A pricing measure for non-tradable assets with mean-reverting dynamics

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AMAMEF & Swiss Quote Conference
EPFL, Laussane
7-10 September 2015

Support from the projects EMMOS and MAWREM is gratefully acknowledged

Non tradable assets

- An asset is non tradable if we cannot build a portfolio with it.
- Energy related examples: non-storable commodities, weather indices, electricity...
- Interested in pricing forwards contracts on these assets.
- There is no buy and hold strategy $\Longrightarrow$ classical non-arbitrage arguments break down.
- Any probability measure Q equivalent to the historical measure P is a valid risk neutral pricing measure.
- The forward price with time to delivery $0 < T < T^*$ at time $0 < t < T$ is given by

$$F_Q(t, T) = \mathbb{E}_Q[S(T)|\mathcal{F}_t]$$

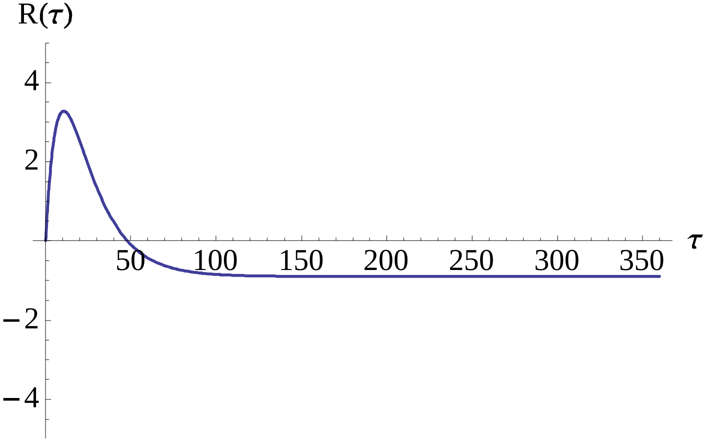
  where $\mathcal{F}_t$ is the information in the market up to time t. Assuming deterministic interest rates and $r = 0$.

Risk premium profile

- The risk premium for forward prices is defined by

$$R_Q^F(t,T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t] - \mathbb{E}_P[S(T)|\mathcal{F}_t].$$

- **Goal**: To be able to obtain more realistic risk premium profiles. For instance:



where $\tau = T - t$ is the time to delivery.

Mathematical model

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]}, P)$ be a filtered probability space satisfying the usual hypothesis, where $T^* > 0$ is a fixed finite time horizon.
- Consider a standard Brownian motion W and a pure jump Lévy subordinator

$$L(t) = \int_0^t \int_0^\infty zN^L(ds, dz), t \in [0, T^*],$$

where $N^L(ds, dz)$ is a Poisson random measure with Lévy measure ℓ satisfying $\int_0^\infty z\ell(dz) < \infty$.
- Let

$$\kappa_L(\theta) \triangleq \log \mathbb{E}_P[e^{\theta L(1)}],$$

and

$$\Theta_L \triangleq \sup\{\theta \in \mathbb{R}_+ : \mathbb{E}_P[e^{\theta L(1)}] < \infty\}.$$

- A minimal assumption is that $\Theta_L > 0$.

Mathematical model

- Consider the Ornstein-Uhlenbeck processes X with Barndorff-Nielsen & Shephard stochastic volatility σ

$$X(t) = X(0) - \alpha \int_0^t X(s)ds + \int_0^t \sigma(s)W(t),$$
$$\sigma^2(t) = \sigma^2(0) - \rho \int_0^t \sigma^2(s))ds + L(t)$$
$$= \sigma^2(0) + \int_0^t (\kappa_L'(0) - \rho\sigma^2(s))ds + \int_0^t \int_0^\infty z\tilde{N}^L(ds, dz),$$

  with $t \in [0, T^*], \alpha, \rho > 0, X(0) \in \mathbb{R}, \sigma^2(0) > 0$ and

$$\tilde{N}^L(ds, dz) = N^L(ds, dz) - \ell(dz)ds.$$

- We model the spot price process by

$$S(t) = \Lambda_g \exp\left(X(t)\right), \quad t \in [0, T^*].$$

The change of measure

- Let $\bar{\beta} = (\beta_1, \beta_2) \in [0,1]^2$ and $\bar{\theta} = (\theta_1, \theta_2) \in \bar{D}_L \triangleq \mathbb{R} \times D_L$, where $D_L \triangleq (-\infty, \Theta_L/2)$.
- Consider the following family of kernels

$$G_{\theta_1,\beta_1}(t) \triangleq \sigma^{-1}(t)\left(\theta_1 + \alpha\beta_1 X(t)\right), \quad t \in [0, T^*],$$

$$H_{\theta_2,\beta_2}(t,z) \triangleq e^{\theta_2 z}\left(1 + \frac{\rho\beta_2}{\kappa_L''(\theta_2)} z\sigma^2(t-)\right), \quad t \in [0, T^*], z \in \mathbb{R}_+.$$

- Next, define the following family of Wiener and Poisson integrals

$$\tilde{G}_{\theta_1,\beta_1}(t) \triangleq \int_0^t G_{\theta_1,\beta_1}(s)dW(s), \quad t \in [0, T^*],$$

$$\tilde{H}_{\theta_2,\beta_2}(t) \triangleq \int_0^t \int_0^\infty \left(H_{\theta_2,\beta_2}(s,z) - 1\right)\tilde{N}^L(ds,dz), \quad t \in [0, T^*],$$

associated to the kernels G_{θ_1,β_1} and H_{θ_2,β_2}, respectively.

The change of measure

- The family of measure changes is given by $Q_{\bar{\theta},\bar{\beta}} \sim P, \bar{\beta} \in [0,1]^2, \bar{\theta} \in \bar{D}_L$, with

$$\left.\frac{dQ_{\bar{\theta},\bar{\beta}}}{dP}\right|_{\mathcal{F}_t} \triangleq \mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(t), \quad t \in [0, T^*],$$

  where $\mathcal{E}(\cdot)$ denotes the stochastic exponential.
- Recall that, if M is a semimartingale, the stochastic exponential of M is the unique strong solution of

$$\begin{aligned} d\mathcal{E}(M)(t) &= \mathcal{E}(M)(t-)dM(t), \quad t \in [0, T^*], \\ \mathcal{E}(M)(0) &= 1, \end{aligned}$$

  which is given by

$$\mathcal{E}(M)(t) = \exp\left(M(t) - \frac{1}{2}\langle M^c, M^c\rangle(t)\right) \prod_{0\leq s\leq t} (1 + \Delta M(s))\, e^{-\Delta M(s)}.$$

The change of measure

- **Yor's Addition Formula**: Let M_1 and M_2 two semimartingales starting at 0. Then,

$$\mathcal{E}(M_1 + M_2 + [M_1, M_2])(t) = \mathcal{E}(M_1)(t)\mathcal{E}(M_2)(t), \quad 0 \le t \le T^*,$$

where

$$[M_1, M_2](t) = \langle M_1^c, M_2^c \rangle(t) + \sum_{s \le t} \Delta M_1(s) \Delta M_2(s).$$

- As $[\tilde{G}_{\theta_1,\beta_1}, \tilde{H}_{\theta_2,\beta_2}] \equiv 0$, we can write

$$\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(t) = \mathcal{E}(\tilde{G}_{\theta_1,\beta_1})(t)\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})(t), \quad t \in [0, T^*].$$

- Conditioning on $\mathcal{F}^L_{T^*}$, we have

$$\begin{aligned} &\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(T^*)] \\ &= \mathbb{E}_P[\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1,\beta_1})(T^*)|\mathcal{F}^L_{T^*}]\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})(T^*)]. \end{aligned}$$

- Hence the problem is reduced to show that $\mathcal{E}(\tilde{G}_{\theta_1,\beta_1})$ and $\mathcal{E}(\tilde{H}_{\theta_2,\beta_2})$ are true martingales with respect to appropriate filtrations.

Sketch of the proof that $\mathcal{E}(M)$ is a martingale

- M can be $\tilde{G}_{\theta_1,\beta_1}$ or $\tilde{H}_{\theta_2,\beta_2}$.
- Localize $\mathcal{E}(M)$ using a reducing sequence $\{\tau_n\}_{n\geq 1}$.
- Check that $1 = \lim_{n\to\infty} \mathbb{E}_P[\mathcal{E}(M)^{\tau_n}(T^*)] = \mathbb{E}_P[\mathcal{E}(M)(T^*)]$.
- Test the uniform integrability of $\{\mathcal{E}(M)^{\tau_n}(T^*)\}_{n\geq 1}$ with $F(x) = x\log(x)$, i.e.

$$\sup_n \mathbb{E}_P[F(\mathcal{E}(M)^{\tau_n}(T^*))] < \infty. \quad (1)$$

- For any $n \geq 1$, $\{\mathcal{E}(M)^{\tau_n}(t)\}_{t\in[0,T^*]}$ is a true martingale and induces a change of measure $\left.\frac{dQ^n}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(M)^{\tau_n}(t)$.
- Condition (1) can be rewritten as

$$\sup_n \mathbb{E}_{Q^n}[\log(\mathcal{E}(M)^{\tau_n}(T^*))] < \infty.$$

- We can get rid of the ordinary exponential in $\mathcal{E}(M)^{\tau_n}(T^*)$.
- The problem is reduced to find a uniform bound for the second moment of X and σ^2 under Q^n.

The dynamics under the new pricing measure

- By Girsanov's theorem for semimartingales, we can write

$$X(t) = X(0) + B^X_{Q_{\bar{\theta},\bar{\beta}}}(t) + \int_0^t \sigma(s) dW_{Q_{\bar{\theta},\bar{\beta}}}(t), \quad t \in [0, T]^*,$$

$$\sigma^2(t) = \sigma^2(0) + B^{\sigma^2}_{Q_{\bar{\theta},\bar{\beta}}}(t) + \int_0^t \int_0^\infty z \tilde{N}^L_{Q_{\bar{\theta},\bar{\beta}}}(ds, dz), \quad t \in [0, T^*],$$

where

$$B^X_{Q_{\bar{\theta},\bar{\beta}}}(t) = \int_0^t (\theta_1 - \alpha(1 - \beta_1)X(s))ds, \quad t \in [0, T^*],$$

and

$$B^{\sigma^2}_{Q_{\bar{\theta},\bar{\beta}}}(t) = \int_0^t \left(\kappa'_L(\theta_2) - \rho(1 - \beta_2)\sigma^2(s)\right) ds, \quad t \in [0, T^*].$$

The dynamics under the new pricing measure

- The $Q_{\bar{\theta},\bar{\beta}}$-compensator measure of σ^2 is given by

$$v^{\sigma^2}_{Q_{\bar{\theta},\bar{\beta}}}(dt, dz) = e^{\theta_2 z}\left(1 + \frac{\rho\beta_2}{\kappa''_L(\theta_2)} z\sigma^2(t-)\right)\ell(dz)dt.$$

- Using integration by parts, we get

$$\begin{aligned} X(T) &= X(t)e^{-\alpha(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha(1-\beta_1)}(1 - e^{-\alpha(1-\beta_1)(T-t)}) \\ &\quad + \int_t^T \sigma(u)e^{-\alpha(1-\beta_1)(T-u)}dW_{Q_{\bar{\theta},\bar{\beta}}}(u), \\ \sigma^2(T) &= \sigma^2(t)e^{-\alpha(1-\beta_2)(T-t)} + \frac{\kappa'_L(\theta_2)}{\alpha(1-\beta_2)}(1 - e^{-\alpha(1-\beta_2)(T-t)}) \\ &\quad + \int_t^T \int_0^\infty e^{-\rho(1-\beta_2)(T-u)} z\tilde{N}^L_{Q_{\bar{\theta},\bar{\beta}}}(du, dz), \end{aligned}$$

where $0 \leq t \leq T \leq T^*$.

Moment condition under the historical measure

- A sufficient condition for S to have finite expectation under P is the following:

Assumption ($\mathcal{P}$)

We assume that $\alpha, \rho > 0$ *and* Θ_L *satisfy*

$$\frac{1}{2\rho}\left(\frac{\rho}{2\alpha}\right)^{\frac{1}{1-\frac{\rho}{2\alpha}}} \leq \Theta_L - \delta,$$

for some $\delta > 0$.

- If $\Theta_L = \infty$ then assumption $\mathcal{P}$ is satisfied.
- If $\Theta_L < \infty$, then if we choose ρ close to zero the value of α must be bounded away from zero, and vice versa, for assumption $\mathcal{P}$ to be satisfied.

Forward price formula

Proposition

The forward price $F_{Q_{\bar\theta,\bar\beta}}(t,T)$ is given by

$$
\begin{aligned}
&F_{Q_{\bar\theta,\bar\beta}}(t,T)\\
&=\Lambda_g(T)\exp\left(X(t)e^{-\alpha(1-\beta_1)(T-t)}+\sigma^2(t)e^{-\rho(1-\beta_2)(T-t)}\frac{1-e^{-(2\alpha-\rho(1-\beta_2))(T-t)}}{2(2\alpha-\rho(1-\beta_2))}\right)\\
&\quad\times\exp\left(\frac{\kappa_L'(\theta_2)}{2\rho(1-\beta_2)}\left(\frac{1-e^{-2\alpha(T-t)}}{2\alpha}-e^{-\rho(1-\beta_2)(T-t)}\frac{1-e^{-(2\alpha-\rho(1-\beta_2))(T-t)}}{(2\alpha-\rho(1-\beta_2))}\right)\right)\\
&\quad\times\exp\left(\frac{\theta_1}{\alpha(1-\beta_1)}(1-e^{-\alpha(1-\beta_1)(T-t)})\right)\\
&\quad\times\mathbb{E}_{Q_{\bar\theta,\bar\beta}}\left[\exp\left(\frac{e^{-2\alpha T}}{2}\int_t^T e^{(2\alpha-\rho(1-\beta_2))s}\left(\int_t^s\int_0^\infty e^{\rho(1-\beta_2)u}z\tilde N_Q^L(du,dz)\right)ds\right)|\mathcal{F}_t\right]
\end{aligned}
$$

In the particular case $Q_{\bar\theta,\bar\beta}=P$, it holds that

$$
\begin{aligned}
F_P(t,T)&=\Lambda_g(T)\exp\left(X(t)e^{-\alpha(T-t)}+\sigma^2(t)e^{-\rho(T-t)}\frac{1-e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)}\right)\\
&\qquad\times\exp\left(\int_0^{T-t}\kappa_L\left(e^{-\rho s}\frac{1-e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)}\right)ds\right).
\end{aligned}
$$

Affine transform formula

Theorem

Let $\bar{\beta} = (\beta_1, \beta_2) \in [0,1]^2, \bar{\theta} = (\theta_1, \theta_2) \in \bar{D}_L$ *and* $T > 0$*. Suppose there exist functions* $\Psi_i^{\bar{\theta},\bar{\beta}}, i = 0, 1, 2$ *belonging to* $C^1([0, T]; \mathbb{R})$*, satisfying the generalised Riccati equation*

$$\begin{aligned} \frac{d}{dt}\Psi_0^{\bar{\theta},\bar{\beta}}(t) &= \theta_1 \Psi_2^{\bar{\theta},\bar{\beta}}(t) + \kappa_L\left(\Psi_1^{\bar{\theta},\bar{\beta}}(t) + \theta_2\right) - \kappa_L(\theta_2), \\ \frac{d}{dt}\Psi_1^{\bar{\theta},\bar{\beta}}(t) &= -\rho \Psi_1^{\bar{\theta},\bar{\beta}}(t) + \frac{(\Psi_2^{\bar{\theta},\bar{\beta}}(t))^2}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}\left(\kappa_L'\left(\Psi_1^{\bar{\theta},\bar{\beta}}(t) + \theta_2\right) - \kappa_L'(\theta_2)\right), \\ \frac{d}{dt}\Psi_2^{\bar{\theta},\bar{\beta}}(t) &= -\alpha(1-\beta_1)\Psi_2^{\bar{\theta},\bar{\beta}}(t), \end{aligned}$$

with initial conditions $\Psi_0^{\bar{\theta},\bar{\beta}}(0) = \Psi_1^{\bar{\theta},\bar{\beta}}(0) = 0$ *and* $\Psi_2^{\bar{\theta},\bar{\beta}}(0) = 1$*.*
Moreover, suppose that the integrability condition

$$\sup_{t \in [0,T]} \kappa_L''\left(\theta_2 + \Psi_1^{\bar{\theta},\bar{\beta}}(t)\right) < \infty,$$

holds.

Affine transform formula

Theorem (Continue)

Then,

$$
\begin{aligned}
&\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(X(T))|\mathcal{F}_t] \\
&= \exp\left(\Psi_0^{\bar{\theta},\bar{\beta}}(T-t) + \Psi_1^{\bar{\theta},\bar{\beta}}(T-t)\sigma^2(t) + \Psi_2^{\bar{\theta},\bar{\beta}}(T-t)X(t)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
R^F_{Q_{\bar{\theta},\bar{\beta}}}(t,T) = \mathbb{E}_P[S(T)|\mathcal{F}_t] \\
&\times \left\{\exp\left(\Psi_0^{\bar{\theta},\bar{\beta}}(T-t) - \int_0^{T-t} \kappa_L \left(e^{-\rho s}\frac{1-e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)}\right) ds\right.\right. \\
&+ \left(\Psi_1^{\bar{\theta},\bar{\beta}}(T-t) - e^{-\rho(T-t)}\frac{1-e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)}\right)\sigma^2(t) \\
&+ \left.\left.\left(\Psi_2^{\bar{\theta},\bar{\beta}}(T-t) - e^{-\alpha(T-t)}\right)X(t)\right) - 1\right\},
\end{aligned}
$$

for $t \in [0,T]$.

Some remarks on the theorem

- The proof follows by applying a result by **Kallsen and Muhle-Karbe** (2010).
- Limited applicability as it is stated.
- Reduction to a one dimensional non autonomous ODE.
  - We have that for any $\bar{\theta} \in \bar{D}_L, \bar{\beta} \in [0,1]^2$, the solution of the last equation is given by $\Psi_2^{\bar{\theta},\bar{\beta}}(t) = \exp(-\alpha(1-\beta_1)t)$.
  - Plugging this solution to the first equation we get the following equation to solve for $\Psi_1^{\bar{\theta},\bar{\beta}}(t)$

$$\frac{d}{dt}\Psi_1^{\bar{\theta},\bar{\beta}}(t) = -\rho\Psi_1^{\bar{\theta},\bar{\beta}}(t) + \frac{e^{-2\alpha(1-\beta_1)t}}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(\Psi_1^{\bar{\theta},\bar{\beta}}(t)+\theta_2) - \kappa_L'(\theta_2)),$$

    with initial condition $\Psi_1^{\bar{\theta},\bar{\beta}}(0) = 0$.
  - The equation for $\Psi_0^{\bar{\theta},\bar{\beta}}(t)$ is solved by integrating $\Lambda_0^{\bar{\theta},\bar{\beta}}(\Psi_1^{\bar{\theta},\bar{\beta}}(t), \Psi_2^{\bar{\theta},\bar{\beta}}(t))$, i.e.,

$$\begin{aligned}\Psi_0^{\bar{\theta},\bar{\beta}}(t) &= \int_0^t \{\theta_1\Psi_2^{\bar{\theta},\bar{\beta}}(s) + \kappa_L(\Psi_1^{\bar{\theta},\bar{\beta}}(s)+\theta_2) - \kappa_L(\theta_2)\}ds \\ &= \theta_1\frac{1-e^{-\alpha(1-\beta_1)t}}{\alpha(1-\beta_1)} + \int_0^t \{\kappa_L(\Psi_1^{\bar{\theta},\bar{\beta}}(s)+\theta_2) - \kappa_L(\theta_2)\}ds.\end{aligned}$$

Sufficient conditions for existence of global solutions

- Study of the equation $\frac{d}{dt}\hat{\Psi}^{\theta_2,\beta_2}(t) = \Lambda^{\theta_2,\beta_2}(\hat{\Psi}^{\theta_2,\beta_2})$, where

$$\Lambda^{\theta_2,\beta_2}(u) = -\rho u + \frac{1}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(u+\theta_2) - \kappa_L'(\theta_2)).$$

- Let

$$\mathcal{D}_b = \{(\theta_2,\beta_2) \in D_L \times (0,1) : \exists u \in [0, \Theta_L - \theta_2) \quad s.t. \quad \Lambda^{\theta_2,\beta_2}(u) \leq 0\}.$$

Theorem

If $(\theta_2,\beta_2) \in \mathcal{D}_b$ and $(\theta_1,\beta_1) \in \mathbb{R} \times [0,1)$ then $\Psi_0^{\bar\theta,\bar\beta}(t), \Psi_1^{\bar\theta,\bar\beta}(t)$ and $\Psi_2^{\bar\theta,\bar\beta}(t))$ are $C^1([0,T];\mathbb{R})$ for any $T > 0$. Moreover,

$$\Psi_0^{\bar\theta,\bar\beta}(t) \longrightarrow \frac{\theta_1}{\alpha(1-\beta_1)} + \int_0^\infty \left\{\kappa_L\left(\Psi_1^{\bar\theta,\bar\beta}(s) + \theta_2\right) - \kappa_L(\theta_2)\right\} ds, \quad t \to \infty,$$

$$(\Psi_1^{\bar\theta,\bar\beta}(t), \Psi_2^{\bar\theta,\bar\beta}(t)) \longrightarrow (0,0), \quad t \to \infty,$$

and

$$t^{-1} \log \left\| (\Psi_1^{\bar\theta,\bar\beta}(t), \Psi_2^{\bar\theta,\bar\beta}(t)) \right\| \to \gamma, \quad t \to \infty,$$

for some negative constant γ.

Risk premium analysis in the geometric BNS model

Lemma

If $(\theta_2, \beta_2) \in \mathcal{D}_b$ *and* $(\theta_1, \beta_1) \in \mathbb{R} \times [0, 1)$, *the sign of the risk premium* $R^F_{Q_{\bar{\theta},\bar{\beta}}}(t, T)$ *is the same as the sign of the function*

$$\Sigma(t, T) \triangleq \Psi_0^{\bar{\theta},\bar{\beta}}(T-t) - \Psi_0^{0,0}(T-t) + (\Psi_1^{\bar{\theta},\bar{\beta}}(T-t) - \Psi_1^{0,0}(T-t))\sigma^2(t) + (\Psi_2^{\bar{\theta},\bar{\beta}}(T-t) - \Psi_2^{0,0}(T-t))X(t).$$

Moreover,

$$\lim_{T-t\to\infty} \Sigma(t, T) = \frac{\theta_1}{\alpha(1-\beta_1)} + \int_0^\infty \int_0^1 \kappa_L' \left(\lambda \Psi_1^{\bar{\theta},\bar{\beta}}(s) + \theta_2\right) d\lambda \Psi_1^{\bar{\theta},\bar{\beta}}(s) ds - \int_0^\infty \int_0^1 \kappa_L' \left(\lambda e^{-\rho s} \frac{1-e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)}\right) d\lambda e^{-\rho s} \frac{1-e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} ds,$$

and

$$\lim_{T-t\to 0} \frac{d}{dT} \Sigma(t, T) = \theta_1 + \alpha\beta_1 X(t).$$

- Analysis of possible risk profiles in **Benth and O.-L.** (2015).

Conclusions

- A pricing measure for the BNS model for non-tradable assets with mean reversion extending Esscher's transform.
- Preserves affine structure of the model.
- Gives control on the speed and level of mean reversion.
- Provides more realistic risk premium profiles.

Further work

- Study the BNS model with jumps

$$X(t) = X(0) - \alpha \int_0^t X(s)ds + \int_0^t \sigma(s)W(t) + \eta L(t),$$

$$\sigma^2(t) = \sigma^2(0) - \rho \int_0^t \sigma^2(s)ds + L(t).$$

- Calibration.
- Pricing more complex derivatives.
- Other models with mean reversion.

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