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Derivative Pricing using a Reduced Basis Method for Parameter Functions

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Overview

Objective: fast derivative pricing

- Solving a pricing model for different parameters and payoff functions
- Application in model calibration process

Method: reduced basis method

- Model Reduction Method
- Partial Differential Equations
- Parameter dependent problems

Example: Heston model

- 2D parabolic PDE
- Affine model

Problem formulation

Consider the Itô diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

and the associated operator

$$\mathcal{A}v(x) = \sum_i b_i(x)\frac{\partial v}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 v}{\partial x_i \partial x_j}.$$

Feynman-Kac Formula

If $v(t,x)$ is the solution of

$$\frac{\partial v}{\partial t} = \mathcal{A}v \; t > 0, \text{ with } v(T,x) = f(x)$$

then

$$v(t,x) = E(f(X_T)|X_t = x).$$

Problem formulation - Heston model

- European options
- Strike price K
- Underlying asset and volatility modelled stochastically $\rightarrow$ **2D problem**
- Existence of closed form solution

Heston, *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options (1993).*

Heston - SDE

$$dS_t = rS_t dt + \sqrt{v_t} S_t dz_1(t), \quad dv_t = \kappa[\theta - v_t]dt + \sigma\sqrt{v_t} dz_2(t)$$

- r: return rate of the asset
- σ: vol. of vol.
- θ: long term variance
- κ: mean reversion rate to θ
- z_1, z_2 Wiener processes with correlation ρ

Heston - PDE

- Transformation in time $t \to T - t$

Expected value given as solution of parabolic PDE in $z = (S, \nu)$

$$\frac{d}{dt}u - \sum_{i,j=1}^{2} A_{i,j} u_{z_i, z_j} - \sum_{i=1}^{2} b_i u_{z_i} + cu = 0$$

$$u(0) = \mu_0 \quad (payoff)$$

$$A = \frac{1}{2}\nu \begin{pmatrix} S^2 & \rho\sigma S \\ \rho\sigma S & \sigma^2 \end{pmatrix}, \quad b = \begin{pmatrix} Sr \\ \kappa\theta - \kappa\nu \end{pmatrix}, \quad c = r.$$

- Parameter $\rho, \sigma, \kappa, \theta$
- r is riskless interest rate (≈ 0)
- Parameter function μ_0

Space-Time Variational Formulation

- Weak formulation of PDE in space AND time
- Convenient for reduced basis method
- Solution $u(t,x)$ is a function in space AND time

Hilbert spaces $V \hookrightarrow H$ dense

$$\mathbb{X} := \{u \in L_2(0,T;V) : u' \in L_2(0,T;V')\}$$
$$\mathbb{Y} := L_2(0,T;V) \times H.$$

For $\mathcal{A} \in \mathcal{L}(V,V')$, $g \in L_2(0,T;V')$ find $u \in \mathbb{X} \subset L_2(0,T;V)$

$$u'(t) + \mathcal{A}u(t) = g \text{ in } L_2(0,T;V'), \quad u(0) = u_0. \tag{1}$$

$$\int_I \langle u'(t), v_1(t)\rangle_{V'\times V}\, dt + \int_I a(u(t), v_1(t))dt + (u(0), v_2)_H \tag{2}$$
$$= \int_I \langle g(t), v_1(t)\rangle_{V'\times V}\, dt + (u_0, v_2)_H \quad \forall\, v = (v_1, v_2) \in \mathbb{Y}.$$

Showalter, *Monotone operations in Banach space and nonlinear partial differential equations (1991)*.

Reduced Basis Method

Two introductions (2015)

For $\mu = (\mu_0, \mu_1) \in \mathcal{D}$

find $u(\mu) \in \mathcal{X}$ such that $b(u(\mu), v; \mu) = f(v; \mu)$ for all $v = (v_1, v_2) \in \mathcal{Y}$

$$b(u(\mu),v;\mu) := \int_I \langle \dot{u}(\mu)(t), v_1(t) \rangle_{V' \times V} dt + \int_I a(u(\mu)(t), v_1(t); \mu_1) dt + (u(\mu)(0), v_2)_H,$$

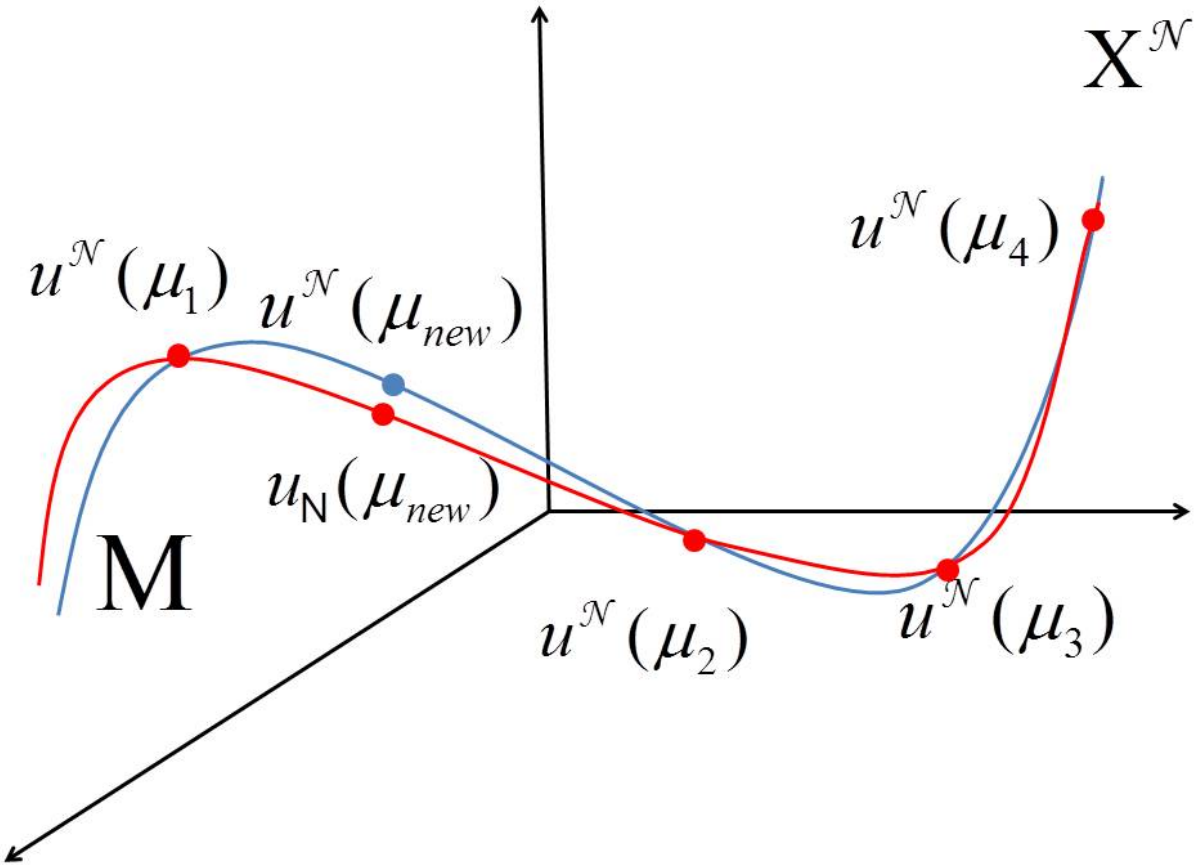
$$f(v;\mu) := (\mu_0, v_2)_H.$$

Standard methods (FEM, FV, FD, ...) $\Rightarrow B_{\mathcal{N}}(\mu)\boldsymbol{u}_{\mathcal{N}}(\mu) = f_{\mathcal{N}}(\mu)$ ($\mathcal{N}$ large)
$\Rightarrow$ computationally expensive for

- multi-query (e.g. calibration/optimisation),
- real time (e.g. pricing)
- parameter variations for every computation.

Approximate $\mathcal{X}_N \subset M = \{u(\mu) | \mu \in \mathcal{D}\} \subset \mathcal{X} \Rightarrow B_N \boldsymbol{u}_N = f_N$ ($N \ll \mathcal{N}$).

Reduced Basis Method



- reduces large linear equation system to small one
- error analysis available
- best approximation results

Reduced Basis Method

- Offline:
  - For sample set $\{\mu_1, \dots, \mu_N\}$ (Greedy procedure) pre-compute (FEM) "snapshots" $u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)$
  - $X_N = \text{span}\{u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)\}$ (Reduced Basis Space) and test space Y_N
- Online: (complexity independent on $\mathcal{N}$)
  - For new $\mu \notin \{\mu_1, \dots, \mu_N\}$ solve

$$u_N(\mu) \in X_N : b(u_N(\mu), v; \mu) = f(v) \quad \forall v \in Y_N$$

Greedy procedure: $M_{\text{train}} \subset \mathcal{D}$. Choose a first μ: $u^{\mathcal{N}}(\mu)$, $X_1 = \{u^{\mathcal{N}}(\mu)\}$.

`WHILE error > tol`

$$\mu^* := \arg\max_{\mu \in M_{\text{train}}} \texttt{error}(u^{\mathcal{N}}(\mu), u_N(\mu)) : \quad u^{\mathcal{N}}(\mu^*),\ X_{N+1} := X_N \cup \{u^{\mathcal{N}}(\mu^*)\}.$$

Space-Time Reduced Basis Method

Urban and Patera, *An Improved Error Bound for Reduced Basis Approximation of Linear Parabolic Problems (2012)*.

- Online: one $N \times N$ linear system (no time stepping!)
- Good error estimator
  - residual based
  - can be efficiently computed
  - to be used in the Greedy procedure offline
  - control of the error online

$$\beta(\mu)\|u - u_N\|_{\mathbb{X}} \leq \sup_{v \in \mathbb{Y}} \frac{b(u - u_N, v; \mu)}{\|v\|_{\mathbb{Y}}} = \sup_{v \in \mathbb{Y}} \frac{f(v) - b(u_N, v; \mu)}{\|v\|_{\mathbb{Y}}}$$

- Greedy works for $\mu \in \mathbb{R}^p$

Reduced basis depends on payoff μ_0!

Space-Time RBM for Parameter Functions

AM and Urban, *A reduced basis method for parabolic PDEs with parameter functions and application to option pricing (to appear).*

$\mu \in H$, Hilbert space

Reduced basis for $\mathcal{D}_0 \times \mathcal{D}_1 \subset H \times \mathbb{R}^p$ parameter domain?

$$b(u(\mu), v; \mu) := \underbrace{\int_I \langle \dot{u}(\mu)(t), v_1(t)\rangle_{V' \times V} \, dt + \int_I a(u(\mu)(t), v_1(t); \mu_1) dt}_{b_1(u, v_1; \mu_1)} + (u(\mu)(0), v_2)_H,$$

$$f(v; \mu) := \underbrace{\int_I \langle g(t), v_1(t)\rangle_{V' \times V} dt}_{g_1(v_1)} + (\mu_0, v_2)_H.$$

Assumption: $\mathcal{D}^{\mathcal{L}} \subset \mathcal{D}$ finite description: ${\mu_0}^{\mathcal{L}} = \sum_{\ell=1}^{\mathcal{L}} d_\ell(\mu_0)\delta_\ell$

Split error estimator:

$$\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathbb{X}} \leq \frac{\|f(\cdot;\mu) - b(u_N(\mu), \cdot; \mu)\|_{\mathbb{Y}'}}{\beta_{\mathrm{LB}}} = \Delta_N(\mu) \leq \Delta_N^0(\mu) + \Delta_N^1(\mu),$$

$$\Delta_N^0(\mu) := \frac{\|u_N(\mu)(0) - \mu_0\|_H}{\beta_{\mathrm{LB}}}, \quad \Delta_N^1(\mu) := \frac{\|g_1(\cdot) - b_1(u_N(\mu), \cdot; \mu)\|_{(L_2(I;V))'}}{\beta_{\mathrm{LB}}}.$$

Space-Time RBM for Parameter Functions - Offline Phase

- $\mathbb{X}^{\mathcal{N}} = E_{\text{time}} \otimes V_{\text{space}}, \quad \mathbb{Y}^{\mathcal{N}} = (F_{\text{time}} \otimes V\text{space}) \times V_{\text{space}}$

$$u(\mu) = u^0(\mu_0) + w(\mu)$$

1st part ($\Delta^0_{N_0} < \text{tol}_0$):

- Only parameter function μ_0
- Find good sample $\mu_0^1, \dots, \mu_0^{N_0}$ (POD, Greedy)
- RB space $H_{N_0} = \{h^1, \dots, h^{N_0}\} \subset H \to \{\sigma^1\} \otimes H_{N_0} \subset \mathbb{X}^{\mathcal{N}}$

  Compute h_i for $i = 1, \dots, N_0$:

$$(h^i, \phi_j)_H = (\mu_0^i, \phi_j)_H \forall\ \phi_j \in V_{\text{space}}, \quad u_i^0 := \sigma^1 \otimes h^i$$

2nd part ($\Delta^1_{N_1} < \text{tol}_1$):

- Find good samples $\{\mu^1, \dots, \mu^{N_1}\} \subset \mathcal{D}_0 \times \mathcal{D}_1$ (Evolution Greedy)
- RB space $\mathbb{W}_{N_1} := \{w_1, \dots, w_{N_1}\}$

  Compute w_j for $\mu^j = (\mu_0^j, \mu_1^j)$:

$$b_1(w_j, z; \mu) = g_1(z) - b_1(u^0_{N_0}(\mu_0^j), z; \mu) \quad \forall\ z \in F_{\text{time}} \otimes V_{\text{space}},$$

Space-Time RBM for Parameter Functions - Online Phase

For new $(\mu_0, \mu_1) \in \mathcal{D}$.

1. Find $\widetilde{h}_{N_0}(\mu_0) \in H_{N_0} : (\widetilde{h}_{N_0}(\mu_0), h^j)_H = (\mu_0, h^j)_H \quad \forall h^j \in H_{N_0}$
2. Find $w^1_{N_1}(\mu) \in \mathbb{W}_{N_1} : b_1(w^1_{N_1}(\mu), v; \mu_1) = g_1(v) - b_1(\sigma^1 \otimes \widetilde{h}_{N_0}(\mu_0), v; \mu_1) \quad \forall v \in \mathbb{Y}^1_{N_1}$
3. $u_N(\mu) = \sigma^1 \otimes \widetilde{h}_{N_0}(\mu_0) + w^1_{N_1}(\mu), \quad N = N_0 + N_1$

- Construct reduced test space $\mathbb{Y}^1_N(\mu)$ for (inf-sup) stability (supremizers).

Numerical Experiments

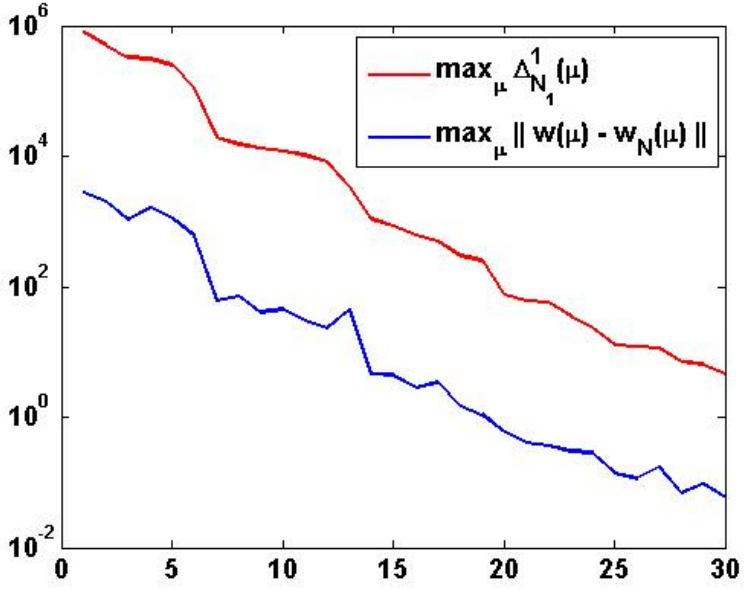
- Heston model: $V = H_0^1(\Omega, \omega)$, $H = L_2(\Omega, \omega)$ (weighted Sobolev Spaces)
- $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^2$
- $\Omega_1 := (0, 140) \cup [140, \infty)$, $\Omega_2 := (0, 1) \cup [1, \infty)$
- $\mu_1 = \rho$, $\kappa = 0.3313, \sigma = 0.6083, \theta = 0.1914$ and $r = 0$
- Parameter domain $[-0.5, 0.5] =: \mathcal{D}_1$
- $T = 1$, $\beta_{\text{LB}} = 0.003$
- $\mathcal{D}_0 \approx \mathcal{D}_0^{\mathcal{L}} = \{\delta_1, \ldots, \delta_{\mathcal{L}}\}$ pw. linear on $\mathcal{T}_{\Omega_1} = \{0, 70, 80, 90, 100, 110, 200\}$ ($\mathcal{L} = 7$), const. on Ω_2 (exact for $K = 70, 80, \ldots$)

All RB calculations were implemented in RBmatlab, see `http://www.morepas.org`.

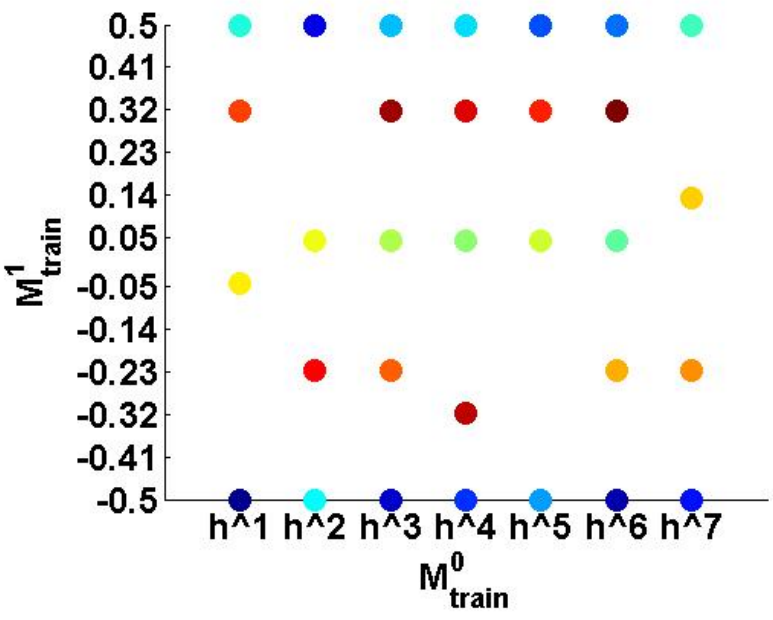
1st part: Initial Value: POD $\rightarrow \{h_1, \ldots, h_7\}$

2nd part: Evolution Greedy

$$M_{\text{train}} := \{h_1, \dots, h_7\} \times \{x_k = -0.5 + k\,\Delta s : k = 0, \dots, 11, \Delta s = \frac{1}{11}\}$$
$$\subset \text{span}\{\delta_1, \dots, \delta_7\} \times [-0.5, 0.5]$$



Evol. Greedy: Maximum error over iterations - error estimate $\Delta^1_{N_1}$ and true error.



Evol. Greedy: Snapshot choice.

Model Calibration

Market price observations $V_1, \dots, V_M$ for pairs $(T_i, \mu_0^i), i = 1, \dots, M$

$$\min_{\mu_1} \sum_{i=1}^{M} (V_i - V(S, \nu, 0; T_i, (\mu_0^i, \mu_1)))^2$$

given $V(S, \nu, 0; T_i, (\mu_0^i, \mu_1)) = u((S, \nu), T_i; (\mu_0^i, \mu_1))$.

fast model calibration

PDE-constraint optimisation using RBM

Grepl Nguyen Veroy Patera Liu, *Certified Rapid Solution of Partial Differential Equations for Real-Time Parameter Estimation and Optimization (2007, in 'Real-Time PDE-Constraint Optimization')*.

Dihlmann Haasdonk, *Certified PDE-constrained parameter optimization using reduced basis surrogate models for evolution problems (2014)*.

PDE-constraint optimisation with Space-Time RBM

Parameter functions:

$$\begin{cases} \min\limits_{\mu_1} J(u^{\mathcal{N}}(\mu_0, \mu_1)) \\ \text{s.t. } u^{\mathcal{N}}(\mu_0, \mu_1) \text{ solves PDE in space-time variational formulation} \end{cases}$$

$$\begin{cases} \min\limits_{\mu_1} J(u_N(\mu_0, \mu_1)) \\ \text{s.t. } u_N(\mu_0, \mu_1) = u^0(\mu_1) + w^1(\mu_0, \mu_1) \text{ solution of two reduced problems} \end{cases}$$

- Parameter dependence $\mu_0(\mu_1)$ possible
- $\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathbb{X}} \leq$ RB-tol and $|J(u^{\mathcal{N}}) - J(u_N)| \leq$ tol
- Gradient via sensitivity PDE

RB only as good as the high dimensional discretisation $\mathbb{X}^{\mathcal{N}}$ and $\mathcal{D}_0^{\mathcal{L}}$!

A first Experiment

- Matlab: `lsqnonlin`: trust region, gradient approximated with finite differences
- NO sensitivity PDE
- Data: Call option on DAX, different time to maturity ($\leq$ 1 year), different strike prices
- Calibration over ρ only
- Initial points in $[-0.5, 0.5]$

	time	number of function calls	residual
FEM-Sol./ Call	240.8082 sec.	6.3	78.1232
RB-Sol./ Call	58.9435 sec.	5	106.5772

- Speedup factor 4

Conclusion

- (parabolic) PDE
- No closed form solution needed
- Initial value is parameter
- Space-time RBM approach: two reduced basis'
- Speedups in e.g. calibration

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Thank you for your attention!