Pathwise Tanaka formulas and local times for typical price paths

David Prömel

Humboldt-Universität zu Berlin

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joint work with Nicolas Perkowski

"Buy-Sell Paradox" (Carr and Jarrow '90)

Consider

- (S_t) a price process for $t \in [0, \infty)$,
- $(S_t - K)^+$ an European call option with strike price K,
- $\pi = \{0 = t_0 < t_1 < \dots\}$ a partition of $[0, \infty)$.

We observe that

$$(S_t - K)^+ = (S_0 - K)^+ + \sum_{j=0}^{\infty} \mathbf{1}_{[K,\infty)}(S_{t_j})[S_{t_{j+1}\wedge t} - S_{t_j \wedge t}] + L_t^{\pi}(S, K).$$

As $|\pi| \to 0$ one gets

$$(S_t - K)^+ = (S_0 - K)^+ + \int_0^t \mathbf{1}_{[K,\infty)}(S_s)\,\mathrm{d}S_s + L_t(S, K).$$

Outline

1. Pathwise Tanaka formulas
2. Local time for typical price paths

Pathwise Itô Formula

Let (π^n) be a sequence of partitions and let $S \in C([0,\infty),\mathbb{R})$.

S has *quadratic variation* along (π^n) if the sequence of measures

$$\sum_{t_j \in \pi^n} (S(t_{j+1}) - S(t_j))^2 \delta_{t_j}$$

converges vaguely to a measure μ. We write $\langle S \rangle(t) := \mu([0,t])$.

Theorem (Föllmer 1981)

Assume that S has quadratic variation along (π^n) and $f \in C^2$. Then, one has the pathwise Itô formula

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\,\mathrm{d}S(s) + \frac{1}{2}\int_0^t f''(S(s))\,\mathrm{d}\langle S \rangle(s).$$

Pathwise Itô Formula

Without probability Föllmer constructed

$$\int_0^t f'(S(s))\,\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j\in\pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j\wedge t)).$$

Two natural (pathwise) extensions:

1. Path-dependent functionals f:
   Cont and Fournié (2010), Imkeller and P. (2015)
2. Less regular functions f:
   Wuermli (1980), Perkowski and P. (2015), Davis, Obłój and Siorpaes (2015)

Applications to robust and model-independent finance:
Bick and Willinger (1994), Lyons 1995, ..., Davis, Obłój and Raval 2014, ...

Pathwise Local Time

Let f' be right-continuous and of locally bounded variation, set $f(x) := \int_0^x f'(y)\,\mathrm{d}y$ for $x \geqslant 0$.

Then we get for $b \geqslant a$ that

$$\begin{aligned} f(b) - f(a) &= f'(a)(b-a) + \int_{(a,b]} (f'(x) - f'(a))\,\mathrm{d}x \\ &= f'(a)(b-a) + \int_{(a,b]} (b-t)\,\mathrm{d}f'(t). \end{aligned}$$

Therefore, for any $S \in C([0,\infty),\mathbb{R})$ and any partition π we have

$$\begin{aligned} f(S(t)) - f(S(0)) = &\sum_{t_j \in \pi} f'(S(t_j \wedge t))(S(t_{j+1} \wedge t) - S(t_j \wedge t)) \\ &+ \int_{-\infty}^{\infty} \sum_{t_j \in \pi} \Big(\mathbf{1}_{(\!| S(t_j \wedge t), S(t_{j+1} \wedge t)]\!]}(u) |S(t_{j+1} \wedge t) - u| \Big)\,\mathrm{d}f'(u). \end{aligned}$$

Pathwise Local Time

Let us define a discrete pathwise local time by setting

$$L_t^\pi(S,u) := \sum_{t_j\in\pi} \mathbf{1}_{(\!| S(t_j\wedge t), S(t_{j+1}\wedge t)]\!]}(u)|S(t_{j+1}\wedge t)-u|,$$

for $(t,u)\in[0,\infty)\times\mathbb{R}$. Hence, we read

$$f(S(t))-f(S(0)) = \sum_{t_j\in\pi} f'(S(t_j\wedge t))(S(t_{j+1}\wedge t)-S(t_j\wedge t)) + \int_{-\infty}^{\infty} L_t^\pi(S,u)\,\mathrm{d}f'(u).$$

Let (π^n) be a sequence of partitions and let $S\in C([0,\infty),\mathbb{R})$ be such that

$$\lim_{n\to\infty} m(S,\pi^n[0,T]) := \lim_{n\to\infty} \max_{t_j\in\pi^n[0,T]} |S(t_j^n)-S(t_{j-1}^n)| = 0, \quad T>0.$$

L^p-Local Time

A function $L(S)\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is called L^p-local time of S along (π^n) if $L_t^{\pi^n}(S,\bullet)$ converge weakly in $L^p(\mathrm{d}u)$ to $L_t(S,\bullet)$ for all $t \in [0,\infty)$.

Theorem (Wuermli (1980), Davis, Obłój and Siorpaes (2015))

For $f \in W^{2,q}$ (Sobolev space) with $1/q + 1/p = 1$, we have the generalized pathwise Itô formula

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\,\mathrm{d}S(s) + \int_{-\infty}^{\infty} f''(u) L_t(S,u)\,\mathrm{d}u.$$

Remark: Existence of L^p-local time implies the quadratic variation along (π^n). The converse is wrong!

Continuous Local Time

S has a continuous local time along (π^n) if

- $L_t^{\pi^n}(S,\bullet)$ converge uniformly to a continuous limit $L_t(S,\bullet)$ for all $t \in [0,\infty)$,
- $(t,u) \mapsto L_t(S,u)$ is jointly continuous.

Theorem (Perkowski and P. (2015))

Let f be absolutely continuous with right-continuous f' of bounded variation.

Then, we have the generalized change of variable formula

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(u))\,\mathrm{d}S(u) + \int_{-\infty}^{\infty} L_t(u)\,\mathrm{d}f'(u).$$

Local Time of Finite p-Variation

Recall the p-variation of $f:\mathbb{R}\to\mathbb{R}$ is given by

$$\|f\|_{p\text{-var}} := \sup\left\{\left(\sum_{k=1}^{n}|f(u_k)-f(u_{k-1})|^p\right)^{1/p} : -\infty < u_0 < \ldots < u_n < \infty\right\}.$$

For $p \geqslant 1$ the set $\mathcal{L}_{c,p}(\pi^n)$ consists of all $S \in C([0,T],\mathbb{R})$

- having a continuous local time $L_t(S,u)$ with
- discrete local times $(L_t^{\pi^n})$ of uniformly bounded p-variation, uniformly in $t \in [0,T]$ for all $T > 0$, i.e.

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|L_t^{\pi^n}(\bullet)\|_{p\text{-var}} < \infty.$$

Pathwise generalized Itô formula

Theorem (Perkowski and P. (2015))

Let $p, q \geqslant 1$ *be such that* $\frac{1}{p} + \frac{1}{q} > 1$. *Let* (π^n) *be a sequence of partitions as above and let* $S \in \mathcal{L}_{c,p}(\pi^n)$.

Let $f : \mathbb{R} \to \mathbb{R}$ *be absolutely continuous with right-continuous derivative* f' *of locally finite* q*-variation. Then for all* $t \in [0, \infty)$ *the generalized change of variable formula*

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\,\mathrm{d}S(s) + \int_{-\infty}^{\infty} L_t(u)\,\mathrm{d}f'(u)$$

holds, where $\mathrm{d}f'(u)$ *denotes Young integration and where*

$$\int_0^t f'(S(s))\,\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j \in \pi^n} f'(S(t_j))(S(t_{j+1} \wedge t) - S(t_j \wedge t)).$$

Outline

Problem: Not every continuous functions possess a local time!

Probabilistic examples: Brownian motion, martingales, semi-martingales, ...

Aim: Robust justification of local times in finance!

We follow Vovk's game-theoretic approach and set $\Omega := C([0,\infty),\mathbb{R})$.

A *simple strategy* $H: \Omega \times [0,\infty) \to \mathbb{R}$ consists of

1. stopping times $0 = \tau_0 < \tau_1 < \dots$ such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many n,
2. $\mathcal{F}_{\tau_n}$-measurable bounded functions $F_n : \Omega \to \mathbb{R}$.

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Vovk's Approach to Finance

Thus,

$$H_s(\omega) = \sum_{n=0}^{\infty} F_n(\omega) 1_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(s).$$

In that case the integral

$$\int_0^t H_s \, \mathrm{d}\omega_s = \sum_{n=0}^{\infty} F_n(\omega)[\omega_{\tau_{n+1}(\omega)\wedge t} - \omega_{\tau_n(\omega)\wedge t}]$$

is well-defined for every $\omega \in \Omega$.

No arbitrage opportunity of the first kind (NA1)

A simple strategy H is called 1-*admissible* if $\int_0^t H_s \, \mathrm{d}\omega_s \geqslant -1$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.

Definition

A set $A \subseteq \Omega$ is a null set if and only if there exists a sequence of 1-admissible simple strategies (H^n) such that

$$\liminf_{n \to \infty} \Big(1 + \int_0^\infty H_s^n \, \mathrm{d}\omega_s\Big) \geqslant \infty \bullet \mathbf{1}_A(\omega),$$

where we set $\infty \bullet 0 = 0$.

Remark: (NA1) is the minimal assumption a market model should fulfill!

(cf. Karatzas and Kardaras (2007), Ruf (2013), Imkeller and Perkowski (2015),...)

Typical price path

A property (P) holds for *typical price paths* if the set where (P) is violated is a null set.

Observations due to Vovk:

- Every null set is a null set under all (local) martingale measures on Ω.
- Typical price paths have quadratic variation along a sequence of partitions (π^n).
- Typical price paths have finite p-variation for $p > 2$.

Theorem (Perkowski and P., 2013)

Typical d-dimensional price paths have an associated Itô rough path in the sense of Lyons.

Construction of Local Time

For $n \in \mathbb{N}$ denote the set of dyadic points by $\mathbb{D}^n := \{k2^{-n} : k \in \mathbb{Z}\}$ and define the partition by the sequence π^n of stopping times

$$\tau_0^n(\omega) := 0, \quad \tau_{k+1}^n(\omega) := \inf\{t \geq \tau_k^n(\omega) : S_t(\omega) \in \mathbb{D}^n \setminus S_{\tau_k^n(\omega)}(\omega)\}.$$

We see

$$L_t^{\pi^n}(S,u) = (S_t - u)^- - (S_0 - u)^- + \sum_{j=0}^{\infty} \mathbf{1}_{(-\infty,u)}(S_{\tau_j^n})[S_{\tau_{j+1}^n \wedge t} - S_{\tau_j^n \wedge t}],$$

where we recall that

$$L_t^{\pi^n}(S,u) = \sum_{j=0}^{\infty} \mathbf{1}_{(\!|S_{\tau_j^n \wedge t}, S_{\tau_{j+1}^n \wedge t}]\!]}(u)|S_{\tau_{j+1}^n \wedge t} - u|.$$

Local Time for Typical Price Paths

Theorem (Perkowski and P. (2015))

Let $T > 0$, $\alpha \in (0, 1/2)$ *and* (π^n) *as defined on the last slide.*

For typical price paths $\omega \in \Omega$,

- *the discrete local time* L^{π^n} *converges uniformly in* $(t, u) \in [0, T] \times \mathbb{R}$ *to a limit* $L \in C([0, T], C^\alpha(\mathbb{R}))$,
- *there exists* $C = C(\omega) > 0$ *such that*

$$||L^{\pi^n} - L||_{L^\infty([0,T]\times\mathbb{R})} \leq C 2^{-n\alpha},$$

- *uniformly bounded* p*-variation of* L^{π^n} *for* $p > 2$, *i.e.*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|L_t^{\pi^n}(\bullet)\|_{p\text{-var}} < \infty.$$

References

Perkowski, N. and Prömel, D. J. (2015). Local times for typical price paths and pathwise Tanaka formulas. *Electron. J. Probab.*, (46):1–15.

Thank you very much for your attention!