# Managing Inventory with Proportional Transaction Costs


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based on joint work with Remy Chicheportiche, Florent Gallien, Julien Hugonnier, Serge Kassibrakis and Semyon Malamud.

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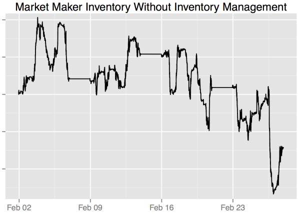
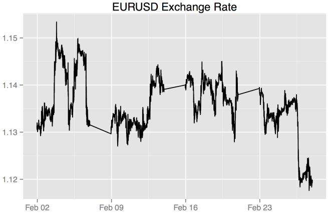
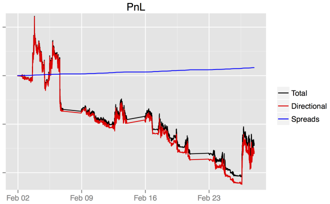
# Introduction

- Market makers are active in several markets. They provide a pair of bid-ask prices to their clients, buying financial instruments at the bid price and selling at the ask price. Their main goal is to benefit from the bid-ask spread.
- Usually, market makers are supposed to trade any amount of assets at the quoted bid and ask prices.
- Their inventory can grow in an unpredictable way, leading to potential loss if the prices move in an unfavorable way $\rightarrow$ **Directional Risk**.

- For instance, consider the inventory of the EURUSD pair of a Forex broker, **without inventory management**:



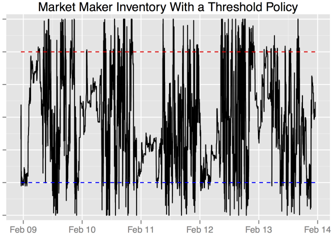




- Huge fluctuations of directional PnL, losses can be larger than profits from the spreads !

- For an optimal market making activity, it is crucial to reduce the directional exposure, controlling the inventory. How to do that?
- Market makers can control the inventory in different ways:
  - Adjusting the quoted bid and ask prices.
  - Trading the inventory in excess with other market participants (active trading). For instance, a Forex broker can trade with other brokers or other participant in the interbank market (Liquidity Providers (LP)).

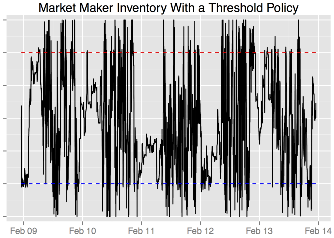
- In the following we will study the case of a market maker that manages his inventory using only active trades with Liquidity Providers.
- Usually, practitioners implement this risk control strategy using a threshold policy:




- Two thresholds are fixed. They encode the risk preference of the market maker.
- A rebalancing trade is executed when the inventory exceed the thresholds.

- The threshold policy is quite natural. Indeed, rebalancing trades imply proportional transaction costs that, as usual, give rise to a no-trade region.
- The strategy is completely characterized by the choice of the thresholds.
- Thresholds are chosen in order to get the ideal trade off between potential spreads revenue and directional risk.
- In the following, we will use the HJB formalism to derive the optimal thresholds.

# Thresholds Euristic




- Based on intuition, we can expect the thresholds to depend on the following parameters:
  - Asset price volatility. (High Vol $\rightarrow$ Narrow non-trading zone)
  - Asset price bid-ask spread. (High spread $\rightarrow$ Wide non-trading zone)
  - Asset price drift. (Shift of the whole non-trading zone, accordingly)
  - Expected imbalance of client orders. (Shift of the thresholds)
- Our formal derivation of the thresholds will confirm all of these intuitive expectations !

# The Mathematical Model

- The market maker manages in continuous time an inventory of a risky asset whose price $P_t$ follows an arithmetic Brownian motion

$$dP_t \;=\; \mu\, dt \;+\; \sigma\, dB_t$$

- The asset value is continuously marked-to-market.
- The inventory of the risky asset is denoted by $q_t$ and it follows

$$dq_t = dN_t + dM_t$$

  - $dN_t$ is the contribution of client orders. They are modeled as a compound Poisson process with intensity $\lambda$ and jump sizes that are drawn from a distribution $d\eta(x)$. A positive (negative) realization of the jump corresponds to a client's sell (buy) order.
  - $dM_t$ are the trades with the liquidity providers.

# Spreads revenue

- We denote the market bid-ask spread as $\alpha_l$. Therefore:
  - $P_t^{\text{LP}} = P_t + \text{sign}(dM_t)\alpha_l$ is the price for an LP trade $dM_t$
  - $\alpha_l|dM_t|$ is the cost of an LP trade $dM_t$
- In agreement with the real world practice, the market maker charge a fee ($m$) to his clients that is proportional to the orders. Defining $\alpha = \alpha_l + m$ as the clients spread, we have:
  - $P_t^{\text{cl}} = P_t - \text{sign}(dN_t)\alpha$ is the price for a client trade $dN_t$
  - $\alpha|dN_t|$ is the profit due to a client trade $dN_t$
- In total, the spreads revenue $d\nu_t$ is given by

$$d\nu_t = \alpha|dN_t| \; - \; \alpha_l|dM_t|$$

- The total cash flows $dC_t$ of the agent can be written as

$$dC_t = dL_t + d\nu_t$$

  - $d\nu_t$ is the spreads revenue
  - $dL_t = q_t dP_t$ directional exposure of the existing inventory due to marking-to-market

- Therefore, the PnL between time $t_1$ and $t_2$ is:

$$\mathrm{PnL}_{(t_1,t_2)} = q_{t_2} P_{t_2} - q_{t_1} P_{t_1} - \int_{t_1}^{t_2} P_t^{\mathrm{cl}} dN_t - \int_{t_1}^{t_2} P_t^{\mathrm{LP}} dM_t$$

  - $P_t^{\mathrm{cl}} = P_t - \mathrm{sign}(dN_t)\alpha$ Clients trades price
  - $P_t^{\mathrm{LP}} = P_t + \mathrm{sign}(dM_t)\alpha_l$ LP trades price

# Utility Function

- We assume that the agent maximizes constant absolute risk aversion (CARA) utility from intermediate consumption $c_t$

$$V(w, q) \; = \max_{\{(c_t, dM_t) | t \in (0,\infty)\}} E\left[\int_0^\infty -e^{-\beta t} e^{-\gamma c_t} dt\right]$$

  - $q = q_{t=0}$ and $w = W_{t=0}$
  - $\gamma$ is the risk aversion parameter

- The wealth $W_t$ satisfies the following budjet constraint

$$dW_t \; = \; (rW_t - c_t)dt \; + \; dL_t \; + \; d\nu_t$$

# HJB Equation

As is common in models with proportional transaction costs, we can decompose the state space into the *no-trade region* and the *trading region*:

- **no-trade region**: only optimization over the consumption rate

$$\sup_c \Big\{ -e^{-\gamma c} + (rw - c + q\mu)V_w + 0.5\sigma^2 q^2 V_{ww} - \beta V + \lambda \int (V(w + \alpha|x|, q + x) - V(w, q)) d\eta(x) \Big\} = 0$$

  - $-e^{-\gamma c}$ is the flow of marginal utility
  - $(rw - c + q\mu)V_w$ is due to the drift in the agent's wealth
  - $0.5\sigma^2 q^2 V_{ww}$ is due to volatility
  - $-\beta V$ is a time discount term
  - the last term is the expected change due to client orders

# HJB Equation

- **trading region**: the agent immediately contacts the liquidity provider and adjusts his inventory position to end up to the border of the no-trade region.

  The value function satisfies the condition

  $$\max_{\bar{q}}(V(w - \alpha_l|q - \bar{q}|, \bar{q}) - V(w, q)) = 0$$

  - $\alpha_l|q - \bar{q}|$ is the cost of the trade with the LP

- It is useful to define the rescaled transaction costs
  - $\varphi \;=\; r\gamma\alpha_l$ paid by the agent to the liquidity provider
  - $\psi \;=\; r\gamma\alpha$ paid by the clients to the agent

# HJB Equation

### Theorem

*The optimal policy is of a single band type: there exist $q_L < q_H$ and a convex function $a(q) \in C^1(\mathbb{R})$ such that*

- *the value function is given by* $V(w,q) = -e^{-r\gamma w + a(q)}$
- *we have*

$$a(q) = a(q_H) + \varphi(q - q_H), \; q > q_H$$
$$a(q) = a(q_L) + \varphi(q_L - q), \; q < q_L$$

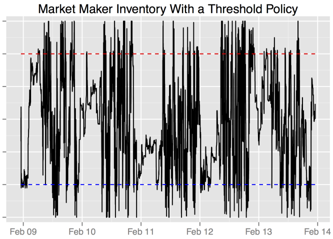
- *for $q \in [q_L, q_H]$, the function $a(q)$ satisfies the HJB equation*

$$-r + r(\log r + a(q)) + r\gamma q\mu - 0.5\sigma^2 q^2 (r\gamma)^2 + \beta$$
$$+\lambda \int_{\mathbb{R}} (-e^{-\psi|x| + a(q+x) - a(q)} + 1) d\eta(x) = 0$$

*together with the boundary conditions* $-a'(q_L) = a'(q_H) = \varphi$.

# HJB Equation

- The optimal inventory policy is characterized by the band $[q_L, q_H]$:




  - $q_H$ is the red line
  - $q_L$ is the blue line

- To get $q_H$ and $q_L$ explicitly we need to solve HJB.

# Zero order arrival intensity

- A simple case: set to zero the order arrival intensity, $\lambda = 0$.
- We denote the corresponding disutility function by $a_0(q)$ and the no-trade region by $[q_L(0), q_H(0)]$.
- It results

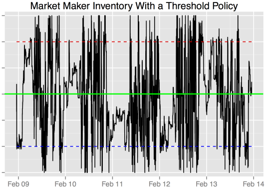
$$a_0(q) \;=\; 1 - \log r + \gamma(-q\mu + 0.5\sigma^2 q^2 r\gamma) - \beta r^{-1}$$

- Solving the boundary conditions $-a_0'(q_L) = a_0'(q_H) = \varphi$, we get

$$q_H(0) \;=\; q_* \;+\; \frac{\alpha_l}{r\gamma\sigma^2}$$
$$q_L(0) \;=\; q_* \;-\; \frac{\alpha_l}{r\gamma\sigma^2}$$

- Thus, the no-trade region is symmetric around $q_*$, the Merton portfolio

$$q^* \;=\; \frac{\mu}{r\gamma\sigma^2}$$




- These results are compatible with previous literature on proportional costs.
- In agreement also with intuitive expectations we already mentioned:
  - no-trade zone width $\propto (1/\sigma^2)$ (High Vol $\rightarrow$ Narrow non-trading zone)
  - no-trade zone width $\propto \alpha_l$ (High spread $\rightarrow$ Wide non-trading zone)
  - no-trade zone centered around Merton portfolio
    (A price drift shifts the whole non-trade zone, accordingly)

# Client Orders

- What is the effect of client orders ($\lambda \neq 0$) ?

**Proposition**

*The presence of inventory shocks always widens the no-trade region. That is, we always have* $q_H \geq q_H(0) > q_L(0) \geq q_L$ .

- We have studied in details the following limit cases:
  - $\lambda \ll 1$. Small Order Arrival Intensity
  - $\lambda \gg 1$. Large Order Arrival Intensity
- In the following we will focus on the case $\lambda \gg 1$, more relevant in reality.

# Large Order Arrival Intensity

- When $\lambda$ is very large, it makes sense to work with the following **rescaled quantities**:
  - $y = q/\lambda$ Rescaled inventory.
  - $\bar{\rho} = \lambda\rho$ where $\rho = 0.5\sigma^2\gamma^2 r$. Basically, rescaled risk aversion.
  - $A(y) = a(\lambda y)/\lambda$ Rescaled disutility function.
  - $y_{H,L} = q_{H,L}/\lambda$ Rescaled boundaries.

- the HJB equation in the no-trade zone becomes:

$$
\begin{aligned}
&-r + r(\log r + \lambda A(y)) + r\gamma\lambda y\mu - r\bar{\rho}\lambda y^2 + \beta \\
&+ \lambda \int_{\mathbb{R}} (-e^{-\psi|x| + \lambda(A(y+x/\lambda) - A(y))} + 1) d\eta(x) \;=\; 0
\end{aligned}
$$

- In the limit $\lambda \to \infty$, the HJB equation reduces to:

$$r(A(y) - \bar{\Phi}(y)) \;-\; \Psi(A'(y)) \;=\; 0$$

  - where:

$$\Psi(z) \;\equiv\; \int_{\mathbb{R}} e^{-\psi|x|+xz} d\eta(x)$$

    is the moment-generating function of the *markup-adjusted order size distribution* $e^{-\psi|x|} d\eta(x)$

  - We also defined: $\bar{\Phi}(y) \;\equiv\; \bar{\rho} y^2 - \gamma\mu y \;-1/r$

- The boundary conditions are: $-A'(y_L) = A'(y_H) = \varphi$

- Solving the optimal inventory management problem for large $\lambda$ reduces to solving an ODE with certain boundary conditions !

### Theorem

- *If* $\Psi'(\varphi) \geq 0 \geq \Psi'(-\varphi)$ *then the no-trade region is given by* $[y_L(0), y_H(0)]$;
- *If* $\Psi'(\varphi) > \Psi'(-\varphi) \geq 0$, *let* $\bar{y}_L$ *be the unique solution to* $A'_+(\bar{y}_L; y_H(0)) = -\varphi$. *Then, the no-trade region is given by* $[\bar{y}_L, y_H(0)]$;
- *If* $0 \geq \Psi'(\varphi) > \Psi'(-\varphi)$, *let* $\bar{y}_H$ *be the unique solution to* $A'_-(\bar{y}_H; y_L(0)) = \varphi$. *Then, the no-trade region is given by* $[y_L(0), \bar{y}_H]$.

- Where $A_\pm(x; b)$ is the unique smooth solution to

$$\begin{aligned} \Psi(A'_\pm(x;b)) &= r(A_\pm(x;b) - \bar{\Phi}(x)) \\ A'_\pm(b;b) &= \pm\varphi \end{aligned}$$

- Different types of optimal strategy are characterized by the sign of $\Psi'(\varphi)$ and $\Psi'(-\varphi)$ where

$$\Psi'(z) = \int_{\mathbb{R}} x e^{-\psi|x|+xz} d\eta(x)$$

  Client orders distribution $\eta$ and the transaction cost $\varphi$ determine the type of optimal strategy.

- Let's explain these results in a couple of cases:
  - client orders are **balanced** $\rightarrow \Psi'(\varphi) \geq 0 \geq \Psi'(-\varphi)$

    Theorem $\rightarrow$ no-trade region is given by $[y_L(0), y_H(0)]$, the same as in the case without client orders !

    Intuition: no need to change strategy. Orders will cancel each other.

- client orders are **negative on average** $\rightarrow 0 \geq \Psi'(\varphi) > \Psi'(-\varphi)$

  Theorem $\rightarrow$ no-trade region is given by $[y_L(0), \bar{y}_H]$.

  Lower bound is $y_L(0)$, the same as in the case without client orders.

  Upper bound is $\bar{y}_H$, that is **larger** than the case without client orders.

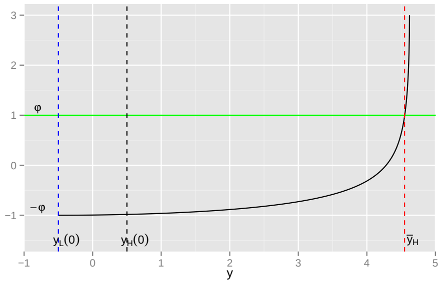
  Intuition: new arriving negative orders will quickly bring the inventory closer to the optimal position. The agent can avoid active trades and spare some transaction costs.

- Let's derive $\bar{y}_H$:
  - First, we solve this ODE:

$$\begin{aligned} \Psi(A'_-(x; y_L(0))) &= r(A_-(x; y_L(0)) - \bar{\Phi}(x)) \\ A'_-(y_L(0); y_L(0)) &= -\varphi \end{aligned}$$

  - Then, $\bar{y}_H$ is given by:

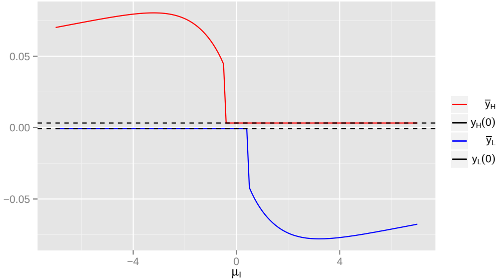
$$A'_-(\bar{y}_H; y_L(0)) = \varphi$$



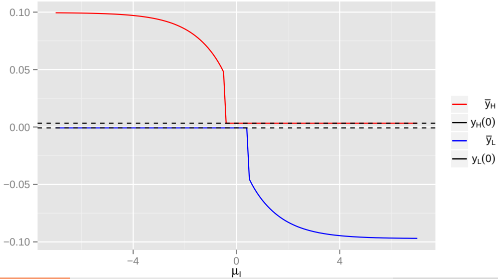

In this plot $\Psi'(\varphi) = -0.008$, $\Psi'(-\varphi) = -0.632$, $y_L(0) = -0.49$, $y_H(0) = 0.5$, $\bar{y}_H = 4.5$.

No-trade zone as a function of client orders imbalance:

- $\varphi < \psi$




- $\varphi = \psi$

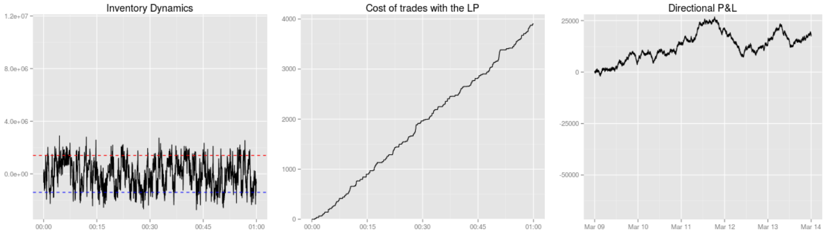


# Monte Carlo simulation

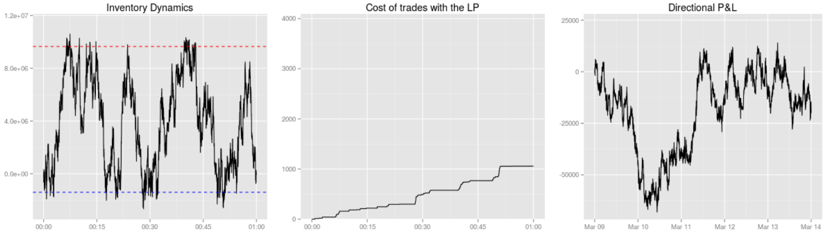
Let's test the results with synthetic data.

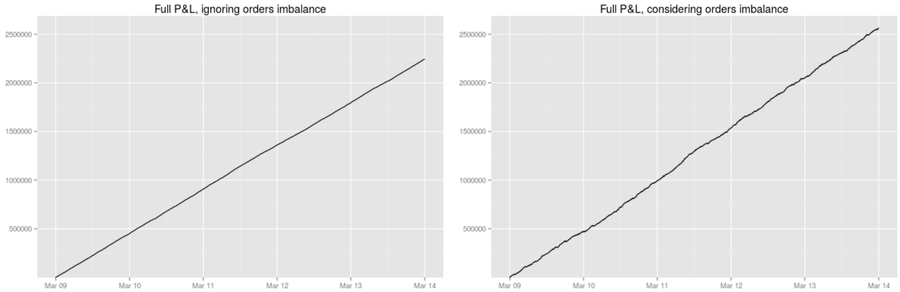
- The asset follows an Arithmetic Brownian Motion with parameters typical of the EURUSD pair:
  - Volatility: vol $\approx 1.3\, x\, 10^{-3}$ USD per Hour
  - Half Spread: $\alpha_I \approx 2.5\, x\, 10^{-5}$ USD
  - Rate Drift: $\mu = 0$ (no drift)
- Client Orders:
  - Client Spread: $\alpha_I = \alpha$ (remove the benefits from markup)
  - Intensity: $\lambda \approx 1$ per second.
  - Normal distribution with mean=-5000 USD and sd=500000 USD.
- Risk aversion parameter chosen to have:
  - $q_H(0) \approx 1.4$ millions
  - $q_L(0) \approx -1.4$ millions

- No-trade zone is $[y_L(0), y_H(0)]$. Order imbalance effect is ignored.




- No-trade zone is $[y_L(0), \bar{y}_H]$. Order imbalance effect is considered.



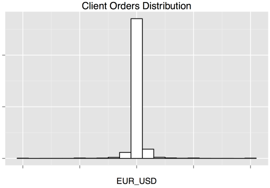


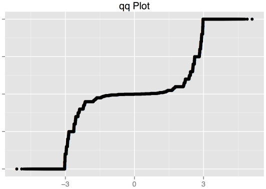

- Taking into account the effect of the order imbalance we improve the total P&L !

# Real data

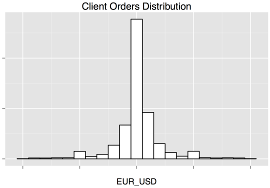
Let's test some of the results using data of EURUSD trades of Swissquote clients.

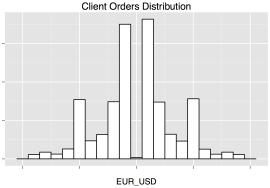
- The distribution of YTD orders has fat tails:



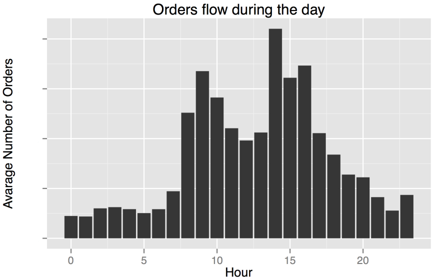




- there is not significant imbalance on a YTD timescale:



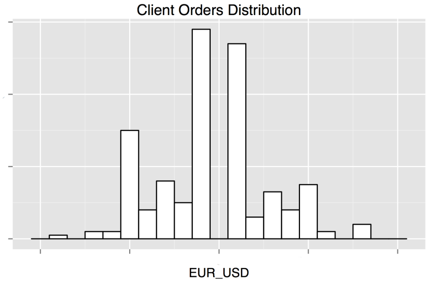



- The flow of orders has a strong intra day seasonality:



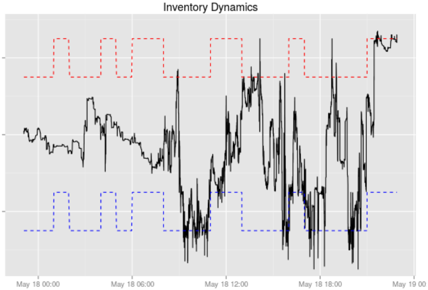

- During peak hours, $\lambda \gg 1$ per Hour.

- The orders distribution for 1 hour can show imbalance. For instance, 2015-02-26 from 13:00 to 14:00:

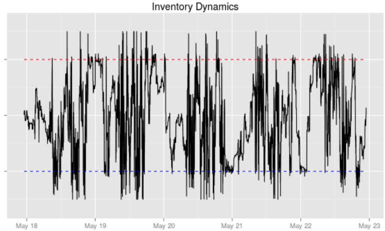


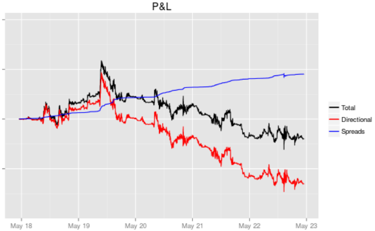

- Therefore, we can try to apply our model, changing the parameters hour by hour.

- We use our model to compute the thresholds, hour by hour.
- Thresholds depend on asset volatility, asset drift, asst bid-ask spread and orders distribution.
- Parameters are computed using simple in sample analysis.
- For practical reasons, we cap the theoretical thresholds between a max and a min value. That is, each thresholds oscillate between two values.

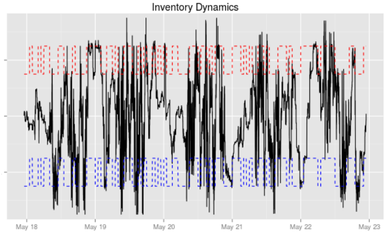


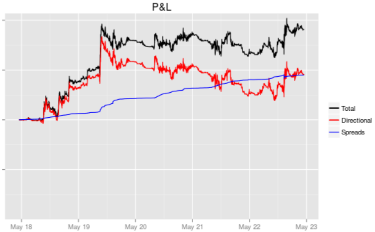
- Static Threshold:




- Dynamical Threshold:




- Significant improvement, also with the cap. Possible out of sample/ reality application !

# Conclusion

- We have built a practical formalism to compute optimal thresholds for market making.
- Our model takes into account several aspects of the asset dynamics and clients behavior.
    - Expectation on clients orders are used to spare risk hedging costs.
    - Possible to combine market making activity with directional trading.
- Future directions:
    - Many assets case: correlations will play an important role.
    - Evolution of parameters in the theoretical model.
    - Correlation between clients behavior and market conditions in the theoretical model.

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