Asymptotic behaviour of multivariate default probabilities and default correlations under stress

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Stress testing of bank portfolios

- Calculate risk measures (expected loss, value-at-risk, economic capital) and regulatory capital under adverse market conditions
- Stress tests are typically conducted within models
- Crucial inputs of any portfolio model:
  - Distribution assumption on portfolio constituents, e.g.
    - normally distributed asset returns
    - fat-tailed asset returns
  - Dependence assumption among portfolio constituents, e.g. correlation

Stress testing of bank portfolios

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- **Questions:**
  - What is the model behaviour under stress?
  - What are model side effects when stress testing?
- In a series of papers we investigate these questions:
  - M. Kalkbrener and N. Packham. Correlation under stress in normal variance mixture models. *Mathematical Finance*, 25:2 (2015), 426–456.
  - M. Kalkbrener and N. Packham. Stress testing of credit portfolios in light- and heavy-tailed models. *J. Risk Management in Financial Institutions*, 8:1 (2015), 34–44.
  - N. Packham, M. Kalkbrener, and L. Overbeck. Asymptotic behaviour of multivariate default probabilities and default correlations under stress. *J. Applied Probability*, 53:1 (2016).

Overview

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- Structural credit portfolio models and stress testing
- Distribution of model variables
- Asset correlations under stress
- Default probabilities and default correlations under stress
- Risk measures
- Conclusion

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Structural credit model

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- Merton-model (Merton, 1974) links default of firm to relationship between assets and liabilities
- Counterparty i in default at T...
  - ... if asset value Z_i ...
  - ... below debt value D_i

Assets	**Liabilities**
Assets	Debt
	Equity

- Default event: $\{Z_i < D_i\}$
- Portfolio loss:

$$L := \sum_{i=1}^{d} l_i \cdot \mathbf{1}_{\{Z_i \leq D_i\}},$$

with l_i loss-at-default of counterparty i

Credit portfolio risk

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- Risk concentrations, correlations:

$$Z_i = \sqrt{R_i^2} \sum_{j=1}^{m} w_{ij} X_j + \sqrt{1 - R_i^2} \varepsilon_i, \quad i = 1, \dots, d,$$

where

- $X_1, \dots, X_m$: systematic factors or risk factors,
- ε_i: firm-specific factor

Credit portfolio risk measures

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- Expected loss: $\mathbb{E}(L) = \sum_{i=1}^{d} l_i \cdot \mathbf{P}(Z_i \leq D_i)$
- Value-at-risk (at level β): β-quantile of L:

$$\mathrm{VaR}_\beta(L) = \inf\{x \in \mathbb{R} : \mathbf{P}(L \leq x) \geq \beta\}$$

- Default correlations as measure of dependence:

$$\mathrm{Corr}(\mathbf{1}_{\{Z_i \leq D_i\}}, \mathbf{1}_{\{Z_j \leq D_j\}}) = \frac{\mathbf{P}(Z_i \leq D_i, Z_j \leq D_j) - p_i\, p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}},$$

where $p_i := \mathbf{P}(Z_i \leq D_i)$, $i = 1, \dots, d$

Stress testing

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- Translate stress scenario into constraints on risk factors
- Truncate risk factor variable Z_0 (which is typically one of the systematic factors X_i):

$$Z_0 \leq C, \quad C \in \mathbb{R} \text{ stress level}$$

- Portfolio risk is evaluated under $\mathbf{P}(\cdot | Z_0 \leq C)$
- Consistent framework that associates severity of stress scenario with probability of stress scenario
- See e.g. Bonti *et al.* (2006); Duellmann and Erdelmeier (2009); Kalkbrener and Packham (2015a)

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Elliptically distributed random variables

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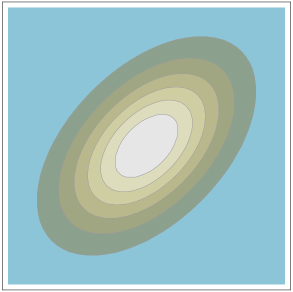
- Standard approach in credit risk portfolio modelling: risk factors and asset variables normally distributed
- Generalisations:
  - normal variance mixture distribution
  - elliptical distribution
- Cover variety of light-tailed to heavy-tailed distribution

Example bivariate density:



Elliptically distributed random variables

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- Let random vector $\mathbf{Z} = (Z_0, \ldots, Z_d)^T$ follow elliptical distribution with representation

$$\mathbf{Z} \stackrel{\mathcal{L}}{=} GA\mathbf{U},$$

  where
  - $G > 0$ is a scalar random variable, the mixing variable,
  - A is a deterministic $(d+1) \times (d+1)$ matrix with $AA^T := \Sigma$, which in turn is a $(d+1) \times (d+1)$ nonnegative definite symmetric matrix of rank $d+1$,
  - $\mathbf{U}$ is a $(d+1)$-dimensional random vector uniformly distributed on the unit sphere $\mathcal{S}_{d+1} := \{\mathbf{z} \in \mathbb{R}^{d+1} : \mathbf{z}^T\mathbf{z} = 1\}$,
  - $\mathbf{U}$ is independent of G.

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Asset correlations

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Asset correlations under stress (Kalkbrener and Packham, 2015a):

- multivariate normal or t-distribution: explicit formulas for asset correlations under stress

$$\mathrm{Corr}^C(Z_i, Z_j)$$

- Normal variance mixture distribution:

$$\lim_{C\to-\infty} \mathrm{Corr}^C(Z_i, Z_j) = \frac{\rho_i\,\rho_j + (\rho_{ij} - \rho_i\,\rho_j)\,(\alpha-1)}{\sqrt{(\rho_i^2 + (1-\rho_i^2)\,(\alpha-1))\,(\rho_j^2 + (1-\rho_j^2)\,(\alpha-1))}},$$

with $\alpha > 2$ the tail index of asset returns and risk factor and $\rho_i = \rho_{0i}$

Asset correlations

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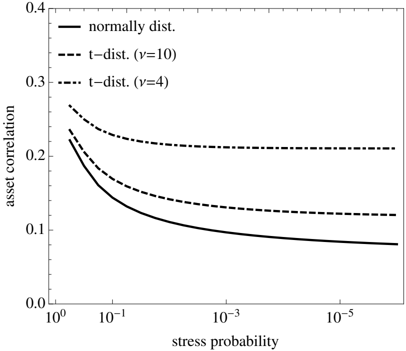
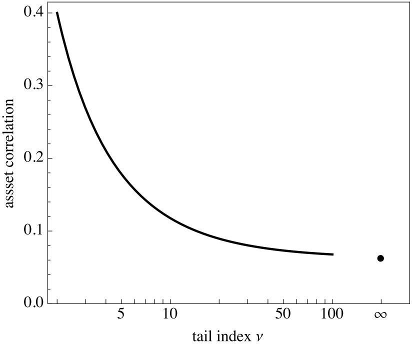
- In a typical scenario, assets de-correlate with increasing stress and increasing tail index
- Helps explain why the relative impact of stress on EL of portfolio is stronger than on VaR





- Left: Conditional asset correlations; right: Asymptotic asset correlations

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Preliminaries

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- Fréchet max domain / regular variation:
  - RV_α: regularly varying functions with index $\alpha \in \mathbb{R}$
  - G in Fréchet max domain iff $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$ for some $\alpha > 0$
- Gumbel max domain / rapid variation:
  - $RV_{-\infty}$: rapidly varying functions
  - If G in Gumbel max-domain, then $\mathbf{P}(G > \cdot) \in RV_{-\infty}$

Preliminaries

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- Gumbel max domain / rapid variation:
  - $RV_{-\infty}$: rapidly varying functions
  - If G in Gumbel max-domain, then $\mathbf{P}(G > \cdot) \in RV_{-\infty}$
- Random vector $\mathbf{Z} = GA\mathbf{U}$ standardised, so that $\Sigma = AA^T$ is the correlation matrix
- Correlations are positive, i.e., $\rho_{ij} > 0$
- $A_{i\cdot}$: i-th row of A
- $F_\mathbf{U}$: uniform distribution on $\mathcal{S}_{d+1}$

Default probabilities

Theorem (Packham *et al.* (2016))

(i) *If* $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$, *then*

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \leq D_1, \dots, Z_d \leq D_d | Z_0 \leq C)$$

$$= \int_{\mathbf{u} \in \mathcal{S}_{d+1}, A_0.\mathbf{u} > 0, \dots, A_d.\mathbf{u} > 0} (A_0.\mathbf{u})^\alpha \, \mathrm{d}F_{\mathbf{U}}(\mathbf{u}) \left(\int_{\mathbf{u} \in \mathcal{S}_{d+1}, A_0.\mathbf{u} > 0} (A_0.\mathbf{u})^\alpha \, \mathrm{d}F_{\mathbf{U}}(\mathbf{u}) \right)^{-1}.$$

(ii) *If* $\mathbf{P}(G > \cdot) \in RV_{-\infty}$, *then*

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \leq D_1, \dots, Z_d \leq D_d | Z_0 \leq C) = 1.$$

- Asymptotic default probabilities do not depend on default thresholds.

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Theorem (Packham *et al.* (2016))

Let $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$.

(i) *For* $d = 1$,

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \le D_1 | Z_0 \le C) = t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,\rho_{01}}{\sqrt{1-\rho_{01}^2}}\right) \in [1/2, 1).$$

(ii) *For* $d = 2$,

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le D_2 | Z_0 \le C)$$

$$= \frac{1}{2} t_{\alpha+1}\left(\frac{\sqrt{(\alpha+1)}\,t}{\sqrt{1-t^2}}\right) + \textit{some long integrals.}$$

Default probabilities

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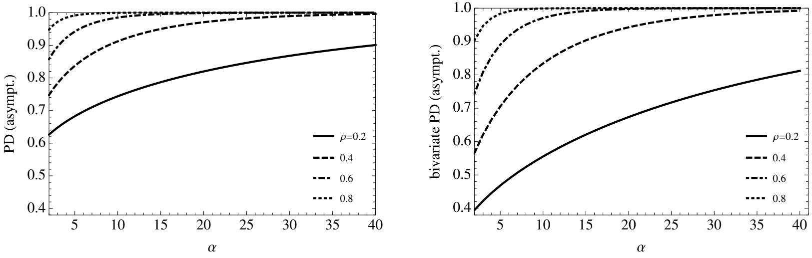


- Asymptotic univariate PD's (left) and bivariate PD's (right) as a function of the tail index.
- Correlations: $\rho_{01} = \rho_{02} = \rho$ and $\rho_{12} = \rho^2$.

Default correlations

- Regularly varying case easily calculated from previous Theorem.

Theorem

Let $\mathbf{P}(G > \cdot) \in RV_{-\infty}$. *Then,*

$$\lim_{C \to -\infty} Corr^C(\mathbf{1}_{\{Z_1 \leq D_1\}}, \mathbf{1}_{\{Z_2 \leq D_2\}}) = 0,$$

where $Corr^C$ *denotes the correlation under* $\mathbf{P}(\cdot | Z_0 \leq C)$.

Default probabilities vs. tail dependence

- Asymptotic default probability:

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \leq D | Z_0 \leq C).$$

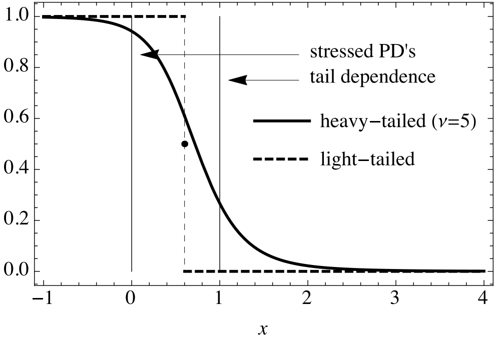
- Coefficient of (lower) tail dependence:

$$\lambda_l(Z_0, Z_1) := \lim_{C \to -\infty} \mathbf{P}(Z_1 \leq C | Z_0 \leq C).$$

- In the light-tailed case, tail dependence is 0, which is in contrast to the asymptotic default probability
- Tail dependence function that captures both:

$$\lambda(Z_0, Z_1, x) := \lim_{C \to -\infty} \mathbf{P}(Z_1 \leq x \cdot C | Z_0 \leq C), \quad x \in \mathbb{R}.$$

Relation to tail dependence



- Closed formula for tail dependence function $\lambda(Z_0, Z_1, x)$ in paper
- Special cases:
  - stressed PD's: $x = 0$
  - tail dependence: $x = 1$

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Risk measures

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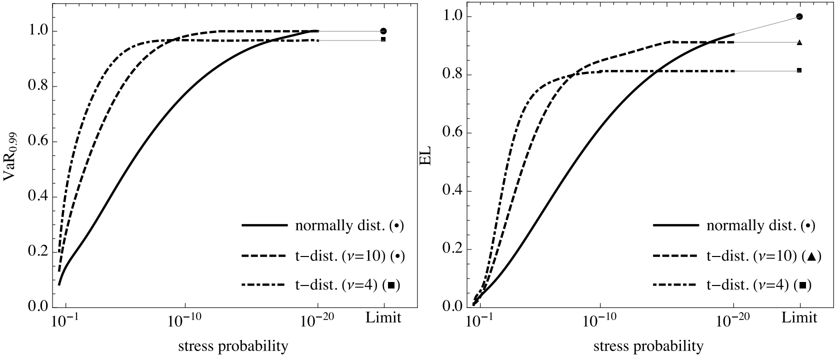
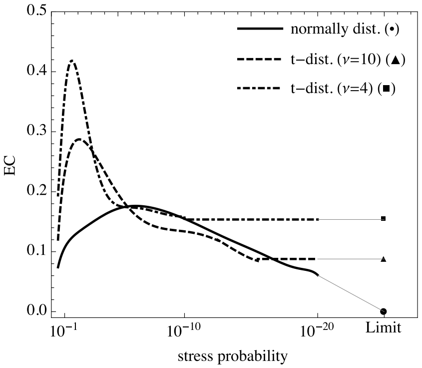


- Risk measures for portfolio consisting of 60 homogeneous counterparties, each with a PD of 1%.
- Left: Value-at-risk at 99% confidence level
- Right: Expected loss

Risk measures



- Risk measures for portfolio consisting of 60 homogeneous counterparties, each with a PD of 1%.
- Economic Capital (VaR-EL)

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Conclusion

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- Stress tests are an integral part of risk management and banking supervision, ...
- ... and the analysis and understanding of risk model behaviour under stress has become ever more important.
- We analyse asset correlations, default probabilities and default correlations under stress in a generalised Merton-type credit portfolio setup covering light- and heavy-tailed distributions.
- It turns out that the model behaviour under stress depends on the heaviness of the tails of the risk factors.
- Contrary to popular belief, light-tailed models show a higher impact in extreme stress scenarios.
- We use our results to study the implications for credit reserves and capital requirements under stress.

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Thank you!

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