The Jacobi Stochastic Volatility model

Sergio Pulido
joint work with Damir Filipovic and Damien Ackerer (EPFL)

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Outline

Heston Stochastic Volatility model

Heston Model (1993)

$$X_t = \text{log-asset price}$$
$$V_t = \text{stochastic volatility process}$$

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dW_{1t} \\ dX_t &= (r - V_t/2)\,dt + \sqrt{V_t}\left(\rho\,dW_{1t} + \sqrt{(1-\rho^2)}\,dW_{2t}\right) \end{aligned} \tag{1}$$

for $\kappa, \theta, \sigma > 0$, interest rate r, correlation parameter $\rho \in [-1, 1]$, and 2-dimensional BM $W = (W_1, W_2)$

- Widely used **Stochastic Volatility model**
- Models **volatility smile + positive mean-reverting volatility**
- **Affine model** - Fourier Transform techniques for pricing

Pricing with the Fourier Transform

- **Closed formula for the Fourier Transform:** Suppose that X_T has a density equal to g_T. Then

$$\widehat{g_T}(u) = \mathbb{E}[e^{iuX_t}] = e^{\Psi_1(t,u)X_0 + \Psi_2(t,u)V_0 + \Phi(t,u)}$$

  where Ψ_1, Ψ_2, Φ solve a Ricatti system of ODEs
- **Pricing with the Fourier Transform:** Suppose that a European type discounted derivative's payoff is $f(X_T)$. Then

$$Price = \pi_f = \mathbb{E}[f(X_T)] = \text{const} \int_{\mathbb{R}} \widehat{f}(u + i\omega)\overline{\widehat{g_T}}(u + i\omega)\, du$$

  for a conveniently chosen ω and where $\widehat{f}$ is the Fourier Transform of f
- Numerical integration simultaneously done for different strikes - **FFT method** of Carr-Madan (1999)

Jacobi Stochastic Volatility model

Define the quadratic function

$$Q(v) = \frac{(v - v_{min})(v_{max} - v)}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2}$$

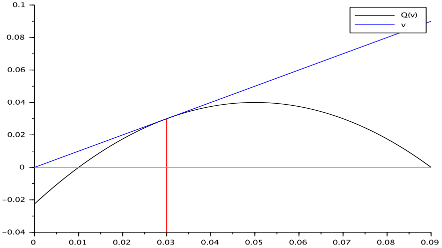
Jacobi Model

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)\, dt + \sigma \sqrt{Q(V_t)}\, dW_{1t} \\ dX_t &= (r - V_t/2)\, dt + \rho \sqrt{Q(V_t)}\, dW_{1t} + \sqrt{V_t - \rho^2\, Q(V_t)}\, dW_{2t} \end{aligned} \tag{2}$$

for $\kappa, \sigma > 0$, $\theta \in [v_{min}, v_{max}]$, interest rate r, $\rho \in [-1, 1]$, and 2-dimensional BM $W = (W_1, W_2)$

Observations

- $v \geq Q(v)$, with equality if and only if $v = \sqrt{v_{min} v_{max}}$, and $Q(v) \geq 0$ for all $v \in [v_{min}, v_{max}]$



- V_t is a Jacobi process and models the instantaneous volatility process,

$$d\langle X, X\rangle_t = V_t \, dt$$

and $V_t \in [v_{min}, v_{max}]$

Observations (cont.)

- The discounted price process $\mathrm{e}^{-rt}S_t = \mathrm{e}^{-rt+X_t}$ is a martingale
- The parameter ρ tunes the instantaneous correlation between price returns and squared volatility increments,

$$\frac{d\langle V, X\rangle_t}{\sqrt{d\langle V, V\rangle_t}\sqrt{d\langle X, X\rangle_t}} = \rho\sqrt{Q(V_t)/V_t}$$

  This correlation is equal to ρ if $V_t = \sqrt{v_{min}v_{max}}$. In general, we have $\sqrt{Q(V_t)/V_t} \leq 1$
- The **JSV is a polynomial model - efficient calculation of moments**
- The Jacobi model (2) converges weakly in the path space to the Heston model (1) as $v_{min} \to 0$ and $v_{max} \to \infty$

The main theorem

We define

$$C_T = \int_0^T \left(V_t - \rho^2 Q(V_t)\right) dt$$

Theorem

Let $\epsilon < 1/(2v_{max}T)$. If $C_T > 0$ then the distribution of X_T admits a density $g_T(x)$ on $\mathbb{R}$ that satisfies

$$\int_{\mathbb{R}} \mathrm{e}^{\epsilon x^2} g_T(x)\, dx < \infty \tag{3}$$

If

$$\mathbb{E}\left[{C_T}^{-1/2}\right] < \infty \tag{4}$$

then $g_T(x)$ and $\mathrm{e}^{\epsilon x^2} g_T(x)$ are uniformly bounded and continuous on $\mathbb{R}$. A sufficient condition for (4) to hold is

$$v_{min} > 0 \text{ and } \rho^2 < 1$$

The case of the Heston model

- The statements of Theorem 1 also hold for the Heston model (1), with $Q(v) = v$, for $\epsilon \leq 0$
- However, the Heston model does not satisfy (3) for any $\epsilon > 0$. Otherwise its Fourier Tranform

$$\widehat{g_T}(u) = \int_{\mathbb{R}} g(x) \mathrm{e}^{iux} \, dx$$

would extend to an entire function in $z \in \mathbb{C}$. But it is well known that $\widehat{g_T}(z)$ becomes infinite for $|Im(z)|$ large enough

A crucial corollary

Corollary

Assume (4) *holds. Then* $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, *where*

$$L^2_w := \left\{ h : \int_{\mathbb{R}} |h(x)|^2 w(x)\, dx \right\}$$

and $w(x)$ *is any Gaussian density with variance* σ^2_w *satisfying*

$$\sigma^2_w > \frac{v_{max}\, T}{2} \tag{5}$$

- (Filipovic, Mayerhofer, Schneider 2013) For the Heston model we have that $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, where $w(x)$ **is a (bilateral) Gamma density**

Density approximation technique - Setup

- **The price:**

$$\pi_f = \text{price of a claim with discounted payoff } f(X_T)$$

- **The weight function:**

$$w(x) = \text{Gaussian density with mean } \mu_w \text{ and variance } \sigma_w^2$$

- **The weighted Hilbert space:**

$$L^2_w = \left\{ f(x) \mid \|f\|^2_w = \int_{\mathbb{R}} f(x)^2 \, w(x) dx < \infty \right\}$$

which is a Hilbert space with scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x) g(x) \, w(x) dx$$

Density approximation technique - Setup (cont.)

- **Orthonormal basis of generalized Hermite polynomials** $H_n(x)$, $n \geq 0$, given by

$$H_n(x) = \frac{1}{\sqrt{n!}} \mathscr{H}_n\left(\frac{x - \mu_w}{\sigma_w}\right)$$

  where $\mathscr{H}_n(x)$ are the standard Hermite polynomials
- In particular, $\deg H_n(x) = n$, and $\langle H_m, H_n \rangle_w = 1$ if $m = n$ and zero otherwise

Density and price approximations

- If the assumption of the main Corollary holds then $\ell(x) = g_T(x)/w(x) \in L^2_w$
- If $f(x) \in L^2_w$ then the price π_f is well defined and equals

$$\pi_f = \int_{\mathbb{R}} f(x) g_T(x)\, dx = \langle f, \ell \rangle_w = \sum_{n \geq 0} f_n \ell_n \tag{6}$$

for the **Fourier coefficients and Hermite moments**

$$f_n = \langle f, H_n \rangle_w, \quad \ell_n = \langle \ell, H_n \rangle_w = \int_{\mathbb{R}} H_n(x) g_T(x)\, dx$$

Density and price approximations (cont.)

- We approximate the price π_f by truncating the series in (6) at some order $N \geq 1$ and write

$$\pi_f^{(N)} = \sum_{n=0}^{N} f_n \ell_n = \sum_{n=0}^{N} \langle f, \ell_n H_n \rangle_w = \int_{\mathbb{R}} f(x) g_T^{(N)}(x)\, dx \quad (7)$$

  where

$$g_T^{(N)}(x) = \sum_{n=0}^{N} \ell_n H_n(x) w(x)$$

  serves as a proxy for the density $g_T(x)$
- While $\pi_f^{(N)} \to \pi_f$ as $N \to \infty$ in general, in some cases the proxy is exact as the following lemma states

Lemma

If $f(x)$ is a polynomial then $\pi_f^{(N)} = \pi_f$ for all $N \geq \deg f(x)$

Calculation of the Hermite-moments ℓ_n (cont.)

Theorem

The coefficients ℓ_n are given by

$$\ell_n = [h_1(V_0, X_0), \ldots, h_M(V_0, X_0)]\, \mathrm{e}^{TG}\, \mathbf{e}_{\pi(0,n)}, \quad 0 \leq n \leq N$$

where $\mathbf{e}_i$ is the i–th standard basis vector in $\mathbb{R}^M$ and $h_0, \ldots, h_M$ is a basis of polynomials. In particular,

$$\ell_0 = 1;$$

$$\ell_1 = \frac{1}{\sigma_w}\left(rT - \frac{\theta}{2}\left(T + \frac{\mathrm{e}^{-\kappa T} - 1}{\kappa}\right) + \frac{\mathrm{e}^{-\kappa T} - 1}{2\kappa} V_0 + X_0 - \mu_w\right)$$

Here

- G is the $(M \times M)$–matrix representing the infinitesimal generator of (V_t, X_t) on Pol_N, which is a sparse matrix

European calls and puts

Theorem

Consider the discounted payoff function for a call option with log strike k*,*

$$f(x) = \mathrm{e}^{-rT} \left(\mathrm{e}^{x} - \mathrm{e}^{k} \right)^{+}$$

Its Fourier coefficients f_n *for* $n \geq 1$ *are given by*

$$f_n = \mathrm{e}^{-rT+\mu_w} \frac{1}{\sqrt{n!}} \sigma_w I_{n-1} \left(\frac{k-\mu_w}{\sigma_w} ; \sigma_w \right)$$

The functions $I_n(\mu;\nu)$ *are defined recursively by*

$$\begin{aligned} I_0(\mu;\nu) &= \mathrm{e}^{\frac{\nu^2}{2}} \Phi(\nu - \mu); \\ I_n(\mu;\nu) &= \mathscr{H}_{n-1}(\mu) \mathrm{e}^{\nu\mu} \phi(\mu) + \nu I_{n-1}(\mu;\nu), \quad n \geq 1 \end{aligned}$$

where $\mathscr{H}_n(x)$ *are the standard Hermite polynomials,* $\Phi(x)$ *denotes the standard Gaussian distribution function, and* $\phi(x)$ *its density*

An example with $v_{max} = (0.3)^2$ - Call option

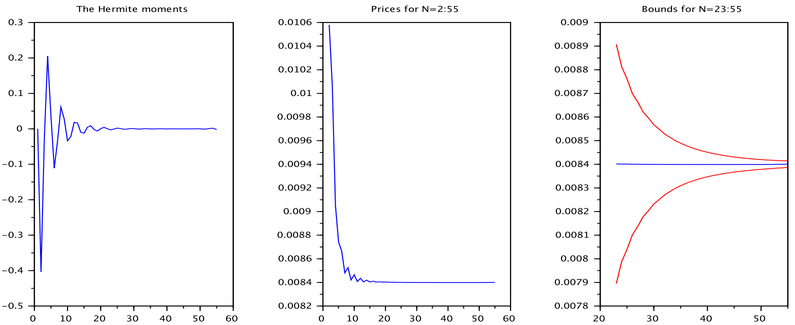


Figure: Here $\kappa = 0.5$, $\theta = (0.20)^2$, $\sigma = 0.15$, $r = 0$, $\rho = -0.5$, $X_0 = 0$, $V_0 = \theta$, $v_{max} = (0.3)^2$, $v_{min} = (0.1)^2$, $\sigma_w = 0.1532971$, $\mu_w = \mathbb{E}[X_T]$, $k = log(1.1)$ and $T = 1/4$

An example with $v_{max} = 1$ - Call option

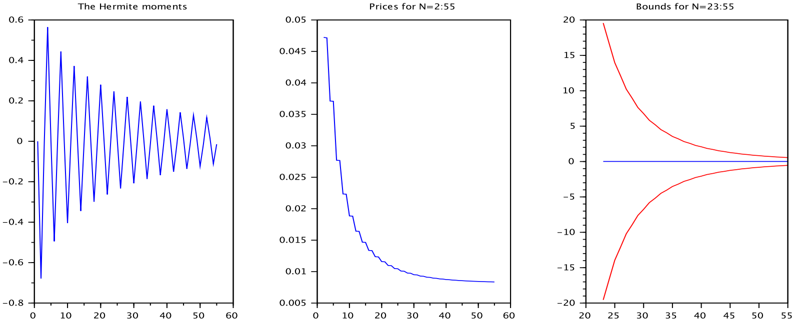


Figure: Here $\kappa = 0.5$, $\theta = (0.20)^2$, $\sigma = 0.15$, $r = 0$, $\rho = -0.5$, $X_0 = 0$, $V_0 = \theta$, $v_{max} = 1$, $v_{min} = (0.1)^2$, $\sigma_w = 0.500999$, $\mu_w = \mathbb{E}[X_T]$, $k = log(1.1)$ and $T = 1/4$

Convergence to the Heston price - Call option

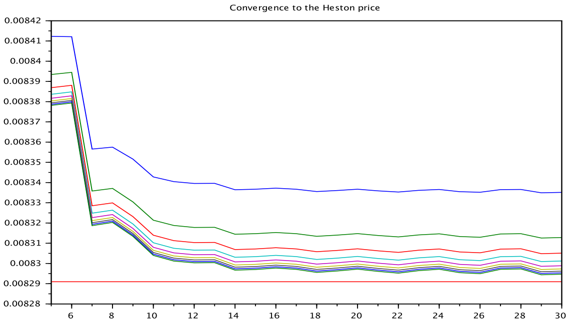


Figure: Here $\kappa = 0.5$, $\theta = (0.20)^2$, $\sigma = 0.15$, $r = 0$, $\rho = -0.5$, $v_{min} = (0.0001)^2$, $X_0 = 0$, $V_0 = \theta$, $k = log(1.1)$, $T = 1/4$ and $v_{max} \in [(0.03)^2, 10^2]$. Heston price ≈ 0.00829

Summary

- The Jacobi Stochastic Volatility **(JSV) model is a polynomial preserving model with possibly bounded (bounded away from zero) volatility**
- The **Heston model is a limiting case** of the JSV model as $[v_{min}, v_{max}] \to [0, \infty)$
- In the JSV model the **log-price density can be approximated in** L^2_w where w is a normal density. In this space **the family of Hermite polynomials is an o.n. basis,** which helps the analysis (calculation of moments, Fourier coefficients and analytical error bounds)
- The **pricing algorithm** using the density approximation is fast because:
    - **Fast calculation of Hermite-moments** (Polynomial Preserving Property)
    - In general once the moments are obtained it **boils down to numerical integration**
    - In some cases (calls and puts) no numerical integration needed

Thank you!