

# Hedging of Life Insurance Liabilities

Thorsten Rheinländer, with Francesca Biagini and Irene Schreiber

Vienna University of Technology and LMU Munich

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# Systematic vs. idiosyncratic risk

- Mortality risk can essentially be split into systematic risk represented by the mortality intensity, i.e. the risk that the mortality rate of an age cohort differs from the one expected at inception, and idiosyncratic or unsystematic risk, i.e. the risk that the mortality rate of the individual is different from that of its age cohort.
- Survivor indices, provided by various investment banks, consist of publicly available mortality data aggregated by population, hence providing a good proxy for the systematic component of the mortality risk.

- Systematic risk may be hedged by investing in a longevity bond, see, e.g., Cairns, Blake & Dowd (2006). These types of bonds pay out the conditional survival probability at maturity as a function of the hazard rate or mortality intensity, which is given by a so-called longevity or survivor index. Here we assume that longevity bonds defined on the same cohort as the insurance portfolio are actively traded on the market.

- We explicitly model the longevity basis risk that arises due to the fact that the hedging instrument is based on an index representing the whole population, not on the insurance portfolio itself. Because of differences in socioeconomic profiles (with respect to e.g., health, income or lifestyle), the mortality rates of the population typically differ from those of the insurance portfolio which is known as selection effect. Hence the hedge will be imperfect, leaving a residual amount of risk, known as basis risk.
- Here we model the mortality intensity of the insurance portfolio together with the intensity of the population in an analytically tractable way by means of a multivariate affine diffusion. The dependency between the two populations is captured by the fact that the intensity of the insurance portfolio is fluctuating around a stochastic drift, which is given by the mortality intensity of the reference index.

- Since it is impossible to completely hedge the financial and mortality risk inherent in the liabilities of the insurance company, even in this setting where we allow for investments in a product representing the systematic mortality risk, the market is incomplete. Here we make use of the popular quadratic risk-minimization method introduced by Föllmer and Sondermann (1986). The idea of this technique is to allow for a wide class of admissible strategies that in general might not necessarily be self-financing, and to find an optimal hedging strategy with “minimal risk” within this class of strategies that perfectly replicates the given claim.
- There exist a number of studies that focus on applications of the quadratic hedging approach in the context of mortality modeling or in related areas such as credit risk, see e.g. Barbarin (2008), Biagini and Cretarola (2009), Dahl and Møller (2006), to name just a few.

# Focus of this study

- We extend these works in several ways: Firstly, we consider a multivariate affine diffusion model where the portfolio mortality intensity is driven by the evolution of the intensity of the population. This captures the basis risk between the insurance portfolio and the longevity index in a concise way, while remaining analytically tractable due to the affine structure.
- We provide explicit computations of risk-minimizing strategies for a portfolio of life insurance liabilities in a complex setting. Thereby we explicitly take into account and model the basis risk between the insurance portfolio and the longevity index and allow for investments in hedging instruments representing the systematic mortality risk.
- Besides that, we allow for a general structure of the insurance products studied and we do not require certain technical assumptions such as the independence of the financial market and the insurance model.

- Let  $T > 0$  be a fixed finite time horizon and  $(\Omega, \mathcal{G}, P)$  a probability space equipped with a filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$  which contains all available information.
- We define  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ , and put  $\mathcal{G} = \mathcal{G}_T$ , where  $\mathbb{H} = (\mathcal{H}_t)_{t \in [0, T]}$  is generated by the death counting processes of the insurance portfolio. The *background filtration*  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  contains all information available except the information regarding the individual survival times.
- Here we define  $\mathcal{F}_t = \sigma\{(W_s, W_s^r, W_s^\mu, W_s^{\bar{\mu}}) : 0 \leq s \leq t\}$ ,  $t \in [0, T]$ , where  $W$ ,  $W^r$ ,  $W^\mu$  and  $W^{\bar{\mu}}$  are independent Brownian motions driving the financial market and the mortality intensities.

- We consider a portfolio of  $n$  lives all aged  $x$  at time 0, with *death counting process*

$$N_t = \sum_{i=1}^n \mathbf{1}_{\{\tau^{x,i} \leq t\}}, \quad t \in [0, T],$$

where  $\tau^{x,i} : \Omega \rightarrow [0, T] \cup \{\infty\}$ , and for convenience in the following we omit the dependency on  $x$ .

- We define  $\mathcal{H}_t = \mathcal{H}_t^1 \vee \dots \vee \mathcal{H}_t^n$ , with  $\mathcal{H}_t^i = \sigma\{H_s^i : 0 \leq s \leq t\}$  and  $H_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$ . We assume that the times of death  $\tau^i$  are totally inaccessible  $\mathbb{G}$ -stopping times, with  $P(\tau^i > t) > 0$  for  $i = 1, \dots, n$  and  $t \in [0, T]$ .



# Hazard process

- An important role is played by the conditional distribution function of  $\tau^i$ , given by

$$F_t^i = P(\tau^i \leq t \mid \mathcal{F}_t), \quad i = 1, \dots, n,$$

and we assume  $F_t^i < 1$  for all  $t \in [0, T]$ .

- Then the *hazard process*  $\Gamma^i$  of  $\tau^i$  under  $P$ ,

$$\Gamma_t^i = -\ln(1 - F_t^i) = -\ln E \left[ \mathbf{1}_{\{\tau_i \geq t\}} \mid \mathcal{F}_t \right],$$

is well-defined for every  $t \in [0, T]$ .

- Since the insurance portfolio is homogenous in the sense that all individuals belong to the same age cohort, we assume

$$\Gamma^i = \Gamma, \quad i = 1, \dots, n,$$

where

$$\Gamma_t = \int_0^t \mu_s ds, \quad t \in [0, T].$$

- The *mortality intensity*  $\mu$  is given as a solution of the following stochastic differential equation:

$$d\mu_t = \mu(t, \mu_t, \bar{\mu}_t) dt + \sigma(t, \mu_t, \bar{\mu}_t) dW_t^\mu, \quad t \in [0, T],$$

where  $\bar{\mu}_t$ ,  $t \in [0, T]$ , is a diffusion driven by the Brownian motion  $W^{\bar{\mu}}$ , such that the two-dimensional process  $(\mu, \bar{\mu})$  follows a multivariate affine diffusion. The process  $\bar{\mu}$  represents the mortality intensity of the equivalent age cohort of the population, and can be derived by means of publicly available data of the *survivor index*

$$S_t^{\bar{\mu}} = \exp\left(-\int_0^t \bar{\mu}_s ds\right), \quad t \in [0, T].$$

# Conditional independence

- We also assume that for  $i \neq j$ ,  $\tau^i, \tau^j$  are conditionally independent given  $\mathcal{F}_T$ , i.e.

$$E \left[ \mathbf{1}_{\{\tau_i > t\}} \mathbf{1}_{\{\tau_j > s\}} \middle| \mathcal{F}_t \right] = E \left[ \mathbf{1}_{\{\tau_i > t\}} \middle| \mathcal{F}_t \right] E \left[ \mathbf{1}_{\{\tau_j > s\}} \middle| \mathcal{F}_t \right], \\ 0 \leq s, t \leq T.$$

- All individuals within the insurance portfolio are subject to idiosyncratic risk factors, as well as common risk factors, given by the information represented by the background filtration  $\mathbb{F}$ . Intuitively, the assumption of conditional independence means that given all common risk factors are known, the idiosyncratic risk factors become independent of each other.

- We consider a financial market consisting of a bank account or numéraire  $B$ , a riskless zero-coupon bond  $P$  with maturity  $T$ , as well as a stock  $S$  and a longevity bond  $P^{\bar{\mu}}$ . We assume that the discounted bond price  $Y$  given by

$$Y_t = E \left[ \frac{1}{B_T} \mid \mathcal{G}_t \right] = E \left[ \frac{1}{B_T} \mid \mathcal{F}_t \right], \quad t \in [0, T],$$

follows the following diffusion:

$$dY_t = \sigma_t^r Y_t dW_t^r, \quad t \in [0, T],$$

with  $Y_0 = y$ . The discounted stock price  $X = S/B$  is assumed to follow the diffusion

$$dX_t = \sigma_t X_t dW_t, \quad t \in [0, T],$$

with  $X_0 = x$ . We assume that  $\sigma$  and  $\sigma^r$  are  $\mathbb{F}$ -adapted or even  $\mathbb{G}$ -adapted processes such that  $\int_0^T \sigma_s^2 ds < \infty$  and  $\int_0^T (\sigma_s^r)^2 ds < \infty$ .

# Longevity bond

- Following Cairns, Blake and Dowd (2006), we assume that  $P^{\bar{\mu}}$  is the price process of a *longevity bond* with maturity  $T$  representing the systematic mortality risk, i.e.  $P^{\bar{\mu}}$  is defined as a zero-coupon-type bond that pays out the value of the survivor or longevity index at  $T$ . This means the discounted value process  $Y^{\bar{\mu}} = P^{\bar{\mu}}/B$  is given by

$$Y_t^{\bar{\mu}} = E \left[ \frac{S_T^{\bar{\mu}}}{B_T} \mid \mathcal{G}_t \right] = E \left[ \frac{\exp(-\int_0^T \bar{\mu}_s ds)}{B_T} \mid \mathcal{F}_t \right], \quad t \in [0, T].$$

- We assume that  $Y^{\bar{\mu}}$  follows the diffusion

$$dY_t^{\bar{\mu}} = Y_t^{\bar{\mu}} (\sigma_t^1 dW_t^{\bar{\mu}} + \sigma_t^2 dW_t^r), \quad t \in [0, T],$$

where  $\sigma^1$  and  $\sigma^2$  are  $\mathbb{F}$ -adapted or even  $\mathbb{G}$ -adapted processes such that  $\int_0^T (\sigma_s^i)^2 ds < \infty$  for  $i = 1, 2$ .

# Hedging portfolio

- Thus the discounted asset prices  $X$ ,  $Y$  and  $Y^{\bar{\mu}}$  are continuous (local)  $(P, \mathbb{F})$ -martingales, i.e. the financial market is arbitrage-free. Note that we allow for mutual influence between the processes  $X$ ,  $Y$  and  $Y^{\bar{\mu}}$  through the volatility processes  $\sigma$ ,  $\sigma^r$ ,  $\sigma^1$  and  $\sigma^2$ . The asset prices may be  $\mathbb{F}$ -adapted, however the trading strategies are allowed to be  $\mathbb{G}$ -adapted, i.e. we will consider (discounted) hedging portfolios

$$V_t(\varphi) = \zeta_t^X X_t + \zeta_t^Y Y_t + \zeta_t^{Y^{\bar{\mu}}} Y_t^{\bar{\mu}} + \zeta_t^0, \quad t \in [0, T],$$

where  $\varphi = (\zeta^X, \zeta^Y, \zeta^{Y^{\bar{\mu}}}, \zeta^0)$  is a  $\mathbb{G}$ -adapted process and  $\zeta^X, \zeta^Y, \zeta^{Y^{\bar{\mu}}}$  and  $\zeta^0$  represent the strategy components with respect to  $X, Y, Y^{\bar{\mu}}$  and the bank account  $B$ .

- This implies that all agents invest according to information incorporating the common risk factors such as the financial market and the mortality intensities, as well as the individual times of death.

# Basic contracts in the combined market

- We consider the extended market  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  and assume that all  $\mathbb{F}$ -local martingales are also  $\mathbb{G}$ -local martingales (Hypothesis (H)).
- A *pure endowment contract* consists of a payoff

$$C^{pe} (n - N_T)$$

at  $T$ , where  $C^{pe}$  is a non-negative square-integrable  $\mathcal{F}_T$ -measurable random variable, i.e. the insurer pays the amount  $C^{pe}$  at the term  $T$  of the contract to every policyholder of the portfolio who has survived until  $T$ .

- The *term insurance contract* is defined as

$$\int_0^T C_s^{ti} dN_s = \sum_{i=1}^n \int_0^T C_s^{ti} dH_s^i = \sum_{i=1}^n \mathbf{1}_{\{\tau^i \leq T\}} C_{\tau^i}^{ti},$$

where  $C^{ti}$  is assumed to be a non-negative square-integrable  $\mathbb{F}$ -predictable process, i.e. the amount  $C_{\tau^i}^{ti}$  is payed at the time of death  $\tau^i$  to every policyholder  $i$ ,  $i = 1, \dots, n$ .

- The *annuity contract* consists of multiple payoffs the insurer has to pay as long as the policyholders are alive. We model these payoffs through their cumulative value  $C_t^a$  up to time  $t$ , where  $C^a$  is assumed to be a right-continuous, non-negative square-integrable increasing  $\mathbb{F}$ -adapted process. The cumulative payment up to time  $T$  is then given by

$$\int_0^T (n - N_s) dC_s^a.$$

- We also provide examples of unit-linked life insurance products where we set  $C^{pe} = f(S_T)$ ,  $C_t^{ti} = f(S_t)$  and  $C_t^a = \int_0^t f(S_u) du$  for a function  $f$  that satisfies sufficient regularity conditions.
- Market incompleteness arises due to the presence of additional sources of randomness, which in our case are represented by mortality risk. In this context, the method of risk-minimization appears as a natural hedging method in hybrid financial and insurance markets.



# Quadratic risk-minimization in incomplete markets

- The risk-minimization method provides strategies whose terminal value will perfectly replicate the final value of the insurance liabilities, but which are not necessarily self-financing and hence might require additional cash injections from the bank account. The optimal strategy is chosen such that the cost is minimal in the  $L^2$ -sense.
- The key to finding the strategy with minimal risk is the well-known Galtchouk-Kunita-Watanabe (GKW) decomposition. For a square-integrable liability  $C_T$ , the expected accumulated total payments may be decomposed by use of the GKW decomposition as

$$E[C_T | \mathcal{F}_t] = E[C_T] + \int_{]0,t]} \zeta_s^A dX_s + L_t^A, \quad t \in [0, T],$$

where  $\zeta^A \in L^2(X)$  and  $L^A$  is a square integrable martingale null at 0 that is strongly orthogonal to the space of square-integrable stochastic integrals with respect to the vector of discounted asset prices  $X$ .

# Choice of pricing measure

- While in our setting the measure  $P$  comprises a martingale measure for the traded assets on the financial market, so far it is not specified how to find the projection of this measure on the filtration  $\mathbb{H}$ . The change of measure should provide a positive market-value margin.
- Two canonical choices are as follows, see e.g. Wüthrich and Merz (2013). In both cases the risk driver could be chosen to be the death counting process. The first measure transform corresponds to the change from the experience basis to the more prudent technical basis, while the second choice is the deflator corresponding to the Esscher premium.

# Discussion of risk-minimizing strategies

- Our main result states explicit formulae for the life insurance risk-minimizing strategies. This result is quite technical, however it provides an interesting insight on several issues.
- Firstly, the optimal strategy is proportional to the number of survivors and the hazard process of the mortality intensity.
- Furthermore, the optimal cost can be decomposed into a part resulting from the hedging error related to the systematic as well as to the unsystematic mortality risk. This decomposition could also be used to choose suitable longevity indices (and longevity bonds), which should be as close as possible (at least in the  $L^2$ -sense) to the portfolio longevity.
- Moreover, in our multivariate affine diffusion model, we can obtain the optimal strategies in terms of Riccati equations.

- One of the main achievements compared to the previous literature is that we deal with the basis risk by letting the portfolio mortality intensity be driven by the mortality intensity of the general population, while taking care of selection effects.
- Besides that we allow for various mutual dependencies, thus working in a very general and flexible setting, yet still having a tractable model by using a multivariate affine framework.
- We also provide a detailed study of unit-linked products where the dependency between the index and the insurance portfolio is explicitly modeled by means of an affine mean-reverting diffusion process with stochastic drift.

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