Discretizing the Heston Model: An Analysis of the Weak Convergence Rate

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Part I: The Numerical Problem

Weak approximation of the Heston model

Multidimensional Heston model

Asset prices (Heston, 1993)

$$S_t^i = \exp(X_t^i), \quad t \in [0,T], \qquad i = 1,\dots,d$$

where

$$\mathrm{d}X_t^i = \left(b_i - \frac{1}{2}V_t^i\right)\mathrm{d}t + \sqrt{V_t^i}\mathrm{d}B_t^i, \qquad X_0^i = x_0^i \in \mathbb{R}$$

$$\mathrm{d}V_t^i = \kappa_i(\lambda_i - V_t^i)\mathrm{d}t + \theta_i\sqrt{V_t^i}\mathrm{d}W_t^i, \qquad V_0^i = v_0^i > 0$$

where $b_i \in \mathbb{R}$, κ_i, λ_i, $\theta_i > 0$, correlated Brownian motions B^i, W^i

Popular, but difficult model

- Moment explosions possible, i.e. $\mathbf{E}|S_T^i|^p$ maybe infinite, depending on the parameters (Andersen, Piterbarg, 2007; Lions, Musiela, 2007)
- SDE with non-Lipschitz coefficients: no standard results from numerical analysis applicable
- Volatility process V^i: CIR process, positive sample paths

Numerical problem

Weak convergence analysis

Task Find implementable approximation x_T^Δ such that

$$\text{(conv)} \qquad \lim_{\Delta\to 0} \mathbf{E} f(x_T^\Delta) = \mathbf{E} f(X_T) \qquad \text{for } f \in \mathcal{F}_{conv}$$

and

$$\text{(rate)} \qquad \left|\mathbf{E} f(x_T^\Delta) - \mathbf{E} f(X_T)\right| \leq c_f \Delta^\alpha \quad \text{for } f \in \mathcal{F}_{rate,\alpha}$$

with

(a) $\mathcal{F}_{conv}, \mathcal{F}_{rate,\alpha} \subset \{f : \mathbb{R}^d \to \mathbb{R}_{\geq 0};\ \text{measurable}\}$

(b) convergence rate $\alpha > 0$

as **large** as possible

Note $f = \text{payoff} \circ \exp$; SDE for (X, V): log-Heston SDE

Known results

- Replace v by $|v|$ or v^+ and discretize with Euler scheme: (conv) for

$$\mathcal{F} = \{f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \text{ bounded}, \lambda^d - a.s. \text{ continuous}\}$$

  Higham, Mao, 2005; Lord et al, 2010; ...

- No mathematical results for (rate) available, "only" simulation studies
  Kahl, Jäckel, 2006; Andersen, 2008; Lord et al, 2010; ...
  Convergence rate seems to deteriorate if $2\kappa_i\lambda_i/\theta_i^2 \ll 1$

Remarks For $d = 1$:

- Exact simulation of X_T possible (Broadie, Kaya, 2006)
- Alfonsi, 2005 & 2010: weak convergence analysis for CIR

Part II: A Warning

Shortcomings of the Euler scheme under non-standard assumptions

Moment explosions

Heston-3/2-model: volatility process

$$\mathrm{d}Z_t = 1.2 Z_t (0.8 - Z_t)\mathrm{d}t + Z_t^{3/2}\mathrm{d}W_t, \quad Z_0 = z_0 = 0.5$$

Euler scheme

$$z_{k+1} = z_k + 1.2 z_k (0.8 - z_k)\Delta + |z_k|^{3/2}\Delta_k W$$

with $\Delta = T/n$, $\Delta_k W = W_{(k+1)T/n} - W_{kT/n}$

Euler based standard Monte Carlo estimator

$$\widehat{p}_{\Delta,N} = \frac{1}{N}\sum_{i=1}^{N} |z_T^{\Delta,(i)}|$$

for $\mathbf{E}|Z_T|$ where $z_T^{\Delta,(i)}$ iid copies of $z_T^{\Delta} = z_n$

Moment explosions

For $T = 4$: $\mathbf{E}|Z_4| = 0.5662...$

Simulation study for $\widehat{p}_{\Delta,N}$:

repetitions / stepsize	$\Delta = 2^0$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}
$N = 10^3$	6.3272	Inf	Inf	0.5502	0.5535	0.5551
10^4	6.8947	Inf	Inf	Inf	0.5627	0.5634
10^5	7.4306	Inf	Inf	Inf	0.5662	0.5671
10^6	7.2274	Inf	Inf	Inf	Inf	0.5658
10^7	7.2792	Inf	Inf	Inf	Inf	Inf

In fact:

$$\lim_{\Delta \to 0} \mathbf{E}|z_T^{\Delta}| = \infty$$

Hutzenthaler et al, 2011, also for more general SDEs with super-linear coefficients

In our context no superlinear coeff's, but exponentially growing payoffs

Weak order: Always $\alpha = 1$?

Kebaier, 2005

$$\mathrm{d}X_t = -\frac{X_t}{2}\mathrm{d}t - Y_t \mathrm{d}W_t, \quad \mathrm{d}Y_t = -\frac{Y_t}{2}\mathrm{d}t + X_t \mathrm{d}W_t$$

with $(x_0, y_0) = (\cos\theta, \sin\theta)$ (solution paths lie on unit circle!)
Euler scheme

$$x_{k+1} = x_k - \frac{x_k}{2}\Delta - y_k \Delta_k W, \quad y_{k+1} = y_k - \frac{y_k}{2}\Delta + x_k \Delta_k W$$

Let $\alpha \in [1/2, 1]$, $f_\alpha(x, y) = \left|x^2 + y^2 - 1\right|^{2\alpha}$. Then

$$\Delta^{-\alpha} \cdot \left|\mathbf{E} f_\alpha(x_T^\Delta, y_T^\Delta) - \mathbf{E} f_\alpha(X_T, Y_T)\right| \to \text{ const}$$

with $x_T^\Delta = x_n, y_T^\Delta = y_n$

In our context Log-Heston SDE takes values in a half space, payoffs typically non-smooth

Part III: Our Results

Weak convergence analysis for a combined Euler & drift-implicit Milstein scheme

Numerical method

Euler scheme for log-asset price:

$$x^i_{k+1} = x^i_k + \Big(b_i - \frac{1}{2}v^i_k\Big)\Delta + \sqrt{v^i_k}\Delta_k B^i$$

Drift-implicit Milstein scheme for volatility:

$$v^i_{k+1} = v^i_k + \kappa_i(\lambda_i - v^i_{k+1})\Delta + \theta\sqrt{v^i_k}\Delta_k W^i + \frac{\theta_i^2}{4}(|\Delta_k W^i|^2 - \Delta)$$

Günther et al, 2008

Note

$$v^i_{k+1} = \frac{1}{1+\kappa_i\Delta}\left(\left(\sqrt{v^i_k} + \frac{\theta_i}{2}\Delta_k W^i\right)^2 + \Big(\kappa_i\lambda_i - \frac{\theta_i^2}{4}\Big)\Delta\right)$$

Thus: v^i_k well defined and positivity preserving for $\kappa_i\lambda_i/\theta_i^2 \geq 1/4$

Weak convergence

Notation/Assumption: $\rho_i = \frac{1}{t}\mathbf{E} B_t^i W_t^i$, $t > 0$

Theorem (Altmayer, N, 2015)

Let $\min_{i=1,\dots,d} \kappa_i \lambda_i / \theta_i^2 > 1/4$.

(i) *(conv) holds for*

$$\mathcal{F} = \{f : \mathbb{R}^d \to \mathbb{R}_{\geq 0};\ \lambda^d - \text{a.s. continuous},\ \mathbf{E} f(X_T) = \infty\}$$

(ii) *If* $\max_{i=1,\dots,d} \rho_i < 0$, *then (conv) holds for*

$$\mathcal{F} = \left\{f : \mathbb{R}^d \to \mathbb{R}_{\geq 0};\ \lambda^d - \text{a.s. continuous},\ f(x) = \mathcal{O}\Big(\sum_{i=1}^d \exp(x_i)\Big)\right\}$$

Remarks

- Case (i): moment explosions are recovered
- Case (ii) applies e.g. to basket calls

Weak convergence

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(ii) *If* $\max_{i=1,\dots,d} \rho_i < 0$, *then (conv) holds for*

$$\mathcal{F} = \left\{f : \mathbb{R}^d \to \mathbb{R}_{\geq 0};\ \lambda^d - a.s.\ continuous,\ f(x) = \mathcal{O}\left(\sum_{i=1}^d \exp(x_i)\right)\right\}$$

Proof

- Step 1: $x_n \to X_T$ in probability, e.g. using Yamada functions for CIR
- Step 2, case (i): Fatou's lemma
- Step 2, case (ii): dominated convergence via representation

$$x_n^i \leq \text{const} - \left(\frac{1}{2} - \frac{\rho_i\kappa_i}{\theta_i}\right)\sum_{k=0}^{n-1} v_k^i\Delta - \sum_{k=0}^{n-1}\frac{\rho_i\theta_i}{4}(\Delta_k W^i)^2 + \sum_{k=0}^{n-1}\sqrt{v_k}\Delta_k(B^i - \rho_i W^i)$$

and conditional normality

Convergence rate

Theorem (Altmayer, N, 2015)

Let $d = 1$, $\kappa\lambda/\theta^2 > 1$, $\varepsilon > 0$. Then:
(rate) for all $\alpha \in (0,1)$ and $\mathcal{F} = \{f \in C^{2,\varepsilon}(\mathbb{R};\mathbb{R}_{\geq 0});\ f \text{ compact support}\}$

Proof Tools:

(1) classical Kolmogorov PDE approach, i.e. $u(t,x,v) = \mathbf{E}f(X^{x,v}_{T-t})$ and $|\mathbf{E}f(X_T) - \mathbf{E}f(x_n)| \leq \sum_{k=1}^{n} |\mathbf{E}u(k\Delta, x_k, v_k) - \mathbf{E}u((k-1)\Delta, x_{k-1}, v_{k-1})|$ (Talay, Tubaro, 1990)

(2) a-priori estimates for log-Heston PDE (Fehan, Pop, 2013) on $\mathbb{R} \times \mathbb{R}_{>0}$

(3) bound for inverse moment of v_k ($\rightarrow$ condition on $\kappa\lambda/\theta^2$)

(4) tail estimates for CIR and numerical scheme

(5) Malliavin integration by parts

Crucial for (2), (3), (5): positivity preserving numerical scheme for CIR

Numerical tests Weak convergence order one also under less restrictive assumptions on $\kappa\lambda/\theta^2$ and $\mathcal{F}$ (as always for CIR / log-Heston ...)

Summary

Results Combined Euler & drift-implicit Milstein for log-Heston SDE:

- Weak convergence for $d > 1$ for large class of testfunctions under mild assumptions on the parameters
- Weak convergence order of (almost) one for $d = 1$, $\kappa\lambda/\theta^2 > 1$ and $C_c^{2,\varepsilon}$-test functions

Related work

- Multilevel quadrature and Malliavin smoothing in the Heston model
  Altmayer, N, 2015: *Multilevel Monte Carlo Quadrature of Discontinuous Payoffs in the Generalized Heston Model using Malliavin Integration by Parts*, SIAM J. Finan. Math., 6(1), 22–52
- Altmayer, 2015: weak convergence order one for measurable and bounded payoffs if $\kappa\lambda/\theta^2 > 7/4$

Work in progress / Open problems Weak convergence order for more irregular payoffs / under weaker condition on $\kappa\lambda/\theta^2$ / for $d > 1$