On mean-variance hedging under partial observations

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Introduction

We know that in a complete financial market a unique equivalent martingale measure exists and every contingent claim is attainable. This means that an agent can always construct a hedging strategy based on available assets. But, in practice, we rarely deal with a complete market, because the number of causes for uncertainty is greater than the number of assets held by the agent. Consequently, the hedging strategy does not always exist in an incomplete market and corresponding portfolio value does not equal to the claim value. In this case, we want to find a strategy that minimizes the expected value of the squared difference of the contingent claim and portfolio value. Such problem is called the problem of mean-variance hedging (MVH).

Since the pioneering work of Föllmer and Sondermann [Follmer, Sondermann], the mean-variance hedging is a permanent area of research in mathematical finance. At the beginning the problem was formulated assuming that the probability measure was a martingale measure. In this context, some results were obtained in the case of full information by Föllmer and Sondermann [Follmer, Sondermann] and Schweizer [Schweizer (2001)]. Under incomplete information this type of hedging was developed later in many papers. See, for example Schweizer [Schweizer (92)], where two correlated Wiener processes were considered and the hedging strategy could depend on both, and Schweizer [Schweizer (94)] where results were obtained using projection techniques. Mention also papers [Schweizer (96)] and [Monat, Stricker]. A similar problem for a short-fall risk minimization was studied by Weisshaupt in his example-oriented paper [Weisshaupt]. Černỳ [Cerny], Schäl [Schal] and Schweizer [Schweizer (95)], derived the mean–variance hedging process in discrete time.

Among the new studies we point out Fujii and Takahashi [Fujii, Takanashi] and Hubalek et al. [Hubalek, Kallsen, Krawchyk]. Also mention paper of Ceci et al. [Ceci], where the authors provide a suitable Galtchouk-Kunita-Watanabe decomposition of the contingent claim that works in a partial information framework, and work of Jeanblanc et al. [Jenblac et al.], where the problem of mean-variance hedging for general semimartingale models is solved via stochastic control methods. In general, the mean-variance problem is very popular and has been used and studied in many examples and contexts. In particular, in stochastic volatility models ([Biagini, Paolo, Maurizio],[Laurent, Pham]), insurance ([Dal, Moller],[Delong, Gerard]), weather derivatives or electricity loads ([Brockett],[Keppo et al]), insider trading ([Biagini, Oksendal],[Kohlman et al.]).

The novelty of our approach consists on considering a mean-variance minimization problem where there are restrictions in the available information and there is a random lower bound on a terminal wealth. More precisely, we consider a mean-variance minimization problem for an unobservable contingent claim under the condition that the observable contingent claim is superhedged. To authors' knowledge such an approach has not been considered in the existing literature.

Consider the financial market with one hedger on it and two contingent claims that will be described below. Compare the information related to both claims. The fact that the hedging strategy for some claim does not always exist can be interpreted as a lack of sufficient information about the market.

Mathematically, the "complete" information is described by the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ and "incomplete" information is described by its subfiltration $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t, t \geq 0\}$. We restrict ourselves to the finite horizon $T > 0$.

The prices of underlying asset are given by $\widetilde{\mathbb{F}}$-adapted square-integrable stochastic process $\widetilde{S} = \{\widetilde{S}_t, t \in [0, T]\}$. We assume that $\widetilde{S}$ is a semimartingale.

A trading strategy is described by $\widetilde{\mathbb{F}}$-predictable square-integrable stochastic processes $\xi = \{\xi_t, t \in [0, T]\}$ such that the stochastic integral $\int_0^T \xi_t d\widetilde{S}_t$ is well-defined. This integral describes the trading gains induced by the self-financing portfolio strategy associated to ξ.

Let a contingent claim $\widetilde{H}$ be a $\widetilde{\mathcal{F}}_T$-measurable square-integrable nonnegative random variable. At time T, a hedger who starts with initial capital x and uses the strategy ξ, has to pay the random amount $\widetilde{H}$, so that portfolio value should not be less than $\widetilde{H}$. This contingent claim $\widetilde{H}$ can be interpreted as a random lower bound of a terminal wealth. At the same time hedger wants approximate a random amount H by portfolio value. In contradistinction to $\widetilde{H}$, we assume that H is a $\mathcal{F}_T$-measurable square-integrable nonnegative random variable. In this context, mean-variance hedging problem with partial observations means solving the optimization problem

$$\text{minimize } \mathbf{E}\left(H - x - \int_0^T \xi_t d\widetilde{S}_t\right)^2 \text{ over all } \xi \in \Xi(x, \widetilde{H}),$$

where

$$\Xi(x, \widetilde{H}) = \{\xi : x + \int_0^T \xi_t d\widetilde{S}_t \geq \widetilde{H} \ a.s.\}$$

Similar problem in the partial case of two correlated Wiener processes was considered by Weisshaupt, but he maximized the probability of successful hedging of the unobservable claim, and we are interested in the mean-variance framework, assuming that there is some reserved capital and trying to minimize its value. This problem is naturally related to the mean-variance hedging. The main challenge in solving mean-variance hedging problem is to find more explicit descriptions of the optimal strategy.

The plan is the following: –Formulate the conditional mean-variance hedging problem under incomplete information in the general martingale setting and reduce it to the simplified statement.

–Prove the auxiliary result concerning the representation of the random variable that is approached and prove the main result that gives the solution of the minimization problem.

–The corresponding results are illustrated with the help of the model with two correlated Wiener processes.

–Numerical illustrations.

–How the problem can be solved in a semimartingale case?

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Preliminaries

Let we have complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ that corresponds to the "complete information".

Suppose that there exists a subfiltration $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t, t \geq 0\}$ that corresponds to the "incomplete information".

Consider square-integrable cadlag martingale $\widetilde{M} = \{\widetilde{M_t}, t \geq 0\}$ adapted to the "incomplete information", or, that is the same for us, $\widetilde{\mathbb{F}}$-adapted.

Moreover, assume that subfiltration $\widetilde{\mathbb{F}}$ is generated by $\widetilde{M}$.

We consider all processes on the interval $[0, T]$. Now we introduce two square-integrable nonnegative random variables, H and $\widetilde{H}$, H being $\mathcal{F}_T$-measurable and $\widetilde{H}$ being $\widetilde{\mathcal{F}}_T$-measurable. We can characterize them as "unobservable" and "observable" random variables or contingent claims, correspondingly.

Problem $[M(x, \widetilde{H})]$

Denote by Ξ_M class of such $\widetilde{\mathcal{F}}$-predictable square-integrable processes $\xi = \{\xi_t, t \in [0, T]\}$ that $\mathbf{E} \int_0^T \xi_s^2 d\langle \widetilde{M} \rangle_s < \infty$. Further, for any $x \in \mathbb{R}$ denote

$$\Xi_M(x, \widetilde{H}) = \{\xi \in \Xi_M : x + \int_0^T \xi_s d\widetilde{M}_s \geq \widetilde{H} \ a.s.\}.$$

Since the underlying price process $\widetilde{M}$ is a martingale, we say that this is a conditional minimization problem in the martingale framework.

Problem $[M(x, \widetilde{H})]$. For fixed $x > 0$ to find

$$\min_{\xi \in \Xi_M(x, \widetilde{H})} \mathbf{E} \left(H - x - \int_0^T \xi_s d\widetilde{M}_s \right)^2,$$

and such $\widetilde{\xi}$ for which

$$\min_{\xi \in \Xi_M(x, \widetilde{H})} \mathbf{E} \left(H - x - \int_0^T \xi_s d\widetilde{M}_s \right)^2 = \mathbf{E} \left(H - x - \int_0^T \widetilde{\xi}_s d\widetilde{M}_s \right)^2.$$

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In order to simplify the process of solving the Problem $[M(x, \widetilde{H})]$, we make the following remarks.

Remark

Random variable $H - \mathbf{E}(H|\widetilde{\mathcal{F}}_T)$ is orthogonal to any $\widetilde{\mathcal{F}}_T$-measurable square-integrable random variable. Therefore, it is sufficient to find

$$\min_{\xi \in \Xi_M(x,\widetilde{H})} \mathbf{E} \left(\mathbf{E}(H|\widetilde{\mathcal{F}}_T) - x - \int_0^T \xi_s d\widetilde{M}_s \right)^2 .$$

Remark

Denote $H_1 = \mathbf{E}(H|\widetilde{\mathcal{F}}_T)$ and $H_2 = H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} + \widetilde{H} \mathbb{1}_{H_1 < \widetilde{H}} \geq \widetilde{H}$.
For any $\xi \in \Xi_M(x, \widetilde{H})$

$$\mathbf{E}\left(H_2 - x - \int_0^T \xi_s d\widetilde{M}_s\right)^2 \leq \mathbf{E}\left(H_1 - x - \int_0^T \xi_s d\widetilde{M}_s\right)^2 \tag{1}$$

$$\leq \mathbf{E}\left(H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} - x - \int_0^T \xi_s d\widetilde{M}_s\right)^2 - \mathbf{E}\left(H_1 \mathbb{1}_{H_1 < \widetilde{H}}\right)^2, \tag{2}$$

and the equalities in (1)–(2) are achieved if and only if $H_1 \geq \widetilde{H}$ a.s. Therefore, we can restrict ourselves to the case $H_1 \geq \widetilde{H}$ a.s. and in other cases apply bounds (1)–(2)

Remark

Now, let $H_1 \geq \widetilde{H}$ a.s. and consider the case when $x = \mathbf{E}H = \mathbf{E}H_1$. Recall that subfiltration $\widetilde{\mathbb{F}}$ is generated by $\widetilde{M}$. Applying Clark-Ocone integral representation theorem to H_1, we get the representation

$$H_1 = x + \int_0^T \xi_s^0 d\widetilde{M}_s$$

for some $\xi^0 \in \Xi_M$. So, we put $\widetilde{\xi} = \xi^0$ and get the trivial zero solution of minimization problem. So, it is necessary consider two cases: $x < \mathbf{E}H$ and $x \geq \mathbf{E}H$. However, since our goal is to solve the minimization problem with minimal initial resources, we suppose in what follows that $x < \mathbf{E}H$.

Remark

At last, assume that $H_1 \geq \widetilde{H}$ a.s., $x < \mathbf{E}H = \mathbf{E}H_1$. Applying Clark-Ocone integral representation theorem to $\widetilde{H}$, we get the representation

$$\widetilde{H} = \widetilde{x} + \int_0^T \widetilde{\xi}_s d\widetilde{M}_s$$

with $\widetilde{x} = \mathbf{E}\widetilde{H}$ and $\widetilde{\xi} \in \Xi_M$. Now, rewrite

$$\mathbf{E}\left(H_1 - x - \int_0^T \xi_s d\widetilde{M}_s\right)^2$$
$$= \mathbf{E}\left(H_1 - \widetilde{H} - \left(x - \widetilde{x} + \int_0^T (\xi_s - \widetilde{\xi}_s) d\widetilde{M}_s\right)\right)^2,$$

denote $G = H_1 - \widetilde{H}$, $g = x - \widetilde{x}$, $\eta_s = \xi_s - \widetilde{\xi}_s$ and note that $G \geq 0$ a.s., $0 < g < \mathbf{E}G$ and $g + \int_0^T \eta_s d\widetilde{M}_s = x - \widetilde{x} + \int_0^T (\xi_s - \widetilde{\xi}_s) d\widetilde{M}_s \geq 0$ a.s.

Problem $[M(g,0)]$

Under our assumptions, Problem $(M(x,\widetilde{H}))$ can be reduced to the following one.

Problem $[M(g,0)]$. For fixed square-integrable nonnegative $\widetilde{F}_T$-measurable random variable G and fixed number $0 < g < \mathbf{E}G$ to find

$$\min_{\xi \in \Xi_M(g,0)} \mathbf{E}\left(G - g - \int_0^T \xi_s d\widetilde{M}_s\right)^2,$$

and such $\widetilde{\xi}$ for which

$$\min_{\xi \in \Xi_M(g,0)} \mathbf{E}\left(G - g - \int_0^T \xi_s d\widetilde{M}_s\right)^2 = \mathbf{E}\left(G - g - \int_0^T \widetilde{\xi}_s d\widetilde{M}_s\right)^2.$$

Remark

Consider the term $\mathbf{E}(x,\widetilde{M}) := \mathbf{E}\left(H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} - x - \int_0^T \xi_s d\widetilde{M}_s\right)^2$ from the right-hand side of (2). We can present it as

$$\mathbf{E}\left((H_1 - \widetilde{H})\mathbb{1}_{H_1 \geq \widetilde{H}} - \left(x + \int_0^T \xi_s d\widetilde{M}_s - \widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}}\right)\right)^2.$$

Applying Clark-Ocone integral representation theorem to $\widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}}$, we get the representation $\widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}} = \widetilde{x}_1 + \int_0^T \widetilde{\gamma}_s d\widetilde{M}_s$, where
$\widetilde{x}_1 = \mathbf{E}\widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}} \leq \mathbf{E}\widetilde{H} \leq x$ and $x + \int_0^T \xi_s d\widetilde{M}_s - \widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}} \geq 0$.
Therefore the minimization of the term $\mathbf{E}(x,\widetilde{M})$ is in the framework of the Problem $[M(g,0)]$ with $g = x - \widetilde{x}_1 \geq 0$ and
$G = (H_1 - \widetilde{H})\mathbb{1}_{H_1 \geq \widetilde{H}} \geq 0$. So, in the general case, when the inequality $H_1 \geq \widetilde{H}$ does not hold a.s., we can minimize LHS of (1) and RHS of (2) in the framework of the Problem $[M(g,0)]$.

To solve Problem $[M(g,0)]$, we take into account Remark 1 and consider the representation

$$G = \mathbf{E}G + \int_0^T \eta_s d\widetilde{M}_s.$$

Let $x \in (0, \mathbf{E}G)$. Consider $z(x)$ that is the solution of equation

$$\mathbf{E}\left(z(x) + \int_0^T \eta_s d\widetilde{M}_s\right)^+ = x. \tag{3}$$

Lemma

Function $z = z(x)$, $x \in (0, \mathbf{E}G)$ is uniquely determined, continuous and strongly increasing on the interval $(0, \mathbf{E}G)$. In addition, $z(x) \leq x$. The range of values of this function is the interval $(r, \mathbf{E}G)$, where $r = -\sup_{\omega\in\Omega} \int_0^T \eta_s d\widetilde{M}_s$.

The solution of the minimization problem $[M(g,0)]$.

Theorem

Let $g \in (0, \mathbf{E}G)$ be fixed. Consider $z(g)$ that is the unique solution of equation $\mathbf{E}\left(z(g) + \int_0^T \eta_s d\widetilde{M}_s\right)^+ = g$ and the Clark–Ocone integral representation of the random variable $\left(z(g) + \int_0^T \eta_s d\widetilde{M}_s\right)^+$:

$$\left(z(g) + \int_0^T \eta_s d\widetilde{M}_s\right)^+ = g + \int_0^T \widetilde{\xi}_s d\widetilde{M}_s. \tag{4}$$

Then $\widetilde{\xi}$ is the solution of the minimization problem $[M(g,0)]$.

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Example

Consider the case when we have two correlated Wiener processes, $W = \{W_t, t \geq 0\}$ and $\widetilde{W} = \{\widetilde{W}_t, t \geq 0\}$.
Let $W_t = \rho \widetilde{W}_t + (1-\rho^2)^{1/2} \widehat{W}_t$,
where the Wiener processes $\widetilde{W}$ and $\widehat{W}$ are independent.
Let filtration $\mathbb{F}$ be generated by W, filtration $\widetilde{\mathbb{F}}$ be generated by $\widetilde{W}$.
Also, let $\widetilde{M} = \widetilde{W}$ and let $G = H(W_T)$ be square-integrable random variable, where $H : \mathbb{R} \to \mathbb{R}_+$ is real-valued non-decreasing measurable function of polynomial growth at infinity.
In order to give the explicit solution of minimization problem $[M(g, 0)]$, we make the following steps.

The model with two correlated Wiener processes

- Calculate $\widetilde{G} = \mathbf{E}(G|\widetilde{\mathcal{F}}_T)$. In this order, introduce the function

$$f(x) = \int_{\mathbb{R}} H(z) \frac{\exp\left(-\frac{(z-\rho x)^2}{2T(1-\rho^2)}\right)}{\sqrt{2\pi T(1-\rho^2)}} dz, \tag{5}$$

  and get that $\widetilde{G} = f(\widetilde{W}_T)$.

- Apply Theorem 2.2 and find Clark–Ocone integral representation of the random variable $\left(z(g) + \int_0^T \eta_t d\widetilde{W}_t\right)^+$, where $z(g)$ is a solution of equation (3) with $x = g$. Denote $K_g := \mathbf{E}G - z(g)$. Let $\Phi(x)$ be the standard normal cumulative distribution function Then K_g is a solution of the following equation

$$g = \int_{f^{-1}(K_g)}^{+\infty} f(x) \frac{\exp\left(-\frac{x^2}{2T}\right)}{\sqrt{2\pi T}} dx - K_g \Phi\left(-\frac{f^{-1}(K_g)}{\sqrt{T}}\right). \tag{6}$$

- It follows from integral representation of $\widetilde{G}$ that

$$\left(z(g)+\int_0^T \eta_t d\widetilde{W}_t\right)^+ = \left(f(\widetilde{W}_T)-K_g\right)^+. \tag{7}$$

  From Theorem 2.2 and relation (7) we have that the solution of the minimization problem $[M(g,0)]$ is the process $\widetilde{\xi}$ such that

$$\left(f(\widetilde{W}_T)-K_g\right)\mathbb{1}\{\widetilde{W}_T \geq f^{-1}(K_g)\} = g + \int_0^T \widetilde{\xi}_t d\widetilde{W}_t.$$

-

$$\widetilde{\xi}_t = \rho\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}\{x\sqrt{T-t}+\widetilde{W}_t \geq f^{-1}(K_g)\}$$
$$\times H(\rho x\sqrt{T-t}+y\sqrt{T(1-\rho^2)}+\rho\widetilde{W}_t)\frac{\exp(-(x^2+y^2)/2)}{2\pi\sqrt{T(1-\rho^2)}}ydydx. \tag{8}$$

- Evaluating $\mathbf{E}\left(G - g - \int_0^T \widetilde{\xi}_s d\widetilde{W_s}\right)^2$,
we get

$$\mathbf{E}\left(G - g - \int_0^T \widetilde{\xi}_s d\widetilde{W_s}\right)^2 = \mathbf{E}\left(G - \widetilde{G}\right)^2 + \mathbf{E}\left(\widetilde{G} - g - \int_0^T \widetilde{\xi}_s d\widetilde{W_s}\right)^2$$

$$= \int_{\mathbb{R}} (H^2(x) - f^2(x)) \frac{\exp\left(-\frac{x^2}{2T}\right)}{\sqrt{2\pi T}} dx$$

$$+ K_g^2 \Phi\left(-\frac{f^{(-1)}(K_g)}{\sqrt{T}}\right) + \int_{-\infty}^{f^{(-1)}(K_g)} f^2(x) \frac{\exp\left(-\frac{x^2}{2T}\right)}{\sqrt{2\pi T}} dx.$$

Example

Specify Example 2.3 for the call option $H(y) = (y - K)^+, y \in \mathbb{R}$.

- The function f from (5) has the following form

$$f(x) = \sqrt{\frac{T(1-\rho^2)}{2\pi}} \exp\left(-\frac{(\rho x - K)^2}{2T(1-\rho^2)}\right) + (\rho x - K)\Phi\left(\frac{\rho x - K}{\sqrt{T(1-\rho^2)}}\right). \tag{9}$$

- The solution of the minimization problem $[M(g,0)]$ is the process $\widetilde{\xi}$

$$\widetilde{\xi}_t = \rho \int_{\{x\sqrt{T-t} \geq f^{-1}(K_g) - \widetilde{W}_t\}} \Phi\left(\frac{\rho x \sqrt{T-t} + \rho \widetilde{W}_t - K}{\sqrt{T(1-\rho^2)}}\right) \times \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx. \tag{10}$$

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Consider a numerical example of solutions of minimization problem $[M(g,0)]$ for the call option. Put $g = 0.05$ and consider the cases $T \in \{5, 10\}$, $K \in \{1, 2, 3\}$ and $\rho \in \{0.2, 0.5, 0.75\}$. Values of solutions of equation (6), minimum values in problem $[M(g,0)]$, and values of $\mathbf{E}G$ are presented in the following tables.

Solutions K_g of equation (6)

T=5	K=1	K=2	K=3
ρ=0.2	0.500	0.196	0.044
ρ=0.5	0.887	0.352	0.066
ρ=0.75	1.562	0.725	0.150

T=10	K=1	K=2	K=3
ρ=0.2	0.948	0.547	0.276
ρ=0.5	1.726	1.055	0.544
ρ=0.75	2.929	2.001	1.168

Minimum values in problem $[M(g,0)]$

T=5	K=1	K=2	K=3
ρ=0.2	1.096	0.447	0.161
ρ=0.5	1.034	0.413	0.149
ρ=0.75	0.938	0.346	0.115

T=10	K=1	K=2	K=3
ρ=0.2	2.824	1.557	0.807
ρ=0.5	2.712	1.475	0.754
ρ=0.75	2.555	1.340	0.650

Table: Values of $\mathbf{E}G$

	K=1	K=2	K=3
T=5	0.479811	0.226874	0.093117
T=10	0.824124	0.505794	0.290238

We simulate the trajectories of the Wiener process $\widetilde{W}_t$ on time interval $[0, 10]$, take one of these trajectories on the intervals $[0, 5]$ and $[0, 10]$ and present it at the bottom of Figure 1. For this trajectory we construct the sample paths of $\widetilde{\xi}_t$, with $K = 1$ and $\rho = 0.2, 0.5, 0.75$. At the top of Figure 1 we present these sample paths, grouped by T.

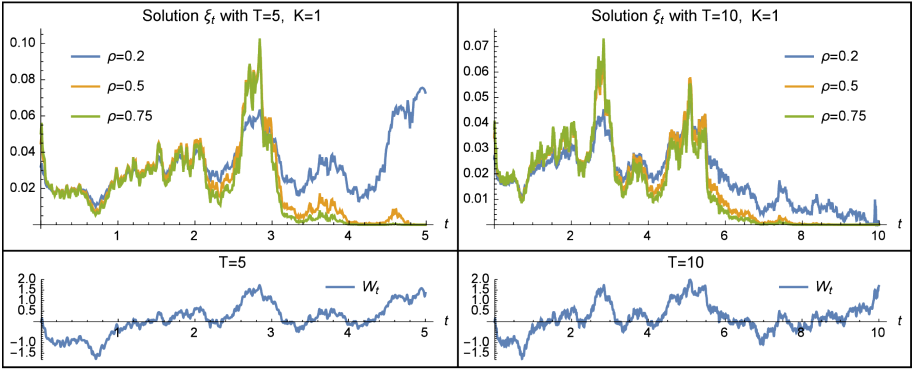

Figure: Sample paths of solutions of minimization problem $[M(g, 0)]$.

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Preliminaries

Consider two continuous risk assets $S = \{S_t, t \geq 0\}$ and $\widetilde{S} = \{\widetilde{S}_t, t \geq 0\}$ such that S_t is $\mathcal{F}_t$-adapted and $\widetilde{S}_t$ is $\widetilde{\mathcal{F}}_t$-adapted so that $S = \{S_t, t \geq 0\}$ is the non-observable asset and $\widetilde{S} = \{\widetilde{S}_t, t \geq 0\}$ is observable asset.

We suppose that non-risky asset $B_t \equiv 1$ and the market $\Sigma = \{1, S_t, \widetilde{S}_t, t \geq 0\}$ is arbitrage-free on $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F}$. Moreover, we suppose that the observable market $\widetilde{\Sigma} = \{1, \widetilde{S}_t, t \geq 0\}$ is complete on $(\Omega, \mathcal{F}, P)$ with filtration $\widetilde{\mathbb{F}}$.

Let $\mathcal{P}$ be the set of all equivalent martingale measures for Σ. Then the restriction $\widetilde{P}$ of any $P^* \in \mathcal{P}$ on $\widetilde{\mathbb{F}}$ is the same unique equivalent martingale measure for the observable market $\widetilde{\Sigma}$ so that $\widetilde{S}$ is $\widetilde{\mathbb{F}}$-martingale w.r.t. $\widetilde{P}$. Denote $D_T = \frac{d\widetilde{P}_T}{dP_T}$ the restriction of $\frac{d\widetilde{P}}{dP}$ on interval $[0, T]$.

Now, let H be a contingent claim on the extended market Σ and $\widetilde{H}$ be a contingent claim on the observable market $\widetilde{\Sigma}$.

In order to remain within the framework of the square-integrable approach, we fix the interval $[0, T]$ and introduce the following assumptions.

$(A1)$ $\widetilde{S} = \{\widetilde{S}_t, t \geq 0\}$ is the semimartingale admitting the representation $\widetilde{S}_t = \widetilde{N}_t + \widetilde{A}_t$, where $\widetilde{N}$ is the square-integrable martingale and $\widetilde{A}$ is the predictable process of square-integrable variation. Suppose that $\widetilde{\mathbb{F}}$ is generated by $\widetilde{N} = \{\widetilde{N}_t, t \geq 0\}$.

$(A2)$ $\mathbf{E}D_T^2 < \infty$, $\mathbf{E}H^2 < \infty$ and $\mathbf{E}\widetilde{H}^2 < \infty$.

These conditions mean, in particular, that we can consider stochastic integral w.r.t. the semimartingale $\widetilde{S}$,

$$I(t,\xi) = \int_0^t \xi_s d\widetilde{S}_s = \int_0^t \xi_s d\widetilde{N}_s + \int_0^t \xi_s d\widetilde{A}_s,\ t \in [0, T]$$

for such $\widetilde{\mathbb{F}}$-predictable processes ξ that $\int_0^T \xi_s^2 d\langle \widetilde{N} \rangle_s < \infty$ and $\int_0^T |\xi_s| d|\widetilde{A}|_s < \infty$ a.s. Denote Ξ_S class of such $\widetilde{\mathbb{F}}$-predictable strategies.

Problem $[S(x, \widetilde{H})]$

Completeness of market $\widetilde{\Sigma}$ together with condition $(A2)$ means that for any initial value $x \geq \mathbf{E}_{\widetilde{P}}\widetilde{H}$ we can construct the superhedge of the contingent claim $\widetilde{H}$ a.s. with the help of such $\xi \in \Xi_S$ that $\mathbf{E}(I(T,\xi))^2 < \infty$. In other words, there exists such $\xi \in \Xi_S$ that

$$x + I(T,\xi) \geq \widetilde{H} \quad a.s.$$

We denote $\Xi_S(x, \widetilde{H})$ class of such strategies. Now we can state a conditional minimization problem in the semimartingale framework.
Problem $[S(x, \widetilde{H})]$. Starting with fixed value $x \geq \mathbf{E}_{\widetilde{P}}\widetilde{H}$, to construct the hedging strategy $\widetilde{\xi} \in \Xi_S(x, \widetilde{H})$ so that

$$\mathbf{E}\left(H - x - \int_0^T \widetilde{\xi}_s d\widetilde{S}_s\right)^2 = \min_{\xi \in \Xi_S(x,\widetilde{H})} \mathbf{E}\left(H - x - \int_0^T \xi_s d\widetilde{S}_s\right)^2,$$

and to find this minimal value.

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Reducing the problem

Remark

Problem $[S(x, \widetilde{H})]$ is reduced to the finding of

$$\min_{\xi \in \Xi_S(x,\widetilde{H})} \mathbf{E} \left(H_1 - x - \int_0^T \xi_s d\widetilde{S}_s \right)^2, \tag{11}$$

and the minimizing $\widetilde{\xi}$, where $H_1 = \mathbf{E}(H|\widetilde{\mathcal{F}}_T)$.

Remark

Consider the expansion $H_1 = H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} + H_1 \mathbb{1}_{H_1 < \widetilde{H}}$ and denote $H_2 = H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} + \widetilde{H} \mathbb{1}_{H_1 < \widetilde{H}} \geq \widetilde{H}$. Then, similarly to (1), for any $\xi \in \Xi_S(x, \widetilde{H})$

$$\begin{aligned} \mathbf{E}\left(H_2 - x - \int_0^T \xi_s d\widetilde{S}_s\right)^2 &\leq \mathbf{E}\left(H_1 - x - \int_0^T \xi_s d\widetilde{S}_s\right)^2 \\ &\leq \mathbf{E}\left(H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} - x - \int_0^T \xi_s d\widetilde{S}_s\right)^2 - \mathbf{E}\left(H_1 \mathbb{1}_{H_1 < \widetilde{H}}\right)^2. \end{aligned} \tag{12}$$

and the equalities in (12) are achieved if and only if $H_1 \geq \widetilde{H}$ a.s. Therefore, we can restrict ourselves to the case $H_1 \geq \widetilde{H}$ a.s. and in other cases apply bounds (12).

Remark

Now, let $H_1 \geq \widetilde{H}$ a.s. and consider the case when $x = \mathbf{E}_{\widetilde{P}} H_1$. It follows from the completeness of the market $\widetilde{\Sigma}$ that we have the representation $H_1 = x + \int_0^T \xi_s^0 d\widetilde{S}_s$ for some $\xi^0 \in \Xi_S(x, \widetilde{H})$. So, we put $\widetilde{\xi} = \xi^0$ and get the trivial zero solution of minimization problem. So, it is reasonable to consider two cases: $x < \mathbf{E}_{\widetilde{P}} H_1$ and $x > \mathbf{E}_{\widetilde{P}} H_1$. However, since our goal is to solve the minimization problem with minimal initial resources, we suppose in what follows that $x < \mathbf{E}_{\widetilde{P}} H_1$.

Remark

Further, let $\mathbf{E}_{\widetilde{P}} \widetilde{H} \leq x < \mathbf{E}_{\widetilde{P}} H_1$. Evidently, $\mathbf{E} \left(H_1 - x - \int_0^T \xi_s d\widetilde{S}_s \right)^2 = \mathbf{E} \left(H_1 - \widetilde{H} - \left(x + \int_0^T \xi_s d\widetilde{S}_s - \widetilde{H} \right) \right)^2$. It follows from the completeness of the market that there exists $\eta \in \Xi_S$ such that $\widetilde{H} = \mathbf{E}_{\widetilde{P}} \widetilde{H} + \int_0^T \eta_s d\widetilde{S}_s$. Denote $g = x - \mathbf{E}_{\widetilde{P}} \widetilde{H} \geq 0$ and let $\zeta_s = \xi_s - \eta_s$, $G = H_1 - \widetilde{H} \geq 0$. Then obviously $0 \leq y < \mathbf{E}_{\widetilde{P}} G$.

Problem $[S(g,0)]$

We reduce Problem $[S(x,\widetilde{H})]$ to the following one.

Problem $[S(g,0)]$. For fixed square-integrable nonnegative $\widetilde{F}_T$-measurable random variable G and fixed number $0 < g < \mathbf{E}_{\widetilde{P}} G$ to find

$$\min_{\xi\in\Xi_S(g,0)} \mathbf{E}\left(G - g - \int_0^T \xi_s d\widetilde{S}_s\right)^2,$$

and such $\widetilde{\xi} \in \Xi_S(g,0)$ for which

$$\min_{\xi\in\Xi_S(g,0)} \mathbf{E}\left(G - g - \int_0^T \xi_s d\widetilde{S}_s\right)^2 = \mathbf{E}\left(G - g - \int_0^T \widetilde{\xi}_s d\widetilde{S}_s\right)^2.$$

Remark

Consider the term $\mathbf{E}(x,\widetilde{S}) := \mathbf{E}\left(H_1 \mathbb{1}_{H_1 \geq \widetilde{H}} - x - \int_0^T \xi_s d\widetilde{S}_s\right)^2$ from the right-hand side of (12). We can present it as

$$\mathbf{E}(x,\widetilde{S}) = \mathbf{E}\left((H_1 - \widetilde{H})\mathbb{1}_{H_1 \geq \widetilde{H}} - \left(x + \int_0^T \xi_s d\widetilde{S}_s - \widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}}\right)\right)^2.$$

$\widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}}$ admits the representation $\widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}} = \widetilde{x}_1 + \int_0^T \widetilde{\gamma}_s d\widetilde{S}_s$, where $\widetilde{x}_1 = \mathbf{E}_{\widetilde{P}}\widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}} \leq \mathbf{E}_{\widetilde{P}}\widetilde{H} \leq x$ and $x + \int_0^T \xi_s d\widetilde{S}_s - \widetilde{H}\mathbb{1}_{H_1 \geq \widetilde{H}} \geq 0$. Therefore, the minimization of the term $\mathbf{E}(x,\widetilde{S})$ is in the framework of the Problem $[S(g,0)]$ with $g = x - \widetilde{x}_1 \geq 0$ and $G = (H_1 - \widetilde{H})\mathbb{1}_{H_1 \geq \widetilde{H}} \geq 0$. So, in the general case, when the inequality $H_1 \geq \widetilde{H}$ does not hold a.s., we can minimize right-hand side of (12) in the framework of the Problem $[S(g,0)]$ and the minimal value of $\mathbf{E}\left(\mathbf{E}(H|\widetilde{\mathcal{F}}_T) - x - \int_0^T \xi_s d\widetilde{M}_s\right)^2$ will be between minimal values of the left- and right-hand sides of (12).

To solve Problem $[S(g,0)]$, denote $D_T = \frac{d\widetilde{P}_T}{dP_T}$ the restriction of $\frac{d\widetilde{P}}{dP}$ on $[0, T]$ and note that

$$\mathbf{E}_{\widetilde{P}}\left(G + v(x)D_T\right)^+ \leq \mathbf{E}D_T G + |v(x)|\mathbf{E}D_T^2 < +\infty.$$

Now, for any $0 < x < \mathbf{E}_{\widetilde{P}}G$ consider equation

$$\mathbf{E}_{\widetilde{P}}\left(G + v(x)D_T\right)^+ = x. \tag{13}$$

Similarly to Lemma 2.1, we can prove the following result.

Lemma

Function $v = v(x),\ x \in (0, \mathbf{E}_{\widetilde{P}}G)$ is uniquely determined, continuous and strongly increasing.

The solution of the minimization problem $[S(g,0)]$

It follows from the completeness of the market that $(G+v(x)D_T)^+$ admits the representation

$$(G+v(x)D_T)^+ = \mathbf{E}_{\widetilde{P}}(G+v(x)D_T)^+ + \int_0^T \widetilde{\xi}_s d\widetilde{S}_s = x + \int_0^T \widetilde{\xi}_s d\widetilde{S}_s \quad (14)$$

with some $\widetilde{\xi} \in \Xi_S$.

Theorem

Let $g \in (0, \mathbf{E}_{\widetilde{P}} G)$ be fixed. Consider $v(g)$ that is the unique solution of equation (13) with $x = g$ and the representation (14) of the random variable $(G+v(g)D_T)^+$:

$$(G+v(g)D_T)^+ = g + \int_0^T \widetilde{\xi}_s d\widetilde{S}_s. \quad (15)$$

Then $\widetilde{\xi}$ is the solution of the minimization problem $[S(g,0)]$.

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Example

Consider the case when we have unobservable semimartingale $S_t = \{W_t + \int_0^t a(s)ds, t \geq 0\}$ and observable semimartingale $\widetilde{S}_t = \{\widetilde{W}_t + at, t \geq 0\}$. Let $\{a(s), s \geq 0\}$ be a deterministic function from $L_1[0, T]$, a be some positive constant and $W_t = \rho\widetilde{W}_t + \sqrt{1-\rho^2}\widehat{W}_t$, where $(\widetilde{W}_t, \widehat{W}_t)$ is two-dimensional Wiener process under measure P. Let filtrations $\mathbb{F}, \widetilde{\mathbb{F}}$ and function H be the same as in Example 2.3. Let $G = H(S_T)$ be a square-integrable random variable. It follows from Girsanov's theorem that $\{\widetilde{S_T}, \widehat{W_t}\}$ is two-dimensional Wiener process under measure $\widetilde{P}$ with Radon-Nikodym derivative

$$D_T = \frac{d\widetilde{P}}{dP}\bigg|_{[0,T]} = \exp\left\{-a\widetilde{S}_T + \frac{a^2 T}{2}\right\}.$$

In order to give the explicit solution of minimization problem $[S(g, 0)]$ we repeat the same steps as in Example 2.3.

- Calculate $\widetilde{G} = \mathbf{E}_{\widetilde{P}}(G|\widetilde{\mathcal{F}}_T)$. Denote $A_T = \frac{1}{\rho}\int_0^T a(s)ds - aT$. We have that

$$\widetilde{G} = \int_{\mathbb{R}} H(u) \frac{\exp\left(-\frac{(u-\rho(\widetilde{S}_T+A_T))^2}{2T(1-\rho^2)}\right)}{\sqrt{2\pi T(1-\rho^2)}} du =: f\left(\widetilde{S}_T + A_T\right).$$

- Define an auxiliary function $h : \mathbb{R} \to \mathbb{R}_+$: $h(x) = f(x + A_T)\exp\left(ax - \frac{a^2 T}{2}\right), x \in \mathbb{R}$. Equation (13) is rewritten in the following form

$$g = \int_{h^{(-1)}(-v(g))}^{+\infty} f(x + A_T)\frac{\exp\left(-\frac{x^2}{2T}\right)}{\sqrt{2\pi T}}dx + v(g)e^{a^2 T}\Phi\left(-\frac{h^{(-1)}(-v(g))}{\sqrt{T}} - a\right).$$

- From the integral representation of $\left(\widetilde{G} + v(g)D_T\right)^+$ get the solution $\widetilde{\xi}_t$ of minimization problem $[S(g,0)]$.

$$\widetilde{\xi}_t = \rho \int_{h^{(-1)}(-v(g))-\widetilde{S}_t}^{+\infty} \int_{\mathbb{R}} H\left(\rho x + y\sqrt{T(1-\rho^2)} + \rho\widetilde{S}_t + \rho A_T\right)$$
$$\times \frac{y\exp(-y^2/2)}{\sqrt{2\pi T(1-\rho^2)}} \frac{\exp\left(-\frac{x^2}{2(T-t)}\right)}{\sqrt{2\pi(T-t)}} dydx$$
$$- v(g)a\exp\left(-a\widetilde{S}_t + \frac{a^2T}{2}\right)\exp\left(\frac{a^2(T-t)}{2}\right)$$
$$\times \Phi\left(-\frac{h^{(-1)}(-v(g)) - \widetilde{S}_t}{\sqrt{T-t}} - a\right).$$

Example

Consider the same problem as in Example 3.3 with specific function $H(y) = (y-K)^+, y \in \mathbb{R}$. We obtain the solution $\widetilde{\xi}_t$ of minimization problem $[(S(g,0)]$:

$$\widetilde{\xi}_t = \rho \int_{h^{-1}(-v(g))-\widetilde{S}_t}^{+\infty} \Phi\left(\frac{\rho x + \rho \widetilde{S}_t + \rho A_T - K}{\sqrt{T(1-\rho^2)}}\right) \frac{\exp\left(-\frac{x^2}{2(T-t)}\right)}{\sqrt{2\pi(T-t)}} dx$$
$$- v(g) a \exp\left(-a\widetilde{S}_t - \frac{a^2 t}{2} + a^2 T\right) \Phi\left(-\frac{h^{(-1)}(-v(g)) - \widetilde{S}_t}{\sqrt{T-t}} - a\right). \tag{16}$$

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Consider a numerical example of minimization problem $[S(g,0)]$ in the case of $g = 0.05$, $a = 0.5$ and $a(s) \equiv 1, s \geq 0$. Let $T \in \{5, 10\}$, $K \in \{1, 2, 3\}$ and $\rho \in \{0.2, 0.5, 0.75\}$.

Solutions of equation (13) with $g = 0.05$

T=5	K=1	K=2	K=3
ρ=0.2	18.2229	12.6102	7.87255
ρ=0.5	23.1506	16.683	10.8804
ρ=0.75	27.9895	20.9392	14.3655

T=10	K=1	K=2	K=3
ρ=0.2	88.0026	73.6978	60.2843
ρ=0.5	102.533	86.995	72.2639
ρ=0.75	116.037	99.5158	83.7536

Minimum values in problem $[S(g,0)]$

T=5	K=1	K=2	K=3
ρ=0.2	22.7261	14.8596	8.99844
ρ=0.5	22.8506	14.8849	8.91728
ρ=0.75	23.0573	15.0116	8.93836

T=10	K=1	K=2	K=3
ρ=0.2	153.068	123.704	97.5674
ρ=0.5	160.729	130.323	103.121
ρ=0.75	168.314	137.029	108.938

Values of $\mathbf{E}_{\widetilde{P}}\widetilde{G}$

T=5	K=1	K=2	K=3
ρ=0.2	3.55638	2.64805	1.83557
ρ=0.5	2.86796	2.02712	1.31678
ρ=0.75	2.3296	1.56514	0.955956

T=10	K=1	K=2	K=3
ρ=0.2	8.00578	7.01487	6.0352
ρ=0.5	6.52311	5.55252	4.6102
ρ=0.75	5.31368	4.38103	3.49984

We simulate the trajectories of the Wiener process $\widetilde{W}_t$ on time interval $[0, 10]$, and so we obtain the trajectories of $\widetilde{S}_t$. We take one of the obtained sample paths of $\widetilde{S}_t$ and presented it at the bottom of Figure 2. By formula (16), for this trajectory we construct the sample paths of $\widetilde{\xi}_t$, with $K = 1$ and $\rho = 0.2, 0.5, 0.75$. At the top of Figure 2 we present these sample paths, grouped by T.

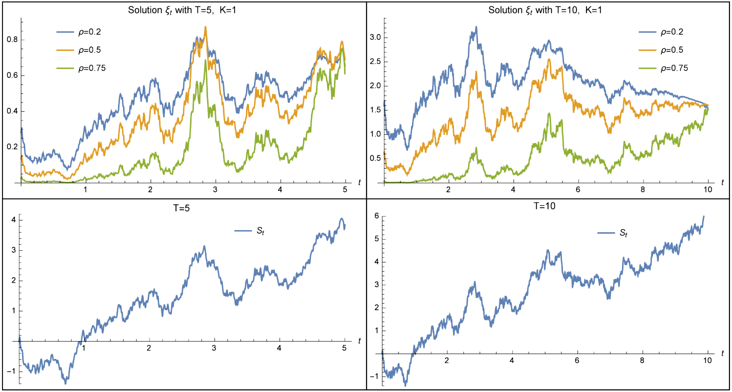

Figure: Sample paths of solutions of problem $[S(g, 0)]$ with $g = 0.05$

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