Pathwise analysis and robustness of hedging strategies for path-dependent options

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Outline

Model Ambiguity and Hedging Issues

Classical Framework: Traded assets $X = (X(t))_{t\in[0,T]}$ modeled as $\mathbb{R}_+^d$-valued semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$

- The choice of $\mathbb{P}$ may be challenged 'a la De Finetti' (Knightian uncertainty)
  $\rightarrow$ Our approach: we set up a probability-free financial model
- The gain process is a stochastic integral, thus
  - it is not necessarily defined for a given path/price scenario
  - scenario analysis and stress tests cannot be performed

  $\rightarrow$ In our setting: for a certain class of trading strategies, the gain process is well-defined path-by-path (as a limit of Riemann sums)
- Robustness analyses are based on the existence of a 'true model', and study the performance of a 'mis-specified model'
  $\rightarrow$ Our analysis: we study the performance and robustness of hedging strategies in given sets of scenarios.

Robustness of Hedging Strategies

Consider a market participant who sells an (exotic) option with payoff H and maturity T on some underlying asset, at a model price given by

$$V(t) = E^{\mathbb{Q}}[H|\mathcal{F}_t]$$

and hedges the resulting Profit/Loss using the hedging strategy derived from the same model (say, Black-Scholes delta hedge). The *actual* dynamics of the underlying asset may, of course, be different from the assumed dynamics.

- How good is the result of the hedging strategy?
- How 'robust' is it to model mis-specification?
- How does the hedging error relate to model parameters and option characteristics?

Robustness of Hedging Strategies

El Karoui, Jeanblanc & Shreve (1998) provided an answer to these important questions, for non-path-dependent options, when the underlying dynamics is

$$\mathrm{d}S(t) = S(t)r(t)\mathrm{d}t + S(t)\sigma(t)\mathrm{d}W(t) \qquad \text{under } \mathbb{Q}$$

such that S is square-integrable. Then a hedging strategy, computed in a (mis-specified) Markovian model

$$\mathrm{d}S(t) = S(t)r(t)\mathrm{d}t + S(t)\sigma_0(t, S(t))\mathrm{d}W(t)$$

with local volatility σ_0 leads to a profit

$$\int_0^T \frac{\sigma_0^2(t, S(t)) - \sigma^2(t)}{2} S(t)^2 e^{\int_t^T r(s)\mathrm{d}s} \overbrace{\partial_{xx}^2 f(t, S(t))}^{\Gamma(t)} \mathrm{d}t$$

where f is the unique solution of the PDE

$$\partial_t f + r(t)x\partial_x^2 f + \sigma_0^2(t,x)x^2\partial_{xx}^2 f/2 = r(t)f \quad f(T,x) = H(x)$$

Notation: Non-Anticipative Functionals

Given $x \in D([0,T],\mathbb{R}^d)$, for all $t \in [0,T]$ we denote:

- $x(t) \in \mathbb{R}^d$ the value of x at t
- $x_t = x(t \wedge \cdot) \in D([0,T],\mathbb{R}^d)$ the path stopped at t
- $x_{t-} = x\mathbb{1}_{[0,t)} + x(t-)\mathbb{1}_{[t,T]} \in D([0,T],\mathbb{R}^d)$
- for $\delta \in \mathbb{R}^d$, $x_t^\delta = x_t + \delta\mathbb{1}_{[t,T]} \in D([0,T],\mathbb{R}^d)$

We define the **space of stopped paths**:

$$\Lambda_T := \Big([0,T] \times D([0,T],\mathbb{R}^d)\Big) / \sim,$$

where $(t,x) \sim (t',x') \iff t = t'$ and $x_t = x'_t$, and the metric

$$\mathrm{d}_\infty((t,x),(t',x')) = \sup_{u\in[0,T]} \left|x(u \wedge t) - x'(u \wedge t')\right| + |t - t'|$$

Definition (Non-anticipative functional)

A **non-anticipative functional** *on* $D([0,T],\mathbb{R}^d)$ *is a map* $F : (\Lambda_T, d_\infty) \to \mathbb{R}$.

Notation: Regularity of $F : \Lambda \to \mathbb{R}$

A non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ is said to be:

- *continuous at fixed times* if for all $t \in [0, T]$

$$F(t, \cdot) : \left(\left(\{t\} \times D([0,T], \mathbb{R}^d) \right) / \sim, ||\cdot||_\infty \right) \mapsto \mathbb{R}$$

  is continuous

- *left-continuous*, $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$, if

$$\forall (t,x) \in \Lambda_T,\ \forall \epsilon > 0,\ \exists \eta > 0 : \ \forall h \in [0,t],\ \forall (t-h, x') \in \Lambda_T,$$
$$\mathrm{d}_\infty((t,x),(t-h,x')) < \eta \quad \Rightarrow \quad |F_t(x) - F_{t-h}(x')| < \epsilon$$

- *boundedness-preserving*, $F \in \mathbb{B}(\Lambda_T)$, if, $\forall K \subset \mathbb{R}^d$ compact, $\forall t_0 \in [0, T]$, $\exists C_{K,t_0} > 0$ s.t. $\forall t \in [0, t_0]$, $\forall (t,x) \in \Lambda_T$

$$x([0,t]) \subset K \Rightarrow |F_t(x)| < C_{K,t_0}$$

Notation: Differentiability of $F : \Lambda \to \mathbb{R}$

Definition (Horizontal derivative)

A non-anticipative functional F is **horizontally differentiable** *at $(t, x) \in \Lambda_T$ if $t \mapsto F(t, x_t)$ is right-differentiable, with derivative denoted by* $\mathcal{D}\mathbf{F}(\mathbf{t}, \mathbf{x})$*; if it holds $\forall (t, x) \in \Lambda_T$, then we denote* $\mathcal{D}\mathbf{F} := (\mathcal{D}F(t, \cdot))_{t \in [0,T)}$

Definition (Vertical derivative)

A non-anticipative functional F is **vertically differentiable at** *$(t, x) \in \Lambda_T$ if the map $\mathbb{R}^d \ni e \mapsto F(t, x_t^e)$ is differentiable at 0; in this case we denote* $\nabla_\omega \mathbf{F}(\mathbf{t}, \mathbf{x}) := (\partial_i F(t, x))_{i=1,\dots,d}$ *, where*

$$\partial_i F(t, x) := \lim_{h \to 0^+} \frac{F(t, x_t^{he_i}) - F(t, x_t)}{h}.$$

If this holds for all $(t, x) \in \Lambda_T$, then $\nabla_\omega \mathbf{F} := (\nabla_\omega F(t, \cdot))_{t \in [0,T]}$

Sets of Smooth Non-Anticipative Functionals

Definition ($\mathbb{C}_b^{1,2}(\Lambda)$ and $\mathbb{C}_{loc}^{1,2}(\Lambda)$)

Denote by $\mathbb{C}_{\mathbf{b}}^{\mathbf{1,2}}(\mathbf{\Lambda_T})$ *the set of non-anticipative functionals* $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ *such that:*

- $\exists \mathcal{D}F$ *continuous at fixed times,*
- $\exists \nabla_\omega^j F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ $j = 1, 2,$
- $\mathcal{D}F, \nabla_\omega F, \nabla_\omega^2 F \in \mathbb{B}(\Lambda_T)\}.$

Denote by $\mathbb{C}_{\mathbf{loc}}^{\mathbf{1,2}}(\mathbf{\Lambda_T})$ *the set of non-anticipative functionals* $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ *such that there exists a sequence of stopping times* $(\tau_k)_{k\geqslant 1}$ *going to* ∞ *and a sequence* $(F^k \in \mathbb{C}_b^{1,2}(\Lambda_T))_{k\geqslant 1}$,

$$F(t, x_t) = \sum_{k\geqslant 1} F^k(t, x_t) \mathbb{1}_{[\tau_k(x), \tau_{k+1}(x))}(t) \quad \forall (t, x) \in \Lambda_T$$

Quadratic Variation for Càdlàg Paths

Fixed sequence of time partitions: $\Pi = \{\pi_n\}_{n \geqslant 1}$,
$\pi_n = (t_i^n)_{i=0,\dots,m(n)}$, $0 = t_0^n < \dots < t_{m(n)}^n = T$, $|\pi_n| \xrightarrow[n\to\infty]{} 0$

Definition (Paths of finite quadratic variation)

- $x \in D([0,T],\mathbb{R})$ *is of* **finite quadratic variation along** Π *if*
$$\forall t \in [0,T], \quad [x](t) := \lim_{n\to\infty} \sum_{t_i^n \leqslant t} \left(x(t_{i+1}^n) - x(t_i^n) \right)^2 < \infty$$
- $x \in D([0,T],\mathbb{R}^d)$ *is of finite quadratic variation along* Π *if,* $\forall 1 \leqslant i,j \leqslant d$, $x^i, x^i + x^j$ *are so. In this case:*
$$[x]_{i,j}(t) \equiv [x^i, x^j](t) = \frac{1}{2}\left([x^i + x^j](t) - [x^i](t) - [x^j](t) \right)$$

Denote $\mathbf{Q}(\mathbf{U},\mathbf{\Pi}) := \{x \in U \subset D([0,T],\mathbb{R}^d),\ x \text{ is of f.q.v. along } \Pi\}$

For every càdlàg path $\omega \in Q(D([0,T],\mathbb{R}^d),\Pi)$, we can always assume that $\sup_{t\in[0,T]-\pi^n} |\Delta\omega(t)| \xrightarrow[n\to\infty]{} 0$.

Change of Variable Formula for Functionals

The piecewise constant approximation

$$\omega^n := \sum_{i=0}^{m(n)-1} \omega(t_{i+1}^n -)\mathbb{1}_{[t_i^n, t_{i+1}^n)} + \omega(T)\mathbb{1}_{\{T\}},$$

converges uniformly to ω

Theorem (Cont, Fournié (2010))

If $F \in \mathbb{C}^{1,2}_{loc}(\Lambda_T)$ and $\omega \in Q(D([0,T],\mathbb{R}^d),\Pi)$, then the limit

$$\int_0^T \nabla_\omega F_t(\omega_{t-}) \mathrm{d}^\Pi \omega := \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F_{t_i^n}(\omega_{t_i^n -}^{n,\Delta\omega(t_i^n)}) \cdot (\omega(t_{i+1}^n) - \omega(t_i^n))$$

exists and

$$F_T(\omega_T) = F_0(\omega_0) + \int_0^T \nabla_\omega F_t(\omega_{t-}) \mathrm{d}^\Pi \omega + \int_0^T \mathcal{D}_t F(\omega_{t-}) \mathrm{d}t + \frac{1}{2}\int_{(0,T]} \mathrm{tr}\left(\nabla_\omega^2 F_t(\omega_{t-}) \mathrm{d}[\omega]^c(t)\right) + \sum_{t\in(0,T]} \left(F_t(\omega_t) - F_t(\omega_{t-}) - \nabla_\omega F_t(\omega_{t-}) \cdot \Delta\omega(t)\right)$$

Functional Itô Formula for Functionals of Semimartingales

Theorem (Cont, Fournié (2013))

If $F \in \mathbb{C}^{1,2}_{loc}(\Lambda_T)$ and X is an $\mathbb{R}^d$-valued semimartingale, then a.s.

$$F(T, X_T) = F(0, X_0) + \int_0^T \nabla_\omega F(t, X_{t-}) \cdot \mathrm{d}X + \int_0^T \mathcal{D}F(t, X_{t-})\mathrm{d}t + \frac{1}{2}\int_{(0,T]} \mathrm{tr}\left(\nabla^2_\omega F(t, X_{t-})\mathrm{d}[X]^c(t)\right) + \sum_{t \in (0,T]} \left(F(t, X_t) - F(t, X_{t-}) - \nabla_\omega F(t, X_{t-}) \cdot \Delta X(t)\right)$$

In particular, Y defined by $Y(t) = F(t, X_t)$ for all $t \in [0, T]$, is a semimartingale

A Pathwise Setting for Continuous-Time Trading

Financial market: $(\Omega, ||\cdot||_\infty)$ metric space, $\Omega := D([0,T],\mathbb{R}^d_+)$,
$\omega \in \Omega$ is a possible trajectory of the (forward) asset prices,
$\mathcal{F}$ is the Borel σ-field and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the canonical filtration

Trading strategies: (V_0, ϕ, ψ), where

V_0	$\Omega \mapsto \mathbb{R}$, $\mathcal{F}_0$-measurable	(initial investment)
ϕ	$\mathbb{R}^d$-valued $\mathbb{F}$-adapted càglàd process	(asset position)
ψ	$\mathbb{R}$-valued $\mathbb{F}$-adapted càglàd process	(bond position)

Portfolio value at time $t \in [0,T]$ along the price path $\omega \in D([0,T],\mathbb{R}^d_+)$:

$$V(t,\omega;\phi,\psi) = \phi(t,\omega)\cdot\omega(t) + \psi(t,\omega)$$

When is a strategy self-financing?
When can we explicitly define its gain?

Simple Trading Strategies

Definition (Simple self-financing trading strategies)

(V_0, ϕ, ψ) *is a* *simple trading strategy* *if* $\phi \in \Sigma(\mathbb{R}^d, \Pi)$ *and* $\psi \in \Sigma(\mathbb{R}, \Pi)$, *where* $\Sigma(U, \Pi) := \bigcup_{n \geqslant 1} \Sigma(U, \pi^n)$,

$$\Sigma(U, \pi^n) := \Big\{ \phi : \ \forall i = 0, \ldots, m(n) - 1, \quad \exists \lambda_i : \Omega \to U \ \mathcal{F}_{t_i^n}\text{-measurable}, \quad \phi(t, \omega) = \sum_{i=0}^{m(n)-1} \lambda_i(\omega) \mathbb{1}_{(t_i^n, t_{i+1}^n]} \Big\}$$

A simple trading strategy (V_0, ϕ, ψ) *is a* *self-financing* *if there exists* $n \geqslant 1$ *such that* $\phi, \psi \in \Sigma(\pi^n)$ *and*

$$\psi(t, \omega) := V_0 - \phi(0+, \omega) \cdot \omega(0) - \sum_{i=1}^{m(n)} \omega(t_i^n \wedge t) \cdot (\phi(t_{i+1}^n \wedge t, \omega) - \phi(t_i^n \wedge t, \omega))$$

Equivalently: $V(t, \omega; \phi) = V_0 + G(t, \omega; \phi)$, *where*

$$G(t, \omega; \phi) := \sum_{i=1}^{m(n)} \phi(t_i^n \wedge t, \omega) \cdot (\omega(t_i^n \wedge t) - \omega(t_{i-1}^n \wedge t))$$

Self-financing Trading Strategies

Definition (Self-financing Trading Strategies on U)

(V_0, ϕ) is a self-financing trading strategy on $U \subset D([0,T], \mathbb{R}^d_+)$ if there exists a sequence $\{(V_0, \phi^n, \psi^n), n \in \mathbb{N}\}$ of self-financing simple trading strategies, such that

$$\forall \omega \in U, \forall t \in [0,T], \quad \phi^n(t,\omega) \xrightarrow[n\to\infty]{} \phi(t,\omega),$$

and any of the following equivalent conditions is satisfied:

(I) $\exists \mathbb{F}$-adapted càdlàg process $G(\cdot,\cdot;\phi)$, $\quad \forall t \in [0,T], \omega \in U$
$$G(t,\omega;\phi^n) \xrightarrow[n\to\infty]{} G(t,\omega;\phi), \quad \Delta G(t,\omega;\phi) = \phi(t,\omega)\Delta\omega(t);$$

(II) $\exists \mathbb{F}$-adapted càdlàg process $\psi(\cdot,\cdot;\phi)$, such that $\forall t \in [0,T], \omega \in U$ $\psi^n(t,\omega) \xrightarrow[n\to\infty]{} \psi(t,\omega;\phi)$ and
$$\psi(t+,\omega;\phi) - \psi(t,\omega;\phi) = -\omega(t)\left(\phi(t+,\omega) - \phi(t,\omega)\right);$$

(III) $\exists \mathbb{F}$-adapted càdlàg process $V(\cdot,\cdot;\phi)$, $\quad \forall t \in [0,T], \omega \in U$
$$V(t,\omega;\phi^n) \xrightarrow[n\to\infty]{} V(t,\omega;\phi), \quad \Delta V(t,\omega;\phi) = \phi(t,\omega)\Delta\omega(t).$$

Gain: Pathwise Construction

Proposition

If there exists $F \in \mathbb{C}^{1,2}_{loc}(\Lambda_T) \cap \mathbb{C}^{0,0}_r(\Lambda_T)$, $\nabla_\omega F \in \mathbb{C}^{0,0}(\Lambda_T)$,

$$\phi(t,\omega) = \nabla_\omega F(t, \omega_{t-}) \quad \forall \omega \in Q(\Omega, \Pi),\ t \in [0, T],$$

Then, ϕ *is the asset position of a self-financing trading strategy on* $Q(\Omega, \Pi)$ *with gain*

$$G(t,\omega;\phi) = \int_0^t \phi(u, \omega_u)\mathrm{d}^\Pi\omega = \lim_{n\to\infty} \sum_{t_i^n < t} \nabla_\omega F(t_i^n, \omega^n_{t_i^n -}) \cdot (\omega(t^n_{i+1}) - \omega(t^n_i))$$

and bond position

$$\psi(t,\omega;\phi) := V_0 - \phi^+(0,\omega) + - \lim_{n\to\infty} \sum_{i=1}^{m^n} \omega(t_i^n \wedge t) \cdot (\phi^n(t^n_{i+1} \wedge t, \omega) - \phi^n(t^n_i \wedge t, \omega))$$

Hedging Error and Super-Strategies

Definition (Hedging error in specific scenarios)

The hedging error *of a self-financing trading strategy* (V_0, ϕ) *on* $U \subset D([0,T], \mathbb{R}^d_+)$ *for a path-dependent derivative with payoff* H *in a scenario* $\omega \in U$ *is the value*

$$V(T, \omega; \phi) - H(\omega) = V_0(\omega) + G(T, \omega; \phi) - H(\omega).$$

(V_0, ϕ) *is called a* super-strategy *for* H *on* U *if its hedging error for* H *is non-negative on* U*, i.e.*

$$V_0(\omega) + G(T, \omega; \phi) \geqslant H(\omega_T) \quad \forall \omega \in U.$$

Given $A \in D([0,T], \mathcal{S})$, $\mathcal{S} := \{M \in \mathbb{R}^{d \times d}, M > 0 \text{ symmetric}\}$,

$$Q_A(\Pi) := \left\{ \omega \in Q(\Omega, \Pi) \text{ such that } [\omega](t) = \int_0^t A(s)\mathrm{d}s \quad \forall t \in [0,T] \right\}$$

Pathwise Replication of (Exotic) Derivatives

Proposition (Pathwise replication of exotic derivatives)

If $H : (\Omega, \|\cdot\|_\infty) \mapsto \mathbb{R}$ is continuous and $F \in \mathbb{C}^{1,2}_{loc}(\mathcal{W}_T) \cap \mathbb{C}^{0,0}(\mathcal{W}_T)$ solves

$$\begin{cases} \mathcal{D}F(t,\omega_t) + \frac{1}{2}\mathrm{tr}\left(\nabla^2_\omega F(t,\omega_t)\cdot A(t)\right) = 0, & t \in [0,T) \\ F(T,\omega) = H(\omega), & \forall \omega \in Q_A(\Pi) \end{cases}$$

Then, the hedging error of the trading strategy $(F_0(\omega_0), \nabla_\omega F)$, self-financing on $Q(C([0,T],\mathbb{R}^d))$, in any scenario $\omega \in Q_{\tilde{A}}(\Pi)$ is

$$\frac{1}{2}\int_0^T \mathrm{tr}\left(\nabla^2_\omega F(t,\omega_t)\cdot (A(t) - \tilde{A}(t))\right)\mathrm{d}t$$

In particular, in all scenarios $\omega \in Q_A(\Pi)$, $(F(0,\omega_0), \nabla_\omega F)$ replicates H and its portfolio value at any time $t \in [0,T]$ is given by $F(t,\omega_t)$.

A Hedging Formula for Path-Dependent Options

Let $\Omega := C([0,T], \mathbb{R}_+)$ and S denote the coordinate process on the canonical space $(\Omega, \mathcal{F}, \mathbb{F})$, i.e. $S(t,\omega) = \omega(t)$, $\forall \omega \in \Omega$, $t \in [0,T]$.

Assumption (Hedger's model assumption)

The market participant assumes that the underlying asset price S evolves according to $\mathrm{d}S(t) = \sigma(t)S(t)\mathrm{d}W(t)$, i.e.

$$S(t) = S(0)e^{\int_0^t \sigma(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t \sigma(u)^2 \mathrm{d}u}, \quad t \in [0,T], \tag{1}$$

where W is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the volatility σ is a non-negative $\mathbb{F}$-adapted process such that $\sigma \neq 0$ $\mathrm{d}t \times \mathrm{d}\mathbb{P}$-almost surely and S is a square-integrable $\mathbb{P}$-martingale.

The discounted *price* at time t of a path-dependent derivative with payoff $H(S_T) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$Y(t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathbb{P}\text{-a.s.}$$

A Universal Hedging Equation

Theorem (Universal hedging formula (Cont-Fournié, 2013))

If $\mathbb{E}^{\mathbb{P}}[|H(S_T)|^2] < \infty$*, then* $Y \in \mathcal{M}^2(\mathbb{F})$ *and the following martingale representation formula holds:*

$$H(S_T) = Y(0) + \int_0^T \nabla_S Y(u) \cdot \mathrm{d}S(u) \quad \mathbb{P}\text{-a.s.}$$

Moreover, if $Y \in \mathbb{C}_b^{1,2}(S)$*, where*

$$\mathbb{C}_b^{1,2}(S) := \{Y : \exists F \in \mathbb{C}_b^{1,2}(\mathcal{W}_T),\ Y(t) = F(t, S_t)\ \mathbb{P}\text{-a.s.}\ \forall t \in [0,T]\},$$

then the hedging strategy for H *is pathwise defined by*
$\nabla_S Y = \nabla_\omega F(\cdot, S_\cdot)\ \mathrm{d}t \times d\mathbb{P} - a.s.$

In case $Y \in \mathcal{M}^2(\mathbb{F})$ but $Y \notin \mathbb{C}_b^{1,2}(S)$, the hedging strategy $\nabla_S Y$ for H is a 'weak vertical derivative', yet it can be uniformly approximated by regular functionals (i.e. Lu-Cont 2015)

A Universal Pricing Equation

Theorem (Pricing equation for path-dependent derivatives)

Consider a path-dependent derivative with payoff $H(S_T)$. If

$$\exists F \in \mathbb{C}_b^{1,2},\ F(t,S_t) = E^{\mathbb{P}}[H(S_T)|\mathcal{F}_t]$$

such that $\mathcal{D}F \in \mathbb{C}_l^{0,0}(\mathcal{W}_T)$*, then* F *is the unique solution of the pricing equation*

$$\mathcal{D}F(t,\omega_t) + \frac{1}{2}\mathrm{tr}\left(\nabla_\omega^2 F(t,\omega_t)\cdot\sigma^2(t)\omega^2(t)\right) = 0, \qquad (2)$$

with the terminal condition $F(T,\omega) = H(\omega)$*, on the topological support of* $(S,\mathbb{P})$ *in* $(C([0,T],\mathbb{R}_+),\|\cdot\|_\infty)$*, i.e.*

$$\forall\omega \in \mathrm{supp}(S) := \Big\{\omega\in\Omega,\ \text{such that } \mathbb{P}(S_T\in V)>0 \\ \forall V \text{ neighborhood of } \omega \text{ in } (\Omega,\|\cdot\|_\infty)\Big\}$$

Pathwise Analysis of Hedging Performance

Problem:

- The hedger sells a path-dependent option with maturity T and payoff $H(S_T)$ such that $\mathbb{E}^{\mathbb{P}}[|H(S_T)|^2] < \infty$
- He computes the price and hedging strategy according to $\mathbb{P}$
- He trades according to that strategy, but taking in input the realized market prices $\Longrightarrow$ in a scenario ω, the final value of the hedging portfolio will differ from $H(\omega_T)$
- What is the performance of the hedging strategy $(Y(0), \nabla_S Y)$ with respect to H?

Robustness of Delta-Hedging Strategies

Definition (Robust delta hedge)

The delta-hedging strategy $(Y(0), \nabla_S Y)$ *is said to be robust for* H *on* $U \subset \Omega$ *if it is a super-strategy for* H *on* U.

Notation: For paths of absolutely continuous finite quadratic variation along Π, we define the local realized volatility as

$$\begin{aligned} \sigma^{\mathrm{real}} : [0,T] \times \mathcal{A} &\rightarrow \mathbb{R}, \\ (t,\omega) &\mapsto \sigma^{\mathrm{real}}(t,\omega) = \frac{1}{\omega(t)}\sqrt{\frac{\mathrm{d}}{\mathrm{d}t}[\omega](t)}, \end{aligned}$$

where

$$\mathcal{A} := \{\omega \in Q(\Omega,\Pi),\ t \mapsto [\omega](t) \text{ is absolutely continuous}\}.$$

The Hedging Error Formula and Robustness

Proposition (The hedging error formula and robustness of delta-hedging)

If there exists $F \in \mathbb{C}_b^{1,2}(\mathcal{W}_T) \cap \mathbb{C}^{0,0}(\mathcal{W}_T)$ *such that* $\mathcal{D}F \in \mathbb{C}_l^{0,0}(\mathcal{W}_T)$, *and*

$$F(t, S_t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-}a.s.$$

Then, the hedging error of $(F(0,\cdot), \nabla_\omega F)$ *for* H *is explicitly given by*

$$\frac{1}{2}\int_0^T \left(\sigma(t,\omega)^2 - \sigma^{\mathrm{real}}(t,\omega)^2\right)\omega^2(t)\nabla_\omega^2 F(t,\omega)\mathrm{d}t$$

In particular, if for all $\omega \in U$ *and Lebesgue-a.e.* $t \in [0, T)$

$$\nabla_\omega^2 F(t,\omega) \geqslant 0 \quad \text{and} \quad \sigma(t,\omega) \geqslant \sigma^{\mathrm{real}}(t,\omega)$$

then the delta hedge for H *is robust on* U.

The Impact of Jumps

Proposition (Impact of jumps on delta hedging)

If there exists a non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ such that

$$F \in \mathbb{C}_b^{1,2}(\Lambda_T) \cap \mathbb{C}^{0,0}(\Lambda_T), \quad \nabla_\omega F \in \mathbb{C}^{0,0}(\Lambda_T), \quad \mathcal{D}F \in \mathbb{C}_l^{0,0}(\mathcal{W}_T)$$

$$F(t, S_t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-a.s.}$$

Then, for any $\omega \in Q(D([0,T], \mathbb{R}_+), \Pi)$ such that $[\omega]^c$ is absolutely continuous, the hedging error of the delta hedge $(F(0,\cdot), \nabla_\omega F)$ for H is explicitly given by

$$\frac{1}{2}\int_0^T \left(\sigma(t,\omega)^2 - \sigma^{\mathrm{real}}(t,\omega)^2\right)\omega^2(t)\nabla_\omega^2 F(t,\omega)\mathrm{d}t$$
$$- \sum_{t \in (0,T]} \left(F(t,\omega_t) - F(t,\omega_{t-}) - \nabla_\omega F(t,\omega_{t-}) \cdot \Delta\omega(t)\right).$$

Vertical Smoothness

The previous results extend *El Karoui, Jeanblanc & Shreve (1998)*:
- to path-dependent options
- to a pathwise setting: it gives the *pathwise* P&L of the strategy and removes unnecessary probabilistic assumptions

Definition (Vertical smoothness)

A functional $h : D([0,T],\mathbb{R}) \mapsto \mathbb{R}$ *is vertically smooth on* $U \subset D([0,T],\mathbb{R})$ *if,* $\forall (t,w) \in [0,T] \times U$*, the real map*

$$g^h(t,\omega;\cdot) : \mathbb{R} \to \mathbb{R}, \quad e \mapsto h(\omega + e\mathbb{1}_{[t,T]})$$

is of class C^2 *on a neighborhood* V *of 0 and there exist* $K, c, \beta > 0$ *such that, for all* $\omega, \omega' \in U$*,* $t, t' \in [0,T]$*,*

$$\left|\partial_e g^h(e;t,\omega)\right| + \left|\partial^2_{ee} g^h(e;t,\omega)\right| \leqslant K, \quad e \in V,$$

$$\left|\partial_e g^h(0;t,\omega) - \partial_e g^h(0;t',\omega')\right| + \left|\partial^2_{ee} g^h(0;t,\omega) - \partial^2_{ee} g^h(0;t',\omega')\right| \leqslant c(\|\omega - \omega'\|_\infty + |t - t'|^\beta).$$

Pricing Functional: Existence and Regularity

Proposition (Pricing functional: existence and regularity)

Let $H : (D([0,T],\mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$ *a locally-Lipschitz payoff functional such that* $\mathbb{E}^{\mathbb{P}}[|H(S_T)|] < \infty$ *and define*

$$h : (D([0,T],\mathbb{R}) \to \mathbb{R}, \quad h(\omega_T) = H(\exp \omega_T),$$

where $\exp \omega_T(t) := e^{\omega(t)}$ *for all* $t \in [0,T]$*.*

If h *is vertically smooth on* $\mathcal{C}([0,T],\mathbb{R}_+)$*, then*

$$\exists F \in \mathbb{C}_b^{0,2}(\mathcal{W}_T) \cap \mathbb{C}^{0,0}(\mathcal{W}_T)$$

such that

$$F(t, S_t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-}a.s.$$

Propagation of 'Vertical Convexity'

Definition (Vertical convexity of non-anticipative functionals)

A non-anticipative functional $G : \Lambda_T \to \mathbb{R}$ is called vertically convex on $U \subset \Lambda_T$ if, for all $(t,\omega) \in U$, there exists a neighborhood $V \subset \mathbb{R}$ of 0 such that the map

$$V \to \mathbb{R}, \quad e \mapsto G\left(t, \omega + e\mathbb{1}_{[t,T]}\right)$$

is convex.

Proposition (Propagation of vertical convexity)

Assume that, for all $(t,\omega) \in \mathbb{T} \times \mathrm{supp}(S,\mathbb{P})$, there exists an interval $\mathcal{I} \subset \mathbb{R}$, $0 \in \mathcal{I}$, such that the map

$$v^H(\cdot;t,\omega) : \mathcal{I} \to \mathbb{R}, \quad e \mapsto v^H(e;t,\omega) = H\left(\omega(1 + e\mathbb{1}_{[t,T]})\right)$$

is convex. If

$$\exists F \in \mathbb{C}_b^{0,2}(\mathcal{W}_T),\ F(t,S_t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-a.s.},$$

then F is vertically convex on $\mathbb{T} \times \mathrm{supp}(S,\mathbb{P})$. In particular:

$$\nabla_\omega^2 F(t,\omega) \geqslant 0, \quad \forall (t,\omega) \in \mathbb{T} \times \mathrm{supp}(S,\mathbb{P}).$$

Example: Discretely-Monitored Exotic Derivatives with Black-Scholes

Lemma (Discretely-monitored path-dependent derivatives with Black-Scholes)

Let $\sigma : [0, T] \to \mathbb{R}_+$ *such that* $\int_0^T \sigma^2(t)\mathrm{d}t < \infty$.

Assume that $H : D([0, T], \mathbb{R}_+)$ *and there exist a partition* $0 = t_0 < t_1 < \ldots < t_n \leqslant T$ *and a function* $h \in C_b^2(\mathbb{R}^n; \mathbb{R}_+)$ *such that*

$$\forall \omega \in D([0, T], \mathbb{R}_+), \quad H(\omega_T) = h(\omega(t_1), \omega(t_2), \ldots, \omega(t_n)).$$

Then,

$$\exists F \in \mathbb{C}^{1,2}_{loc}(\mathcal{W}_T), \; F(t, S_t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-a.s.},$$

and the horizontal and vertical derivatives of F are given in a closed form.

Example: Hedging Asian Options with Black-Scholes

Consider an arithmetic Asian call option

$$H(S_T) = \left(\frac{1}{T} \int_0^T \mathcal{S}(u)\mathrm{d}u - K \right)^+$$

and assume $\sigma : [0, T] \to \mathbb{R}_+$ such that $\int_0^T \sigma^2(t)\mathrm{d}t < \infty$.

The pricing functional F is given by: $\forall (t, \omega) \in \mathcal{W}_T$,

$$F(t, \omega_t) = f(t, a(t), \omega(t)), \quad a(t) = \int_0^t \omega(s)\mathrm{d}s,$$

where $f \in \mathcal{C}^{1,1,2}([0.T) \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}([0.T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ solves

$$\begin{cases} \frac{\sigma^2(t)x^2}{2} \partial_{xx}^2 f(t, a, x) + x\partial_a f(t, a, x) + \partial_t f(t, a, x) = 0, \\ f(T, a, x) = g\left(\frac{a}{T}\right) \end{cases} \tag{3}$$

Note that (3) turns out to be a particular case of the universal pricing equation.

Example: Robustness of Black-Scholes Delta-Hedging for Asian Options

Corollary (Robustness of delta-hedging for Asian options)

If the Black-Scholes volatility term structure over-estimates the local realized volatility on $\mathcal{A} \cap \mathrm{supp}(S, \mathbb{P})$, i.e.

$$\sigma(t) \geqslant \sigma^{\mathrm{real}}(t, \omega) \quad \forall \omega \in \mathcal{A} \cap \mathrm{supp}(S, \mathbb{P}),$$

Then the Black-Scholes-delta hedge for the arithmetic Asian call option is **robust** *on $\mathcal{A} \cap \mathrm{supp}(S, \mathbb{P})$. Moreover, the hedging error is given by*

$$\frac{1}{2} \int_t^T \left(\sigma(u)^2 - \sigma^{\mathrm{real}}(t, \omega)^2 \right) \omega^2(u) \partial_{xx}^2 f(u, a(u), \omega(u)) \mathrm{d}u$$

where f is the solution to the Cauchy problem (3)*.*

Example: Hedging Asian Options with Hobson-Rogers

$$\mathrm{d}S(t) = S(t)\sigma^n(O(t))\mathrm{d}W(t),$$

where

$$\mathrm{d}O(t) = \sigma^n(O(t))\mathrm{d}W(t) - \frac{1}{2}(\sigma^n(O(t))^2 + \lambda O(t))\mathrm{d}t.$$

Consider a geometric Asian call option

$$H(S_T) = \left(e^{M(T)} - K\right)^+, \quad M(T) = \frac{1}{T}\int_0^T \log S(u)\mathrm{d}u$$

The pricing functional F is given by: $\forall (t,\omega) \in \mathcal{W}_T$,

$$F(t,\omega) = u(T-t, \log\omega(t), \log\omega(t) - o(t,\omega), g(t,\omega)),$$

where u is the classical solution of the following Cauchy problem on $[0.T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$\begin{cases} \frac{1}{2}\sigma^n(x_1 - x_2)^2(\partial^2_{x_1x_1}u - \partial_{x_1}u) + \lambda(x_1 - x_2)\partial_{x_2}u + x_1\partial_{x_3}u - \partial_t u = 0, \\ u(0, x_1, x_2, x_3) = \Psi^G(e^{x_1}, \frac{x_3}{T}). \end{cases} \quad (4)$$

(4) is another particular case of the universal pricing equation

Hedging Asian Options with Path-Dependent Models

Corollary

If $\sigma^n(o(t,\omega)) \geqslant \sigma^{\mathrm{real}}(t,\omega) \quad \forall\omega\in\mathcal{A}\cap\mathrm{supp}(S,\mathbb{P})$, *then the Hobson-Rogers delta hedge for* H *is robust on* $\mathcal{A}\cap\mathrm{supp}(S,\mathbb{P})$. *Moreover, the hedging error at maturity is given by*

$$\frac{1}{2}\int_0^T \left(\sigma^n(o(t,\omega))^2 - \sigma^{\mathrm{mkt}}(t,\omega)^2\right)\omega^2(t)$$
$$\partial^2_{xu}(T-t,\log\omega(t),\log\omega(t)-o(t,\omega),g(t,\omega))\mathrm{d}t,$$

where u *is the solution of the Cauchy problem* (4)*.*

Other models that generalize Hobson-Rogers and allow to derive finite-dimensional Markovian representation for S and its arithmetic mean are given by

- Pascucci,Foschi 2006,
- Salvatore, Tankov 2014.

They thus guarantee the existence of a smooth pricing functional for arithmetic Asian options, then robustness of the delta hedge can be proved the same way as we showed in the Black-Scholes