp\left(\frac{cu(1-e^{-at})}{a(d-u)}\right) \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right), & \text{if } d = d_0 \end{cases}$$

where $d_0 = \dfrac{2a}{\sigma^2}$.
Remember: if $f(t,x,y)$ is the density of CIR -process, then

$$\int_{\mathbb{R}_+} f(t,x,y)e^{uy}dy = \left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right).$$

In [Jin, R., Trabelsi, submitted] **we find an explicit formula for the density** $p(t,x,y)$ of the BAJD -process, which in particular has the following form: in all cases we get

$$p(t,x,y) = L(t)f(t,x,y) + \int_0^y f(t,x,y-z)h(t,z)dz, \qquad (2)$$

where $f(t,x,y)$ is the density of the CIR process, and where $L(t)$ is continuous in $t>0$, $0<L(t)<1$ and $h(t,z)$ is non-negative, continuous in $(t,z)\in(0,\infty)\times[0,\infty)$ and $\int_R h(t,z)dz = 1-L(t)$. (see also Filipovic 2001, in case of particular choice of parameter) Moreover we prove **Theorem** [Jin, R., Trabelsi, submitted]

Theorem

The BAJD- process is positive Harris recurrent.

Definition

A continuous-time Markov process (X_t) with state space $(S, \mathcal{S})$ is said to be Harris recurrent if for some σ-finite measure ρ

$$P_x\Big(\int_0^\infty \mathbf{1}_{\{X_s \in A\}} ds = \infty\Big) = 1,$$

for any $x \in S$ and $A \in \mathcal{S}$ with $\rho(A) > 0$.

Definition

A Markov process (X_t) is said to be uniformly transient if

$$\sup_x E_x\Big[\int_0^\infty \mathbf{1}_K(X_t) dt\Big] < \infty$$

for every compact $K \subset S$.

Theorem (Kunita)

A regular Feller Process is either Harris Recurrent or Uniformly transient

Result [Jin, R., Trabelsi submitted

Theorem

The BAJD process is Harris recurrent.

Sketch of the proof

Lemma

The BAJD process is not uniformly transient.

Regularity Property

Lemma

BAJD *process is a regular Feller process on* $\mathbb{R}_+$.

Definition: An $\mathbb{R}_+$ valued Feller process X is said to be regular if there exists a locally finite measure ρ on $\mathbb{R}_+$ and a continuous function $(t, x, y) \mapsto p_t(x, y) > 0$ on $(0, \infty) \times \mathbb{R}_+^2$ such that

$$P_x\{X_t \in B\} = \int_B p_t(x, y)\rho(dy), \quad x \in S,\ B \in \mathbb{R}_+,\ t > 0.$$

The measure ρ is called "supporting measure".

Sketch of the proof: We show that the BAJD process has same supporting measure ρ as the CIR process. (see Jin,Mandrekar, R. Trabelsi COSA 2013 for the CIR process) In fact the supporting measure ρ for the CIR process can not be the Lesbegues measure dy: the behaviour of the density $f(t,x,y)$ of the CIR process violates the regularity condition w.r.t dy at point $y=0$.
if $\frac{2b}{\sigma^2}<1$, $f(t,x,0):=\lim_{y\to 0} f(t,x,y)=\infty$ and if $\frac{2b}{\sigma^2}>1$, then $f(t,x,0):=\lim_{y\to 0} f(t,x,y)=0$
To overcome this difficulty, we define a new measure ρ on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ as

$$\rho(dx):=h(x)dx,$$

where

$$h(x)=\begin{cases} x^{\frac{2b}{\sigma^2}-1}, & 0\le x\le 1,\\ 1, & x>1. \end{cases}$$

Then the transition densities of the BAJD process with respect to the new measure ρ is given by

$$\tilde{p}(t,x,y) = \frac{p(t,x,y)}{h(y)}, \quad t>0,\ x\geq 0,\ y>0. \tag{3}$$

and are regular with supporting measure ρ

- Harris recurrence guarantees the existence of a unique (up to multiplication by a constant) invariant measure for the Markov process.

- If this invariant measure is finite, then the process is called "positive Harris recurrent"

- The BAJD process has unique probability invariant measure π [keller -Ressel, Math. Finance 2011, or Keller -Ressel , Steiner , Finance Stoch. 2014]. The characteristic function of π was given in [Keller -Ressel , Steiner , Finance Stoch. 2014]

- Its unique invariant probability measure π is absolute continuous with respect to the Lebesgue measure and thus has a density function $l(\cdot)$, which can be explicitely defined, i.e. $\pi(dx) = l(dx)dx$ [Jin,R. Trabelsi, submitted]

The BAJD is positive Harris recurrent. Using [Theorem 20.21, Kallenberg's book] we obtain

Corollary

Let $X = (X_t)_{t \geq 0}$ be the BAJD. Then for any $f \in \mathcal{B}_b(\mathbb{R}_+)$ we have

$$\frac{1}{t} \int_0^t f(X_s) ds \;\rightarrow\; \int_{\mathbb{R}_+} f(x) \pi(dx) \quad a.s.$$

as $t \rightarrow \infty$, where π is the unique invariant probability measure of the BAJD.

Using Ergodicity for calibration

Idea: Let π be the invariant measure of an ergodic process X_s.

$$\frac{1}{t}\int_0^t f(X_s)ds \rightarrow \int f(x)\pi(dx)\,; \quad \text{time average} \rightarrow \text{ average w.r.t.}\pi$$

Remark:

- the process depends on some constant a,θ,σ, as well as the invariant measure.
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Introduction to some Application of the insurance company DeBeKa

Calibration of a model describing the credit migration model of Hurd and Kuznetsov. Consider the finite state space $\{1, 2, \cdots, 8\}$, which can be identified with Moody's rating classes via the mapping:

$$\{1, 2, \cdots, 8\} \leftrightarrow \{\text{AAA, AA, A}, \cdots, \text{default}\}.$$

The credit migration matrix $P(s,t), 0 \leq s \leq t$, is a stochastic 8×8 matrix and describes all possible transition probabilities between rating classes from time s to time t, namely

$$P(s,t) = \Big(p_{ij}(s,t)\Big)_{1 \leq i,j \leq 8}$$

where $p_{ij}(s,t)$ represents the transition probability from state i to state j from time s to time t.

Hurd and Kuznetsov (2007) assumed that the migration matrix $P(s,t)$ is given through the CIR process Z_r by

$$P(s,t) = \exp\left(\left(\int_s^t Z_r dr\right) \cdot P\right) \tag{4}$$

$$\begin{aligned} P(s,t) &= \exp\left(\left(\int_s^t Z_s ds\right) \cdot P\right) \\ &= G \cdot \left(e^{-D_{ii} \int_s^t Z_s ds}\right) \cdot G^{-1} \end{aligned}$$

2009 One-Year Letter Migration Rates

FROM/TO:	Aaa	Aa	A	Baa	Ba	B	Caa	Ca_C	DEFAULT	WR
Aaa	62.42%	33.76%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	3.82%
Aa	0.00%	70.98%	22.62%	1.04%	0.15%	0.00%	0.00%	0.00%	0.00%	5.21%
A	0.00%	0.18%	80.20%	12.61%	0.44%	0.53%	0.00%	0.00%	0.18%	5.86%
Baa	0.00%	0.09%	0.93%	85.38%	5.12%	0.84%	0.09%	0.00%	0.74%	6.80%
Ba	0.00%	0.00%	0.00%	3.85%	71.54%	13.27%	0.77%	0.58%	2.31%	7.69%
B	0.00%	0.00%	0.00%	0.00%	2.88%	68.35%	13.46%	0.41%	6.99%	7.91%
Caa	0.00%	0.00%	0.00%	0.00%	0.00%	7.59%	48.81%	6.51%	28.20%	8.89%
Ca-C	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	4.76%	20.63%	65.08%	9.52%

- Migration matrix from time s to t as

$$
\begin{aligned}
P(s,t) &= \exp\left(\left(\int_s^t Z_s ds\right)\cdot P\right) \\
&= G\cdot\begin{pmatrix} e^{-D_{11}\int_s^t Z_s ds} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & e^{-D_{88}\int_s^t Z_s ds} \end{pmatrix}\cdot G^{-1}
\end{aligned}
$$

• For each $1 \le i \le 8$ we can apply our main result by taking $g(x) = D_{ii}x$ and $f(x) = e^{-x}$ and we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{-D_{ii}\int_j^{j+1} Z_s ds} = \mathrm{E}_\mu\left[e^{-D_{ii}\int_0^1 Z_s ds}\right]$$

and the convergence holds almost surely.
• Since the Laplace transform of $\int_0^1 Z_s ds$ is well-known and we can identify the constant:

$$E_\mu\left[e^{-D_{ii}\int_0^1 Z_s ds}\right] = e^{aD_{ii}A(0,D_{ii},1)}\left(\frac{2a}{2a-\sigma^2 D_{ii}B(0,D_{ii},1)}\right)^{\frac{2a}{\sigma^2}}$$

$$-D_{Moody,ii} = aD_{ii}A(0,D_{ii},1) + \frac{2a}{\sigma^2}\log\left(\frac{2a}{2a-\sigma^2 D_{ii}B(0,D_{ii},1)}\right).$$

for each $1 \le i \le 8$.

Strong ergodicity result we used for CIR process, when we calibrated credit migration model of Hurd and Kuznetsov for Moody rating classes depending on CIR

Theorem (P.Jin, V. Mandrekar, B.R., C. Trabelsi, COSA 2013)

Suppose that $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}_+$ is measurable. Then for any $z \in \mathbb{R}_+$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g(Z_s)(\omega) ds \Big) = E_\mu \Big[f\Big(\int_0^1 g(Z_s)(\omega) ds \Big) \Big]$$

for P_z-almost all $\omega \in \Omega$, where μ is the unique invariant probability measure for the CIR process.

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