Insider games with asymmetric information

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1. Introduction

In this talk we present a general method for finding Nash equilibria of *insider control games with asymmetric information*, i.e. stochastic differential games where one or several of the controllers have access to information about the future of the system.

This *inside information* in the control process puts the problem outside the context of semimartigale theory, and we therefore apply general *anticipating white noise calculus*, including *forward integrals* and *Hida-Malliavin calculus* to study it. Combining this with the *Donsker delta functional* for the random variable Y which represents the inside information, we are able to prove both sufficient and necessary maximum principles for the Nash equilibria of such games.

As illustrations of this machinery we apply it to
(i) the problem of optimal consumption under model uncertainty for an insider in a jump-diffusion financial market,
(ii) the problem of optimal portfolio for an insider in an insider influenced market.

We now explain this in more detail:

The system we consider, is described by a stochastic differential equation driven by a Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}(dt, d\zeta)$, jointly defined on a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ satisfying the usual conditions. We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the inside filtration $\mathbb{H}^i$ for player i has the form

$$\mathbb{H}^i = \{\mathcal{H}_t^i\}_{t \geq 0}, \text{ where } \mathcal{H}_t^i = \mathcal{F}_t \vee Y_i \tag{1.1}$$

for all t, where Y_i is a given $\mathcal{F}_{T_0}$-measurable random variable, for some $T_0 > T$ (both constants), representing the inside information of player i. For simplicity we consider only the case with 2 players. We put

$$Y = (Y_1, Y_2).$$

We assume that the values at time t of our two insider control processes $u_1(t), u_2(t)$ are allowed to depend on both $\mathcal{F}_t$ and Y_1, Y_2, respectively. In other words, u_i is assumed to be $\mathbb{H}^i$-adapted. Therefore it has the form

$$u_i(t,\omega) = \tilde{u}_i(t, Y_i, \omega) \tag{1.2}$$

for some function $\tilde{u}_i : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $u_i(t, y)$ is $\mathbb{F}$-adapted for each $y \in \mathbb{R}$. For simplicity (albeit with some abuse of notation) we will in the following write u_i in stead of $\tilde{u}_i, i = 1, 2$.

To explain our approach we will in the following first discuss the case with just one insider and her stochastic control problem:

Consider a controlled stochastic process $X(t) = X^u(t)$ of the form (1.3)

$$\begin{cases} dX(t) = b(t, X(t), u(t), Y)dt + \sigma(t, X(t), u(t), Y)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u(t), Y, \zeta)\tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x, \quad x \in \mathbb{R}, \end{cases}$$

where $u(t) = u(t, y)_{y=Y}$ is our insider control and the (anticipating) stochastic integrals are interpreted as *forward integrals*, as introduced in Russo & Vallois (1993) [RV] (Brownian motion case) and in Di Nunno et al (2006) [DMØP1] (Poisson random measure case). A motivation for using forward integrals in the modelling of insider control is given in F. Biagini & Ø. (2005) [BØ].

Let $\mathcal{A}$ be a given family of admissible $\mathbb{H}-$adapted controls u. The *performance functional* $J(u)$ of a control process $u \in \mathcal{A}$ is defined by

$$(1.4) \qquad J(u) = \mathbb{E}[\int_0^T f(t, X(t), u(t))dt + g(X(T))]$$

We consider the problem to find $u^\star \in \mathcal{A}$ such that

$$(1.5) \qquad \sup_{u \in \mathcal{A}} J(u) = J(u^\star).$$

We use the Donsker delta functional of Y to transform this anticipating system into a classical (albeit parametrised) adapted system with a non-classical performance functional. Then we solve this transformed system by using modified maximum principles.

2. The Donsker delta functional

Definition

Let $Z : \Omega \to \mathbb{R}$ be a random variable which also belongs to $(\mathcal{S})^*$. Then a continuous functional

$$\delta_Z(.) : \mathbb{R} \to (\mathcal{S})^* \tag{2.1}$$

is called a *Donsker delta functional* of Z if it has the property that

$$\int_{\mathbb{R}} g(z)\delta_Z(z)dz = g(Z) \quad a.s. \tag{2.2}$$

for all (measurable) $g : \mathbb{R} \to \mathbb{R}$ such that the integral converges.

The Donsker delta functional is related to the *regular conditional distribution*. The connection is the following:

Define the *regular conditional distribution* with respect to $\mathcal{F}_t$ of a given real random variable Y, denoted by $Q_t(dy) = Q_t(\omega, dy)$, by the following properties:

- For any Borel set $\Lambda \subseteq \mathbb{R}$, $Q_t(\cdot, \Lambda)$ is a version of $\mathbb{E}[\mathbf{1}_{Y\in\Lambda}|\mathcal{F}_t]$
- For each fixed ω, $Q_t(\omega, dy)$ is a probability measure on the Borel subsets of $\mathbb{R}$.

It is well-known that such a regular conditional distribution always exists. See e. g. Breiman (1968) [B], page 79.
From the required properties of $Q_t(\omega, dy)$ we get the following formula

$$\int_{\mathbb{R}} f(y) Q_t(\omega, dy) = \mathbb{E}[f(Y)|\mathcal{F}_t] \tag{2.3}$$

Comparing with the definition of the Donsker delta functional, we obtain the following representation of the regular conditional distribution:

Theorem

Suppose $Q_t(\omega, dy)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Then the Donsker delta functional of Y, $\delta_Y(y)$, exists and we have

$$\frac{Q_t(\omega, dy)}{dy} = \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \tag{2.4}$$

A general expression, in terms of Wick calculus, for the Donsker delta functional of an Itô diffusion with non-degenerate diffusion coefficient can be found in the amazing paper Lanconelli & Proske (2004) [LP]. See also Meyer-Brandis & Proske (2006) [MP]. In the following we present more explicit formulas the Donsker delta functional and its conditional expectation and Hida-Malliavin derivatives, for Itô - Lévy processes.

List of notation:

- λ denotes Lebesgue measure on $\mathbb{R}$, or on $[0, T]$, depending on the situation.
- $F \diamond G$ = the Wick product of random variables F and G.
- $F^{\diamond n} = F \diamond F \diamond F \ldots \diamond F$ (n times). (The n'th Wick power of F).
- $\exp^{\diamond}(F) = \Sigma_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n}$ (The Wick exponential of F.)
- $D_t F$ = the Hida-Malliavin derivative of F at t with respect to $B(\cdot)$.
- $D_{t,z} F$ = the Hida-Malliavin derivative of F at (t, z) with respect to $N(\cdot, \cdot)$.
- $D_{.}(\varphi^{\diamond}(F)) = ((\varphi)')^{\diamond}(F) \diamond D_{.}F$, where $D_{.} = D_t$ or $D_{.} = D_{t,z}$.(The Wick chain rule.)
- $(\mathcal{S}), (\mathcal{S})^*$ = the Hida stochastic test function space and stochastic distribution space, respectively.
- $(\mathcal{S}) \subset L^2(P) \subset (\mathcal{S})^*$.

2.1. The Donsker delta functional for a class of Itô - Lévy processes

Consider the special case when Y is a first order chaos random variable of the form

$$Y = Y(T_0);\ \text{where}$$

(2.5)

$$Y(t) = \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(ds,d\zeta); t \in [0, T_0]$$

for some deterministic functions β, ψ satisfying

$$\int_0^{T_0} \{\beta^2(t) + \int_{\mathbb{R}} \psi^2(t,\zeta)\nu(d\zeta)\}dt < \infty \text{ a.s.} \quad (2.6)$$

We also assume that the following holds thoughout this paper: For every $\epsilon > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} e^{\rho\zeta} d\nu(\zeta) < \infty. \quad (2.7)$$

In this case it is well known (see e.g. Mataramvura et al (2004) [MØP], Di Nunno & Ø. (2007) ([DiØ1], [DiØ2]), and Di Nunno et al. (2009) [DØP]) that the Donsker delta functional exists in $(\mathcal{S})^*$ and is given by

$$\begin{aligned}\delta_Y(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^\diamond \Big[\int_0^{T_0}\int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1)\tilde{N}(ds, d\zeta) \\ + \int_0^{T_0} x\beta(s)dB(s) \\ + \int_0^{T_0} \{\int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta) \\ + \frac{1}{2}x^2\beta^2(s)\}ds - ixy\Big]dx. \end{aligned} \tag{2.8}$$

Taking conditional expectation brings us from $\mathcal{S}^*$ down to $L^2(\lambda \times P)$. In fact, we have the following result:

Lemma

$$
\begin{aligned}
&\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\{\exp\Big[\int_0^t\int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s) \\
&+ \int_t^{T_0}\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds \\
&+ \int_t^{T_0}\frac{1}{2}x^2\beta^2(s)ds - ixy\Big]\}dx
\end{aligned} \tag{2.9}
$$

Next, we need the following:

Lemma

(2.10)

$$
\begin{aligned}
&\mathbb{E}[D_{t,z}\delta_Y(y)|\mathcal{F}_t] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\{ \exp\Big[\int_0^t \int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s) \\
&+ \int_t^{T_0}\int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds + \int_t^{T_0} \frac{1}{2}x^2\beta^2(s)ds - ixy\Big] \\
&\times (e^{ix\psi(t,z)} - 1)\}dx.
\end{aligned}
$$

Finally, we need the following result:

Lemma

(2.11)

$$
\begin{aligned}
&\mathbb{E}[D_t \delta_Y(y) | \mathcal{F}_t] \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \{\exp\Big[\int_0^t \int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s) \\
&+ \int_t^{T_0} \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds \\
&+ \int_t^{T_0} \frac{1}{2} x^2 \beta^2(s)ds - ixy\Big] x\beta(t)\} dx.
\end{aligned}
$$

2.2. The Donsker delta functional for a Gaussian process

Consider the special case when Y is a Gaussian random variable of the form

$$(2.12) \quad Y = Y(T_0); \text{ where } Y(t) = \int_0^t \beta(s)dB(s), \text{ for } t \in [0, T_0]$$

for some deterministic function $\beta \in \mathbf{L}^2[0, T_0]$ with

$$(2.13) \qquad \|\beta\|^2_{[0,T]} := \int_t^T \beta(s)^2 ds > 0 \text{ for all } t \in [0, T].$$

In this case it is well known that the Donsker delta functional is given by

$$(2.14) \qquad \delta_Y(y) = (2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y-y)^{\diamond 2}}{2v}]$$

where we have put $v := \|\beta\|^2_{[0,T_0]}$. See e.g. Aase et al (2001) [AaØU], Proposition 3.2.

Using the Wick rule when taking conditional expectation, using the martingale property of the process $Y(t)$ and applying Lemma 3.7 in [AaØU] we get

$$
\begin{aligned}
&\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \\
&= (2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\mathbb{E}[\frac{(Y(T_0)-y)^{\diamond 2}}{2v}|\mathcal{F}_t]] \\
&= (2\pi\|\beta\|^2_{[0,T_0]})^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y(t)-y)^{\diamond 2}}{2\|\beta\|^2_{[0,T_0]}}] \\
&= (2\pi\|\beta\|^2_{[t,T_0]})^{-\frac{1}{2}} \exp[-\frac{(Y(t)-y)^2}{2\|\beta\|^2_{[t,T_0]}}]. \qquad (2.15)
\end{aligned}
$$

Similarly, by the Wick chain rule and Lemma 3.8 in [AaØU] we get, for $t \in [0, T]$,

$$\mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t]$$

$$= -\mathbb{E}[(2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y(T_0)-y)^{\diamond 2}}{2v}] \diamond \frac{Y(T_0)-y}{v}\beta(t)|\mathcal{F}_t]$$

$$= -(2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y(t)-y)^{\diamond 2}}{2v}] \diamond \frac{Y(t)-y}{v}\beta(t)$$

(2.16)

$$= -(2\pi\|\beta\|^2_{[t,T_0]})^{-\frac{1}{2}}) \exp[-\frac{(Y(t)-y)^2}{2\|\beta\|^2_{[t,T_0]})}]\frac{Y(t)-y}{\|\beta\|^2_{[t,T_0]}}\beta(t).$$

2.4 The Donsker delta functional for a Poisson process

Next, assume that $Y = Y(T_0)$, such that

$$Y(t) = \tilde{N}(t) = N(t) - \lambda t,$$

where $N(t)$ is a Poisson process with intensity $\lambda > 0$.

In this case the Lévy measure is $\nu(d\zeta) = \lambda\delta_1(d\zeta)$ since the jumps are of size 1. Comparing with (2.8) and by taking $\beta = 0$ and $\psi = 1$, we obtain

(2.17)

$$\delta_Y(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{\diamond}\left[(e^{ix} - 1)\tilde{N}(T_0) + \lambda T_0(e^{ix} - 1 - ix) - ixy\right]dx$$

By using the general expressions in Section 2.1, we get:
(2.18)
$$\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[ix\tilde{N}(t) + \lambda(T_0 - t)(e^{ix} - 1 - ix) - ixy\right] dx$$

and

$$\mathbb{E}[D_{t,1}\delta_Y(y)|\mathcal{F}_t]$$
(2.19)
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[ix\tilde{N}(t) + \lambda(T_0 - t)(e^{ix} - 1 - ix) - ixy\right](e^{ix} - 1) dx$$

3.The forward integral with respect to Brownian motion

The forward integral with respect to Brownian motion was first defined in the seminal paper Russo & Vallois (1993) [RV] and further studied in subsequent papers by the same authors. This integral was introduced in the modeling of insider trading in F. Biagini & Ø. (2005) [BØ] and then applied by several authors in questions related to insider trading and stochastic control with advanced information.

Definition

We say that a stochastic process $\phi = \phi(t), t \in [0, T]$, is *forward integrable* (in the weak sense) over the interval $[0, T]$ with respect to B if there exists a process $I = I(t), t \in [0, T]$, such that

$$\sup_{t \in [0,T]} \left| \int_0^t \phi(s) \frac{B(s+\epsilon) - B(s)}{\epsilon} ds - I(t) \right| \to 0, \quad \epsilon \to 0^+ \tag{3.1}$$

in probability. In this case we write

$$I(t) := \int_0^t \phi(s) d^- B(s), t \in [0, T], \tag{3.2}$$

and call $I(t)$ the *forward integral* of ϕ with respect to B on $[0, t]$.

The following results give a more intuitive interpretation of the forward integral as a limit of Riemann sums.

Lemma

Suppose ϕ is càglàd and forward integrable. Then

$$\int_0^T \phi(s)d^- B(s) = \lim_{\triangle t \to 0} \sum_{j=1}^{J_n} \phi(t_{j-1})(B(t_j) - B(t_{j-1})) \tag{3.3}$$

with convergence in probability. Here the limit is taken over the partitions
$0 = t_0 < t_1 < ... < t_{J_n} = T$ *of* $t \in [0, T]$ *with*
$\triangle t := \max_{j=1,...,J_n}(t_j - t_{j-1}) \to 0, n \to \infty$.

Remark
From the previous lemma we can see that, if the integrand ϕ is $\mathcal{F}$-adapted, then the Riemann sums are also an approximation to the Itô integral of ϕ with respect to the Brownian motion. Hence in this case the forward integral and the Itô integral coincide. In this sense we can regard the forward integral as an extension of the Itô integral to a nonanticipating setting.

We now give some useful properties of the forward integral. The following result is an immediate consequence of the definition.

Lemma

Suppose ϕ is a forward integrable stochastic process and G a random variable. Then the product $G\phi$ is forward integrable stochastic process and

$$\int_0^T G\phi(t)d^- B(t) = G \int_0^T \phi(t)d^- B(t) \tag{3.4}$$

As a consequence of the above we get the following useful result:

Lemma

Let $\varphi(t, y)$ be an $\mathbb{F}$-adapted process for each $y \in \mathbb{R}$ such that the classical Itô integral

$$\int_0^T \phi(t, y) dB(t)$$

exists for each $y \in \mathbb{R}$. Let Y be a random variable. Then $\varphi(t, Y)$ is forward integrable and

$$\int_0^T \varphi(t, Y) d^- B(t) = \int_0^T \varphi(t, y) dB(t)_{y=Y}. \tag{3.5}$$

The next result shows that the forward integral is an extension of the integral with respect to a semimartingale.

Lemma

Let $\mathbb{G} := \{\mathcal{G}_t, t \in [0, T]\}(T > 0)$ be a given filtration. Suppose that

1. *B is a semimartingale with respect to the filtration $\mathbb{G}$.*
2. *ϕ is $\mathbb{G}$-predictable and the integral*

$$\int_0^T \phi(t)dB(t), \tag{3.6}$$

with respect to B, exists.

Then ϕ is forward integrable and

$$\int_0^T \phi(t)d^- B(t) = \int_0^T \phi(t)dB(t), \tag{3.7}$$

We now turn to the Itô formula for forward integrals. In this connection it is convenient to introduce a notation that is analogous to the classical notation for Itô processes.

Definition

A *forward process* (with respect to B) is a stochastic process of the form

$$(3.8) \qquad X(t) = x + \int_0^t u(s)ds + \int_0^t v(s)d^- B(s), \quad t \in [0, T],$$

(x constant), where $\int_0^T |u(s)|ds < \infty$, $\mathbf{P}$-a.s. and v is a forward integrable stochastic process. A shorthand notation for (3.8) is that

$$(3.9) \qquad d^- X(t) = u(t)dt + v(t)d^- B(t).$$

Theorem

The one-dimensional Itô formula for forward integrals.
Let

$$(3.10) \qquad d^{-}X(t) = u(t)dt + v(t)d^{-}B(t)$$

be a forward process. Let $f \in \mathbf{C}^{1,2}([0,T] \times \mathbb{R})$ *and define*

$$(3.11) \qquad Y(t) = f(t, X(t)), \quad t \in [0,T].$$

Then $Y(t), t \in [0,T]$*, is also a forward process and*
(3.12)

$$d^{-}Y(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))d^{-}X(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))v^2(t)dt.$$

Similar definitions and results can be obtained in the Poisson random measure case. See Di Nunno et al (2006) [DMØP1].

4. The general insider stochastic differential game

In this section, we formulate and prove a sufficient and a necessary maximum principle for general stochastic differential games (not necessarily zero-sum games) for insiders. The system we consider, is described by a stochastic differential equation driven by a Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}(dt, d\zeta)$, jointly defined on a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ satisfying the usual conditions. We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the two inside filtrations $\mathbb{H}^1, \mathbb{H}^2$ representing the information flows available to player 1 and player 2, respectively, have the form

$$(4.1) \qquad \mathbb{H}^i = \{\mathcal{H}_t^i\}_{t \geq 0}, \text{ where } \mathcal{H}_t^i = \mathcal{F}_t \vee Y_i, \quad i = 1, 2$$

for all t, where Y_i is a given $\mathcal{F}_T$-measurable random variable, for some fixed $T > t$.

Here the insider control process $u(t) = (u_1(t), u_2(t))$, where $u_i(t)$ is the control of player i; i=1,2. Thus we assume that the value at time t of our insider control process $u_i(t)$ is allowed to depend on both Y_i and $\mathcal{F}_t; i = 1, 2$. In other words, u_i is assumed to be $\mathbb{H}^i$-adapted for $i = 1, 2$.
Therefore they have the form

$$u_i(t, \omega) = \overline{u}_i(t, Y_i, \omega) \tag{4.2}$$

for some function $\overline{u}_i : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\overline{u}_i(t, y_i)$ is $\mathbb{F}$-adapted for each $y_i \in \mathbb{R}$. For simplicity (albeit with some abuse of notation) we will in the following write u_i in stead of $\overline{u}_i; i = 1, 2$. Consider a controlled stochastic process $X(t) = X^u(t)$ of the form
(4.3)

$$\begin{cases} dX(t) = dX^u(t) = b(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dt \\ +\sigma(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u_1(t), u_2(t), Y_1, Y_2, \zeta)\tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x, \quad x \in \mathbb{R}, \end{cases}$$

where $u_i(t) = u_i(t, y_i)_{y_i = Y_i}$ is the control process of insider i; $i = 1, 2$, and the (anticipating) stochastic integrals are interpreted as *forward integrals*. Let $\mathcal{A}_i$ denote a given set of admissible $\mathbb{H}^i-$adapted controls u_i of player i, with values in $\mathbf{A}_i \subset \mathbb{R}^d, d \geq 1; i = 1, 2$. Denote $\mathbb{U} = \mathbf{A}_1 \times \mathbf{A}_2$. Then $X(t)$ is $\mathbb{F} \vee Y_1 \vee Y_2$-adapted, and hence using the definition of the Donsker delta functional $\delta_{(Y_1, Y_2)}(y_1, y_2)$ of (Y_1, Y_2) we get

$$\begin{aligned} X(t) &= x(t, Y_1, Y_2) = x(t, y_1, y_2)_{y_1 = Y_1, y_2 = Y_2} \\ &= \int_{\mathbb{R}^2} x(t, y_1, y_2)\delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 \end{aligned} \tag{4.4}$$

for some y_1, y_2-parametrized process $x(t, y_1, y_2)$ which is $\mathbb{F}$-adapted for each y_1, y_2.

Then, again by the definition of the Donsker delta functional and the properties of forward integration, we can write

$$
\begin{aligned}
X(t) &= x + \int_0^t b(s, X(s), u(s), Y)ds + \int_0^t \sigma(s, X(s), u(s), Y)dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s), u(s), Y, \zeta)\tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, Y), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2)ds \\
&+ \int_0^t \sigma(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2)dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2, \zeta)\tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)_{y_1 = Y_1, y_2 = Y_2} ds \\
&+ \int_0^t \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)_{y_1 = Y_1, y_2 = Y_2} dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta)_{y_1 = Y_1, y_2 = Y_2} \tilde{N}(ds, d
\end{aligned}
$$

(4.5)

$$
\begin{aligned}
&= x + \int_0^t \int_{\mathbb{R}^2} b(s, x(s, y), u(s, y), u_2(s, y), y)\delta_Y(y)dy_1dy_2ds \\
&+ \int_0^t \int_{\mathbb{R}^2} \sigma(s, x(s, y), u(s, y), y)\delta_Y(y)dy_1dy_2dB(s) \\
&+ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}} \gamma(s, x(s, y), u(s, y), y, \zeta)\delta_Y(y)dy_1dy_2\tilde{N}(ds, d\zeta) \\
&= x + \int_{\mathbb{R}^2} [\int_0^t b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)ds \\
&+ \int_0^t \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)dB(s)
\end{aligned}
$$

(4.6)

$$
+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y), u(s, y), y, \zeta)\tilde{N}(ds, d\zeta)]\delta_Y(y_1, y_2)dy_1dy_2.
$$

Comparing (4.4) and (4.5) we see that (4.4) holds if we choose $x(t,y)$ for each $y = (y_1, y_2)$ as the solution of the classical SDE

$$dx(t,y_1,y_2) = b(t,x(t,y_1,y_2),u_1(t,y_1),u_2(t,y_2),y_1,y_2)dt$$
$$+\sigma(t,x(t,y_1,y_2),u_1(t,y_1),u_2(t,y_2),y_1,y_2)dB(t)$$

(4.7)

$$+\int_{\mathbb{R}} \gamma(t,x(t,y_1,y_2),u_1(t,y_1),u_2(t,y_2),y_1,y_2,\zeta)\tilde{N}(dt,d\zeta); \quad t \geq 0$$

(4.8)

$$x(0,y) = x, \quad x \in \mathbb{R},$$

The *performance functional* $J_i(u); u = (u_1, u_2)$ of player i is defined by

$$J_i(u) = \mathbb{E}[\int_0^T f_i(t,X(t),u_1(t),u_2(t),Y)dt + g_i(X(T),Y)]$$
$$= \mathbb{E}[\int_{\mathbb{R}^2}\{\int_0^T f_i(t,x(t,y),u(t,y),y)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]dt$$

(4.9)

$$+ g_i(x(T,y_1,y_2),y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T]\}dy_1dy_2]; \quad i = 1,2.$$

A *Nash equilibrium* for the game (4.3)-(4.9) is a pair $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\sup_{u_1 \in \mathcal{A}_1} J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \tag{4.10}$$

and

$$\sup_{u_2 \in \mathcal{A}_2} J_2(\hat{u}_1, u_2) \leq J_2(\hat{u}_1, \hat{u}_2). \tag{4.11}$$

We can rewrite this problem in terms of a classical (i.e. $\mathbb{F}$-adapted) stochastic differential game with parameter $y = (y_1, y_2)$ as follows:

For each given $y = (y_1, y_2) \in \mathbb{R}^2$ define

$$J_i(u(\cdot, y))$$
$$= \int_0^T f_i(t, x(t, y), u(t, y), y, \mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_t]dt$$
$$+ g_i(x(T, y_1, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_T]; \quad i = 1, 2. \tag{4.12}$$

Then we see that

$$J_i(u) = \int_{\mathbb{R}^2} J_i(u(\cdot, y_1, y_2))dy_1 dy_2. \tag{4.13}$$

Therefore the problem is, in many situations, reduced to solving a classical stochastic differential game for each y.

5. A sufficient maximum principle

The problem (4.10)-(4.11) is a stochastic differential game with a standard (albeit parametrized) stochastic differential equation (4.7) for the state process $x(t, y_1, y_2)$, but with a non-standard performance functional given by (4.9). We can solve this problem by a modified maximum principle approach, as follows:
Define the *Hamiltonians*
$H_i : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \to \mathbb{R}$ by

$$H_i(t, x, y_1, y_2, u_1, u_2, p, q, r) = H(t, x, y_1, y_2, u_1, u_2, p, q, r, \omega)$$
$$= \mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_t] f_i(t, x, u_1, u_2, y_1, y_2) + b(t, x, u_1, u_2, y_1, y_2)p$$
(5.1)
$$+ \sigma(t, x, u_1, u_2, y_1, y_2)q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1, y_2, \zeta) r(\zeta) \nu(d\zeta); i = 1, 2.$$

Here $\mathcal{R}$ denotes the set of all functions $r(.) : \mathbb{R} \to \mathbb{R}$ such that the last integral above converges.

For $i = 1, 2$ we define the *adjoint* processes $p_i(t, y_1, y_2), q_i(t, y_1, y_2), r_i(t, y_1, y_2, \zeta)$ as the solution of the y_1, y_2-parametrised BSDEs

(5.2)

$$\begin{cases} dp_i(t, y_1, y_2) = -\frac{\partial H_i}{\partial x}(t, y_1, y_2)dt + q_i(t, y_1, y_2)dB(t) \\ + \int_{\mathbb{R}} r_i(t, y_1, y_2, \zeta)\tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ p_i(T, y_1, y_2) = g_i'(x(T, y_1, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_T] \end{cases}$$

Let $J_i(u(., y_1, y_2))$ be defined by

$$J_i(u(., y_1, y_2))$$

$$= \mathbb{E}[\int_0^T f_i(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_t]dt$$

(5.3)

$$+ g_i(x(T, y_1, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_T]]$$

Then we see that

(5.4) $$J_i(u_1, u_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} J_i(u(., y_1, y_2))dy_1 dy_2$$

The insider game problem can therefore be written as follows:

Problem

For each $y_1, y_2 \in \mathbb{R}$, *find* $(u_1^\star(.,y_1), u_2^\star(.,y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ *such that*

$$\sup_{u_1(.,y_1)\in\mathcal{A}_1} \int_{\mathbb{R}}\int_{\mathbb{R}} J_1(u_1(.,y_1), u_2^*(.,y_2))dy_1 dy_2$$

$$(5.5) \qquad \leq \int_{\mathbb{R}}\int_{\mathbb{R}} J_1(u_1^*(.,y_1), u_2^*(.,y_2))dy_1 dy_2$$

and

$$\sup_{u_2(.,y_2)\in\mathcal{A}_2} \int_{\mathbb{R}}\int_{\mathbb{R}} J_2(u_1^*(.,y_1), u_2(.,y_2))dy_1 dy_2$$

$$(5.6) \qquad \leq \int_{\mathbb{R}}\int_{\mathbb{R}} J_2(u_1^*(.,y_1), u *_2 (.,y_2))dy_1 dy_2$$

To study this problem we present two maximum principles for the corresponding games. The first is the following:

Theorem

[Sufficient maximum principle]
Let $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ *with associated solution* $\hat{x}(t, y_1, y_2), \hat{p}_i(t, y_1, y_2), \hat{q}_i(t, y_1, y_2), \hat{r}_i(t, y_1, y_2, \zeta)$ *of* (4.7) *and* (5.2); *i=1,2. Assume that the following hold:*

- $x \to g_i(x)$ *is concave;* $i = 1, 2$
- *The functions*

(5.7)

$$\hat{\mathcal{H}}_1(x) = \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H_1(t, x, y, u_1, \hat{u}_{2,t}(y_2), \widehat{p}_{1,t}(y), \widehat{q}_{1,t}(y), \hat{r}_{1,t}(y)) dy_2$$

and

(5.8)

$$\hat{\mathcal{H}}_2(x) = \sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H_2(t, x, y, \hat{u}_1(t, y_1), u_2, \widehat{p}_{2,t}(y), \widehat{q}_{2,t}(y) \hat{r}_{2,t}(y)) dy_1$$

are concave for all t, y_1, y_2

- $$\sup_{u_1 \in \mathbf{A}_1} \int_{\mathbb{R}} H_1\big(t, \widehat{x}(t,y), u_1, \hat{u_2}(t,y_2), \widehat{p}_1(t,y), \widehat{q}_1(t,y), \hat{r}_1(t,y)\big)\, dy_2$$
$$= \int_{\mathbb{R}} H_1\big(t, \widehat{x}(t,y), \widehat{u}_1(t,y_1), \hat{u_2}(t,y_2), \widehat{p}_1(t,y), \widehat{q}_1(t,y), \hat{r}_1(t,y)\big)\, dy_2$$
(5.9)
for all t, y_1.

- $$\sup_{u_2 \in \mathbf{A}_2} \int_{\mathbb{R}} H_2\big(t, \widehat{x}(t,y), \hat{u}_1(t,y_1), u_2, \widehat{p}_2(t,y), \widehat{q}_2(t,y), \hat{r}_2(t,y)\big)\, dy_1$$
$$= \int_{\mathbb{R}} H_2\big(t, \widehat{x}(t,y), \widehat{u}_1(t,y_1), \hat{u_2}(t,y_2), \widehat{p}_2(t,y), \widehat{q}_2(t,y), \hat{r}_2(t,y)\big)\, dy_1$$
(5.10)
for all t, y_2.

Then $(u_1^*(.,y_1), u_2^*(.,y_2)) := (\widehat{u}_1(.,y_1), \widehat{u}_2(.,y_2))$ is a Nash equilibrium for the problem (5.5)-(5.6).

Theorem

[Sufficient maximum principle with only one insider]
Suppose $Y_2 = 0$, *i.e. player number 2 has no inside information.*
Let $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ *with associated solution* $\hat{x}(t, y_1), \hat{p}_i(t, y_1), \hat{q}_i(t, y_1), \hat{r}_i(t, y_1, \zeta)$ *of* (4.7) *and* (5.2); *i=1,2.*
Assume that the following hold:

- $x \to g_i(x)$ *is concave;* $i = 1, 2$
- *(The Arrow conditions.) The functions*

  (5.11)
  $$\hat{\mathcal{H}}_1(x) = \sup_{u_1 \in \mathcal{A}_1} H_1(t, x, y_1, u_1, \hat{u}_2(t), \widehat{p}_1(t, y_1), \widehat{q}_1(t, y_1), \hat{r}_1(t, y_1, \cdot))$$

  and

  $$\hat{\mathcal{H}}_2(x) =$$
  (5.12)
  $$\sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H_2(t, x, y_1, \hat{u}_1(t, y_1), u_2, \widehat{p}_2(t, y_1), \widehat{q}_2(t, y_1), \hat{r}_2(t, y_1)) dy_1$$

  are concave for all t.

Moreover, assume that

1.

$$\sup_{u_1 \in \mathbf{A}_1} H_1\big(t, \widehat{x}(t,y_1), u_1, \hat{u_2}(t), \widehat{p}_1(t,y_1), \widehat{q}_1(t,y_1), \hat{r}_1(t,y_1,\cdot)\big)$$
$$= H_1\big(t, \widehat{x}(t,y_1), \widehat{u}_1(t,y_1), \hat{u_2}(t), \widehat{p}_1(t,y_1), \widehat{q}_1(t,y_1), \hat{r}_1(t,y_1,\cdot)\big) \tag{5.13}$$

for all t.

2.

$$\sup_{u_2 \in \mathbf{A}_2} \int_{\mathbb{R}} H_2\big(t, \widehat{x}(t,y_1), \hat{u}_1(t,y_1), u_2, \widehat{p}_2(t,y_1), \widehat{q}_2(t,y_1), \hat{r}_2(t,y_1)\big)\, dy_1$$
$$= \int_{\mathbb{R}} H_2\big(t, \widehat{x}(t,y_1), \widehat{u}_1(t,y_1), \hat{u_2}(t), \widehat{p}_2(t,y_1), \widehat{q}_2(t,y_1), \hat{r}_2(t,y_1)\big)\, dy_1 \tag{5.14}$$

for all t.

Then $(u_1^*(.,y_1), u_2^*(.)) := (\widehat{u}_1(.,y_1), \widehat{u}_2(.))$ is a Nash equilibrium for the problem (5.5)-(5.6).

6. A necessary maximum principle

We proceed to establish a corresponding necessary maximum principle. For this, we do not need concavity conditions, but in stead we need the following assumptions about the set of admissible control values:

- A_1. For all $t_0 \in [0, T], y_i \in \mathbb{R}$ and all bounded $\mathcal{F}_{t_0}$-measurable random variables $\alpha_i(y_i, \omega)$, the control $\theta_i(t, y_i, \omega) := \mathbf{1}_{[t_0, T]}(t)\alpha_i(y_i, \omega)$ belongs to $\mathcal{A}_i$ for $i = 1, 2$.
- A_2. For all $u_i; \beta_0^i \in \mathcal{A}_i$ with $\beta_0^i(t, y_i) \leq K < \infty$ for all t, y_i define

  (6.1) $$\delta_i(t, y_i) = \frac{1}{2K} dist((u_i(t, y_i), \partial\mathbb{A}_i) \wedge 1 > 0$$

  and put

  (6.2) $$\beta_i(t, y_i) = \delta_i(t, y_i)\beta_0^i(t, y_i).$$

  Then the control

  $$\widetilde{u}_i(t, y_i) = u_i(t, y_i) + a\beta_i(t, y_i); \quad t \in [0, T]$$

  belongs to $\mathcal{A}_i$ for all $a \in (-1, 1)$ for $i = 1, 2$.

- $A3$. For all β_i as in (6.2) the derivative processes

$$\chi_1(t,y_1,y_2) := \frac{d}{da} x^{(u_1+a\beta_1,u_2)}(t,y_1,y_2)|_{a=0}$$

and

$$\chi_2(t,y_1,y_2) := \frac{d}{da} x^{(u_1,u_2+a\beta_2)}(t,y_1,y_2)|_{a=0}$$

exist, and belong to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

(6.3)

$$\begin{cases} d\chi_1(t,y_1,y_2) = [\frac{\partial b}{\partial x}(t,y_1,y_2)\chi_1(t,y_1,y_2) + \frac{\partial b}{\partial u_1}(t,y)\beta_1(t,y_1)]dt \\ +[\frac{\partial \sigma}{\partial x}(t,y_1,y_2)\chi_1(t,y) + \frac{\partial \sigma}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)]dB(t) \\ + \int_{\mathbb{R}}[\frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)\chi_1(t,y_1,y_2) + \frac{\partial \gamma}{\partial u_1}(t,y_1,y_2,\zeta)\beta_1(t,y_1)]\tilde{N}(dt, \\ \chi_1(0,y_1,y_2) = \frac{d}{da} x^{(u_1+a\beta_1,u_2)}(0,y_1,y_2)|_{a=0} = 0. \end{cases}$$

and

(6.4)

$$\begin{cases} d\chi_2(t,y) = [\frac{\partial b}{\partial x}(t,y_1,y_2)\chi_2(t,y_1,y_2) + \frac{\partial b}{\partial u_2}(t,y_1,y_2)\beta_2(t,y_2)]dt \\ +[\frac{\partial \sigma}{\partial x}(t,y_1,y_2)\chi_2(t,y_1,y_2) + \frac{\partial \sigma}{\partial u_2}(t,y_1,y_2)\beta_2(t,y_2)]dB(t) \\ + \int_{\mathbb{R}}[\frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)\chi_2(t,y_1,y_2) + \frac{\partial \gamma}{\partial u_2}(t,y_1,y_2,\zeta)\beta_2(t,y_2)]\tilde{N}(dt, \\ \chi(0,y_1,y_2) = \frac{d}{da} x^{(u_1,u_2+a\beta_2)}(0,y_1,y_2)|_{a=0} = 0. \end{cases}$$

Theorem

[Necessary maximum principle]

Let $(u_1, u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$. *Then the following are equivalent:*

1. $\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 =$
$\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_2(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0,$
for all bounded $\beta_i \in \mathcal{A}_i$ *of the form (6.2).*

2. $$\int_{\mathbb{R}} \frac{\partial H_1}{\partial v_1}(t, x(t,y), v_1, u_2(t,y_2), p_1(t,y), q_1(t,y), r_1(t,y))dy_2|_{v_1=u_1(t,y_1)}$$
$$= \int_{\mathbb{R}} \frac{\partial H_2}{\partial v_2}(t, x(t,y), u_1(t,y_1), v_2, p_2(t,y), q_2(t,y), r_2(t,y))dy_1|_{v_2=u_2(t,y_2)}$$
(6.5)
$$= 0 \quad ; t \in [0, T].$$

7. The zero-sum case

In the zero-sum case we have

(7.1)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(.,y_1), u_2(.,y_2)) + J_2(u_1(.,y_1), u_2(.,y_2)) dy_1 dy_2 = 0.$$

Then the Nash equilibrium $(\hat{u}_1(.,y_1), \hat{u}_2(.,y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying (5.5)-(5.6) becomes a saddle point for

(7.2)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1(.,y_1), u_2(.,y_2)) dy_1 dy_2 := \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(.,y_1), u_2(.,y_2)) dy_1 dy_2.$$

Hence we want to find $(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\begin{aligned}
&\sup_{u_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 \\
&= \inf_{u_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} J(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \qquad (7.3)
\end{aligned}$$

where

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J(u(.,y_1,y_2))dy_1dy_2 =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\Big[\int_0^T f(t,x(t,y),u_1(t,y_1),u_2(t,y_2),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt$$

(7.4)

$$+ g(x(T,y),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]\Big] dy_1dy_2$$

One of the players is maximising and the other is minimising the performance functional

Let us now look at the problem with one performance functional common to both players, but where one of the players is maximising and the other is minimising it. Then we get just one Hamiltonian and just one BSDE, which is simpler to deal with. In this case the Hamiltonian H, is given by:

$$H(t, x, y_1, y_2, u_1, u_2, p, q, r) = H(t, x, y_1, y_2, u_1, u_2, p, q, r, \omega)$$
$$= \mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_t] f(t, x, u_1, u_2, y_1, y_2) + b(t, x, u_1, u_2, y_1, y_2)p$$

(7.5)

$$+ \sigma(t, x, u_1, u_2, y_1, y_2)q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1, y_2) r(t, \zeta) \nu(d\zeta)$$

Moreover, there is only one triple (p, q, r) of adjoint processes, given by the BSDE

$$(7.6) \quad \begin{cases} dp(t, y_1, y_2) = -\frac{\partial H}{\partial x}(t, y_1, y_2)dt + q(t, y_1, y_2)dB(t) \\ + \int_{\mathbb{R}} r(t, y_1, y_2, \zeta)\tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ p(T, y) = g'(x(T, y_1, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2)|\mathcal{F}_T] \end{cases}$$

We can now state the corresponding sufficient maximum principle for the zero-sum game:

Theorem

(Sufficient maximum principle for the zero-sum game)
Let $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ *with associated solution* $\hat{x}(t, y_1, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \zeta)$ *of* (4.7) *and* (7.6). *Assume that the following holds:*

1. *The function* $x \to g(x)$ *is affine*

2. $$\sup_{u_1 \in \mathbf{A}_1} \int_{\mathbb{R}} H\big(t, \widehat{x}(t,y), u_1, \hat{u}_2(t,y_2), \widehat{p}(t,y), \widehat{q}(t,y), \hat{r}(t,y)\big)\, dy_2 = \int_{\mathbb{R}} H\big(t, \widehat{x}(t,y), \widehat{u}_1(t,y_1), \hat{u}_2(t,y_2), \widehat{p}(t,y), \widehat{q}(t,y), \hat{r}(t,y)\big)\, dy_2$$

(7.7)

   for all t, y_1.

3. $\inf_{u_2 \in \mathbf{A}_2} \int_{\mathbb{R}} H\big(t, \widehat{x}(t,y), \hat{u}_1(t,y_1), u_2, \widehat{p}(t,y), \widehat{q}(t,y), \hat{r}(t,y)\big)\, dy_1$

$$= \int_{\mathbb{R}} H\big(t, \widehat{x}(t,y), \hat{u}_1(t,y_1), \hat{u}_2(t,y_2), \widehat{p}(t,y), \widehat{q}(t,y), \hat{r}(t,y)\big)\, dy_1$$

(7.8)

for all t, y_2.

4. The function

(7.9)

$$\hat{\mathcal{H}}(x) = \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H(t, x, y, u_1, \hat{u_2}(t,y_2), \widehat{p}(t,y), \widehat{q}(t,y), \hat{r}(t,y,\cdot))dy_2$$

is concave for all t, y_1

and the function

(7.10)

$$\underline{\mathcal{H}}(x) = \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H(t, x, y, \hat{u}_1(t, y_1), u_2, \widehat{p}(t, y), \widehat{q}(t, y), \hat{r}(t, y, \cdot)) dy_1$$

is convex for all t, y_2.

Then $\hat{u}(t, y_1, y_2) = (\hat{u}_1(t, y_1), \hat{u}_2(t, y_2))$ is a saddle point for $J(u_1, u_2)$.

Let us now state the corresponding necessary maximum principle for the zero sum game problem:

Theorem

[Necessary maximum principle for zero-sum games]
Assume the conditions of the previous Theorem hold. Then the following are equivalent:

1. $\frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} J(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = \frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} J(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0$ *for all bounded* $\beta_i \in \mathcal{A}_i$ *of the form (6.2).*

2. $$\int_{\mathbb{R}} \frac{\partial H}{\partial v_1}(t, x(t,y), v_1, u_2(t,y_2), p_1(t,y), q_1(t,y), r_1(t,y))dy_2|_{v_1=u_1(t,y_1)}$$
$$= \int_{\mathbb{R}} \frac{\partial H}{\partial v_2}(t, x(t,y), u_1(t,y_1), v_2, p_2(t,y), q_2(t,y), r_2(t,y))dy_1|_{v_2=u_2(t,y_2)}$$
(7.11)
$$= 0 \quad \forall t \in [0,T]$$

8. Application 1: Optimal consumption for an insider under model uncertainty

Suppose we have a cash flow with consumption, modelled by the process $X(t, Y) = X^{c,\mu}(t, Y)$ defined by:

$$\begin{cases} dX(t, Y) = (\alpha(t, Y) + \mu(t, Y_2) - c(t, Y_1))X(t, Y)dt \\ +\beta(t, Y)X(t, Y)dB(t) + \int_{\mathbb{R}} \gamma(t, Y, \zeta)X(t, Y)\tilde{N}(dt, d\zeta) \\ X(0) = x > 0 \end{cases}$$

Here $\alpha(t, Y), \beta(t, Y), \gamma(t, Y)$ are given coefficients, while $c(t, Y_1) > 0$ is the relative consumption rate chosen by the consumer (player number 1) and $\mu(t, Y_2)$ is a perturbation of the drift term, representing the model uncertainty chosen by the environment (player number 2).

Define the performance functional by

$$J(c,\mu) = \mathbb{E}[\int_0^T \log(c(t)X(t)) + \frac{1}{2}\mu^2(t)dt + \theta \log X(T)] \tag{8.1}$$

where $\theta > 0$ is a given constant and $\frac{1}{2}\mu^2(t)$ represents a penalty rate, penalizing μ for being away from 0. We assume that c is $\mathbb{H}^1$-adapted, while μ is $\mathbb{H}^2$-adapted.
We want to find $c^* \in \mathcal{A}_1$ and $\mu^* \in \mathcal{A}_2$ such that

$$\sup_{c\in\mathcal{A}_1} \inf_{\mu\in\mathcal{A}_2} J(c,\mu) = J(c^*,\mu^*). \tag{8.2}$$

As before we rewrite this problem as a classical stochastic differential game with two parameters y_1, y_2. Thus we define, for $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$,

(8.3)

$$\begin{cases} dx(t,y) & = (\alpha(t,y) + \mu(t,y_2) - c(t,y_1))x(t,y)dt + \beta(t,y)x(t,y)dB(\\ & + \int_{\mathbb{R}} \gamma(t,y,\zeta)x(t,y)\tilde{N}(dt,d\zeta) \\ x(0,y) & = x > 0 \end{cases}$$

and

$$J(c(.,y_1),\mu(.,y_2)) = \mathbb{E}[\int_0^T \{\log(c(t,y_1)x(t,y))$$
$$+ \frac{1}{2}\mu^2(t,y_2)\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt$$
$$+ \theta \log x(T,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] \tag{8.4}$$

The Hamiltonian for this problem is

$$H(t,x,y,c,\mu,p,q,r) = \{\log(cx) + \frac{1}{2}\mu^2\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]$$

(8.5)

$$+ (\alpha(t,y) + \mu - c)xp + \beta(t,y)xq + x\int_{\mathbb{R}} \gamma(t,y,\zeta)r(\zeta)d\nu(\zeta)$$

and the BSDE for the adjoint processes p, q, r is

$$\begin{cases} dp(t,y) & = -[\frac{1}{x(t,y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \\ & +(\alpha(t,y) + \mu(t,y_2) - c(t,y_1))p(t,y) \\ & +\beta(t,y)q(t,y) + \int_{\mathbb{R}} \gamma(t,y,\zeta)r(\zeta)d\nu(\zeta)]dt \\ & +q(t,y)dB(t) + \int_{\mathbb{R}} r(t,y)\tilde{N}(dt,d\zeta); 0 \leq t \leq T \\ p(T,y) & = \frac{\theta}{x(T,y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases}$$

Define

(8.6) $$h(t,y) = p(t,y)x(t,y).$$

Then by the Itô formula we get

$$dh(t,y) = x(t,y)[-\frac{1}{x(t,y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]$$
$$-(\alpha(t,Y)+\mu(t,Y_2)-c(t,Y_1))p(t,y)-\beta(t,y)q(t,y)$$
$$-\int_{\mathbb{R}}\gamma(t,y,\zeta)r(t,\zeta)d\nu(\zeta)]dt$$
$$+p(t,y)(\alpha(t,Y)+\mu(t,Y_2)-c(t,Y_1))x(t,y)dt+p(t,y)\beta(t,y)x(t,y)d$$
$$+q(t,y)\beta(t,y)x(t,y)dt$$
$$+\int_{\mathbb{R}}[(x(t,y)+\gamma(t,y,\zeta)x(t,y))(p(t,y)+r(t,y,\zeta))-p(t,y)x(t,y)-p$$

(8.7)

$$+\int_{\mathbb{R}}[(x(t,y)+\gamma(t,y,\zeta)x(t,y))(p(t,y)+r(t,y,\zeta))-p(t,y)x(t,y)]\tilde{N}($$

(8.8)

$$=dF(t,y)+h(t,y)\beta(t,y)dB(t)+h(t,y)\int_{\mathbb{R}}\gamma(t,y,\zeta)\tilde{N}(dt,d\zeta)),$$

where

$$
\begin{aligned}
&dF(t,y) = \\
&-\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + x(t,y)q(t,y)dB(t) \\
&+ x(t,y)\int_{\mathbb{R}} r(t,y,\zeta)(1+\gamma(t,y,\zeta))\tilde{N}(dt,d\zeta).
\end{aligned} \tag{8.9}
$$

To simplify this, we define the process $k(t,y)$ by the equation

$$
dk(t,y) = k(t,y)\Big[b(t,y)dB(t) + \int_{\mathbb{R}} c(t,y,\zeta)\tilde{N}(dt,d\zeta)\Big] \tag{8.10}
$$

for suitable processes b, c (to be determined).

Then again by the Itô formula we get

$$d(h(t,y)k(t,y)) = h(t,y)k(t,y)\Big[b(t,y)dB(t) + \int_{\mathbb{R}} c(t,y,\zeta)\tilde{N}(dt,d\zeta)\Big]$$

$$+ k(t,y)\Big[dF(t,y) + h(t,y)\beta(t,y)dB(t) + h(t,y)\int_{\mathbb{R}} \gamma(t,y,\zeta)\tilde{N}(dt,d\zeta)\Big]$$

$$+ (h(t,y)\beta(t,y) + x(t,y)q(t,y))k(t,y)b(t,y)dt$$

$$+ \int_{\mathbb{R}} \Big(h(t,y)\gamma(t,y,\zeta)$$

$$+ x(t,y)r(t,y,\zeta)(1+\gamma(t,y,\zeta))\Big)k(t,y)c(t,y,\zeta)\tilde{N}(dt,d\zeta)$$

$$+ \int_{\mathbb{R}} \Big(h(t,y)\gamma(t,y,\zeta)$$

(8.11)

$$+ x(t,y)r(t,y,\zeta)(1+\gamma(t,y,\zeta))\Big)k(t,y)c(t,y,\zeta)d\nu(\zeta)dt.$$

Define

$$u(t,y) := h(t,y)k(t,y).$$

(8.12)

Then the equation above can be written

$$\begin{aligned}
du(t,y) &= u(t,y)\Big[\int_{\mathbb{R}} \gamma(t,y,\zeta)c(t,y,\zeta)d\nu(\zeta)dt \\
&+ \{\beta(t,y)+b(t,y)\}dB(t) + \beta(t,y)b(t,y)dt \\
&+ \int_{\mathbb{R}} \{c(t,y,\zeta)+\gamma(t,y,\zeta)+c(t,y,\zeta)\gamma(t,y,\zeta)\}\tilde{N}(dt,d\zeta)\Big] \\
&+ k(t,y)\Big[dF(t,y) + x(t,y)q(t,y)b(t,y)dt \\
&+ \int_{\mathbb{R}} x(t,y)r(t,y,\zeta)c(t,y,\zeta)(1+\gamma(t,y,\zeta))d\nu(\zeta)dt
\end{aligned}$$

(8.13)

$$+ \int_{\mathbb{R}} x(t,y)r(t,y,\zeta)c(t,y,\zeta)(1+\gamma(t,y,\zeta))\tilde{N}(dt,d\zeta)\Big].$$

Choose

$$b(t,y) := -\beta(t,y)$$

$$c(t,\zeta) := -\frac{\gamma(t,y,\zeta)}{1+\gamma(t,y,\zeta)} \tag{8.14}$$

Then (8.13) reduces to

$$\begin{aligned} du(t,y) &= f(t,y)dt + k(t,y)x(t,y)q(t,y)dB(t) \\ &+ \int_{\mathbb{R}} \{x(t,y)r(t,y,\zeta)(1+\gamma(t,y,\zeta))[k(t,y) \\ &+ k(t,y)c(t,y,\zeta)]\}\tilde{N}(dt,d\zeta), \end{aligned} \tag{8.15}$$

where

$$
\begin{aligned}
& f(t,y) = -k(t,y)E[\delta_Y(y)|\mathcal{F}_t] \\
& + u(t,y)[\int_{\mathbb{R}} \gamma(t,y,\zeta)c(t,\zeta)d\nu(\zeta) + \beta(t,y)b(t,y)] \\
& + k(t,y)x(t,y)q(t,y)\beta(t,y) \\
(8.16) \quad & + k(t,y)\int_{\mathbb{R}} x(t,y)r(t,y,\zeta)c(t,y,\zeta)(1+\gamma(t,y,\zeta))d\nu(\zeta)
\end{aligned}
$$

Now define

$$v(t,y) := k(t,y)x(t,y)q(t,y)$$
$$w(t,y) := k(t,y)x(t,y)r(t,y,\zeta). \tag{8.17}$$

Then from (8.11) and (8.14) we get the following BSDE in the unknowns u, v, w:

$$du(t,y) =$$
$$\Big(- k(t,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] - u(t,y)[\int_{\mathbb{R}} \frac{\gamma^2(t,y,\zeta)}{1+\gamma(t,y,\zeta)} d\nu(\zeta) + \beta^2(t,y)]$$
$$- \beta(t,y)v(t,y) - \int_{\mathbb{R}} \gamma(t,y,\zeta)w(t,y,\zeta)d\nu(\zeta)\Big)dt$$
$$+ v(t,y)dB(t) + \int_{\mathbb{R}} w(t,y,\zeta)\tilde{N}(dt,d\zeta); \quad 0 \le t \le T$$

(8.18)

$$u(T,y) = \theta k(T,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]$$

This is a linear BSDE which has a unique solution $u(t,y) = p(t,y)x(t,y)k(t,y), v(t,y), w(t,y,\zeta)$. In particular, we may regard

$$p(t,y)x(t,y) = \frac{u(t,y)}{k(t,y)} \tag{8.19}$$

as known.

Maximizing $\int_{\mathbb{R}} H dy_2$ with respect to c gives the first order equation

$$\int_{\mathbb{R}} \{\frac{1}{c(t,y_1)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] - x(t,y)p(t,y)\}dy_2 = 0, \tag{8.20}$$

i.e.,

$$c(t,y_1) = \hat{c}(t,y_1) = \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dy_2}{\int_{\mathbb{R}} x(t,y)p(t,y)dy_2}. \tag{8.21}$$

Minimizing $\int_{\mathbb{R}} H dy_1$ with respect to μ gives the first order equation

$$\int_{\mathbb{R}} \{\mu(t,y_2)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + x(t,y)p(t,y)\}dy_1 = 0, \tag{8.22}$$

i.e.

$$\mu(t,y_2) = \hat{\mu}(t,y_2) = \frac{\int_{\mathbb{R}} x(t,y)p(t,y)dy_1}{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dy_1}. \tag{8.23}$$

We can now verify that $\hat{c}, \hat{\mu}$ satisfies all the conditions of the sufficient maximum principle, and hence we conclude the following:

Theorem (Optimal consumption for an insider under model uncertainty)

The solution (c^*, μ^*) *of the stochastic differential game* (8.2) *is given by*

$$c^*(t, Y_1) = \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_2|_{y_1=Y_1}}{\int_{\mathbb{R}} x(t,y)p(t,y) dy_2|_{y_1=Y_1}}. \tag{8.24}$$

and

$$\mu^*(t, Y_2) = \frac{\int_{\mathbb{R}} x(t,y)p(t,y) dy_1|_{y_2=Y_2}}{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_1|_{y_2=Y_2}}, \tag{8.25}$$

where $h(t,y) = x(t,y)p(t,y)$ *is given by* (8.18)-(8.19)*.*

8.2 Optimal portfolio for an insider under model uncertainty

Consider a financial market with two investment possibilities:

- (i) A risk free investment possibility with unit price $S_0(t) = 1$ for all $t \in [0, T]$
- (ii) A risky investment, where the unit price $S(t) = S(t, Y)$ is modelled by the (forward) SDE
  (8.26)
  $dS(t, Y) = S(t, Y)[(\alpha(t, Y) + \mu(t))dt + \beta(t, Y)dB(t)]; S(0) > 0.$

Here $\alpha(t, Y), \beta(t, Y) \neq 0$ are given $\mathbb{H}$-adapted coefficients, while $\mu(t)$ is a perturbation of the drift term, representing the model uncertainty chosen by the environment (player number 2).

Suppose the wealth process $X(t, Y) = X^{\pi,\mu}(t, Y)$ associated to an insider portfolio $\pi(t, Y)$ (representing the fraction of the wealth invested in the risky asset) is given by:

(8.27)

$$\begin{cases} dX(t,Y) = \pi(t,Y)X(t,Y)[(\alpha(t,Y)+\mu(t))dt + \beta(t,Y)]dB(t) \\ X(0) = x > 0 \end{cases}$$

Define the performance functional by

$$J(\pi, \mu) = \mathbb{E}[\int_0^T \frac{1}{2}\mu^2(t)dt + \theta \log X(T)], \tag{8.28}$$

where $\theta > 0$ is a given constant and $\frac{1}{2}\mu^2(t)$ represents a penalty rate, penalizing μ for being away from 0. We assume that π is $\mathbb{H}$-adapted, while μ is $\mathbb{F}$-adapted, i.e. has no inside information. We want to find $\pi^* \in \mathcal{A}_1$ and $\mu^* \in \mathcal{A}_2$ such that

$$\sup_{\pi\in\mathcal{A}_1} \inf_{\mu\in\mathcal{A}_2} J(\pi, \mu) = J(\pi^*, \mu^*). \tag{8.29}$$

We rewrite this problem as a classical stochastic differential game with one parameter y_1. Thus we define, setting $y = y_1 \in \mathbb{R}$,

(8.30)

$$\begin{cases} dx(t,y) & = \pi(t,y)x(t,y)[\{\alpha(t,y) + \mu(t)\}dt + \beta(t,y)dB(t)] \\ x(0,y) & = x(y) > 0 \end{cases}$$

and

$$J(\pi(.,y),\mu(.))$$

(8.31)

$$= \mathbb{E}[\int_0^T \frac{1}{2}\mu^2(t)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + \theta \log x(T,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]]$$

The Hamiltonian for this problem is

(8.32)

$$H(t, x, y, \pi, \mu, p, q) = \frac{1}{2}\mu^2 \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + \pi x(\alpha(t, y) + \mu)p + \pi x \beta(t, y)q$$

and the BSDE for the adjoint processes p, q, r is

$$\begin{cases} dp_t(y) = -[\pi_t(y)\{(\alpha_t(y) + \mu_t)p_t(y) + \beta_t(y)q_t(y)\}]dt + q_t(y)dB(t); \\ p_T(y) = \frac{\theta \mathbb{E}[\delta_Y(y)|\mathcal{F}_T]}{x_T(y)}. \end{cases}$$

Minimizing H with respect to π gives the first order equation

$$x(t,y)[(\alpha(t,y)+\mu(t))p(t,y)+\beta(t,y)q(t,y)]=0. \tag{8.33}$$

Since $x(t,y)>0$ and $\beta(t,y)\neq 0$, we deduce that

$$(\alpha(t,y)+\mu(t))p(t,y)+\beta(t,y)q(t,y)=0 \tag{8.34}$$

and

$$q(t,y)=-\frac{\alpha(t,y)+\mu(t)}{\beta(t,y)}p(t,y). \tag{8.35}$$

Hence (8.31) reduces to

$$\begin{cases} dp(t,y)=-\frac{\alpha(t,y)+\mu(t)}{\beta(t,y)}p(t,y)dB(t) \\ p(T,y)=\frac{\theta\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]}{x(T,y)}. \end{cases} \tag{8.36}$$

Define

$$h(t, y) = p(t, y)x(t, y). \tag{8.37}$$

Then by the Itô formula we get

$$\begin{cases} dh(t, y) &= (\pi(t, y)\beta(t, y) - \frac{\alpha(t,y)+\mu(t)}{\beta(t,y)})h(t, y)dB(t) \\ h(T, y) &= p(T, y)x(T, y) = \theta E[\delta_Y(y)|\mathcal{F}_T]. \end{cases} \tag{8.38}$$

This BSDE has the solution

$$h(t, y) = \theta E[\delta_Y(y)|\mathcal{F}_t]. \tag{8.39}$$

Moreover, by the generalized Clark-Ocone formula we have
(8.40)

$$(\pi(t,y)\beta(t,y) - \frac{\alpha(t,y)+\mu(t)}{\beta(t,y)})h(t,y) = D_t h(t) = \theta E[D_t \delta_Y(y)|\mathcal{F}],$$

from which we get the following expression for our candidate $\hat{\pi}(t,y)$ for the optimal portfolio

(8.41)
$$\hat{\pi}(t,y) = \frac{\alpha(t,y)+\hat{\mu}(t)}{\beta^2(t,y)} + \frac{E[D_t \delta_Y(y)|\mathcal{F}_t]}{\beta(t,y)E[\delta_Y(y)|\mathcal{F}_t]},$$

where $\hat{\mu}(t)$ is the corresponding candidate for the optimal perturbation.

Minimizing $\int_{\mathbb{R}} H dy$ with respect to μ gives the following first order equation for the optimal $\hat{\mu}(t)$:

(8.42)
$$\int_{\mathbb{R}} \{\hat{\mu}(t)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + \hat{\pi}(t,y)\hat{x}(t,y)\hat{p}(t,y)\} dy = 0,$$

i.e.,

(8.43)
$$\hat{\mu}(t) = -\frac{\int_{\mathbb{R}} \hat{\pi}(t,y)\hat{x}(t,y)\hat{p}(t,y)dy}{\int_{\mathbb{R}} \hat{x}(t,y)\hat{p}(t,y)dy} = -\int_{\mathbb{R}} \hat{\pi}(t,y)E[\delta_Y(y)|\mathcal{F}_t]dy.$$

We can now verify that $(\hat{\pi}, \hat{\mu})$ satisfies all the conditions of the sufficient maximum principle, and hence we conclude the following:

Theorem (Optimal portfolio for an insider under model uncertainty)

The saddle point $(\pi^*(t,Y), \mu^*(t))$, *where* $\pi^*(t,Y) = \pi^*(t,y)|_{y=Y}$, *of the stochastic differential game* (8.27) *is given by the solution of the following coupled system of equations*

$$\pi^*(t,y) = \frac{\alpha(t,y) + \mu^*(t)}{\beta^2(t,y)} + \frac{E[D_t\delta_Y(y)|\mathcal{F}_t]}{\beta(t,y)E[\delta_Y(y)|\mathcal{F}_t]}, \tag{8.44}$$

and

$$\mu^*(t) = -\int_{\mathbb{R}} \pi^*(t,y)E[\delta_Y(y)|\mathcal{F}_t]dy. \tag{8.45}$$

Remark. This result is an extension to insider trading of a result in Sulem & Ø. (2014) [ØS4].

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