Utility maximization with random horizon: a BSDE approach

To appear in International Journal of Theoretical and Applied Finance

Thibaut Mastrolia

Joint work with Monique Jeanblanc, Dylan Possamaï and Anthony Réveillac.

CEREMADE

Université Paris Dauphine

7th General AMaMeF and Swissquote Conference, September 2015, Lausanne

Introduction: pricing and hedging problems in finance

Financial market model:

- $W := (W_t)_{t\in[0,T]}$ a Brownian motion defined on the probability space $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- Risk-free asset $S^0 := (S^0_t)_{t\in[0,T]}$,

$$dS^0_t = S^0_t r\, dt.$$

  In the following, $r = 0$.
- Asset $S := (S_t)_{t\in[0,T]}$,

$$dS_t = S_t\Big(\mu_t\, dt + \sigma_t dW_t\Big),$$

  where μ, σ are predictable and bounded. Let $\theta := \mu/\sigma$.

Introduction: pricing and hedging problems in finance

Financial market model:

- $W := (W_t)_{t\in[0,T]}$ a Brownian motion defined on the probability space $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- Risk-free asset $S^0 := (S^0_t)_{t\in[0,T]}$,

$$dS^0_t = S^0_t r\, dt.$$

  In the following, $r = 0$.
- Asset $S := (S_t)_{t\in[0,T]}$,

$$dS_t = S_t\Big(\mu_t\, dt + \sigma_t dW_t\Big),$$

  where μ, σ are predictable and bounded. Let $\theta := \mu/\sigma$.
- Investing strategy: $(x, (\Pi_t)_t)$ such that the associated wealth process denoted $(X^{x,\Pi}_t)_t$ and defined for all $t \in [0, T]$ by:

$$X^{x,\Pi}_t := x + \int_0^t \Pi_u \frac{dS_u}{S_u} = x + \int_0^t \Pi_u \sigma_u (dW_u + \theta_u du).$$

Motivation: pricing and hedging problems in finance

Let ξ be an $\mathcal{F}_T$ measurable random variable (the liability of the investor).

$$(\mathcal{P}) \quad V(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X_T^{x,\Pi} - \xi)],$$

Motivation: pricing and hedging problems in finance

Let ξ be an $\mathcal{F}_T$ measurable random variable (the liability of the investor).

$$(\mathcal{P}) \quad V(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X_T^{x,\Pi} - \xi)],$$

$$" \Longleftrightarrow "$$

El Karoui and Rouge (2000), Hu, Imkeller, Müller (2005)

$$Y_t = \xi + \int_t^T h(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad Y_T = \xi$$

with an explicit formula for the generator h, where (Y, Z) is a pair of adapted processes "*regular enough*".

Motivation: pricing and hedging problems in finance

Let ξ be an $\mathcal{F}_T$ measurable random variable (the liability of the investor).

$$(\mathcal{P}) \quad V(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X_T^{x,\Pi} - \xi)],$$

$$" \Longleftrightarrow "$$

El Karoui and Rouge (2000), Hu, Imkeller, Müller (2005)

$$Y_t = \xi + \int_t^T h(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad Y_T = \xi$$

with an explicit formula for the generator h, where (Y, Z) is a pair of adapted processes "*regular enough*".

- Let $U(x) := -e^{-\alpha x}$, $\alpha > 0$,
- the value is given by $V(x) = -e^{-\alpha(x - Y_0)}$,
- optimal strategies are characterized by Z_t.

Let τ be a default time. The problem becomes

$$(\mathcal{P}^\tau) \quad V^\tau(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X^{x,\Pi}_{T\wedge\tau} - \xi)].$$

Utility maximization problem with random horizon

Let τ be a default time. The problem becomes

$$(\mathcal{P}^\tau) \quad V^\tau(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X^{x,\Pi}_{T\wedge\tau} - \xi)].$$

- The case "τ is an $\mathbb{F}$ stopping time" was studied by Karatzas and Wang (2000) (among others), the general case was studied in *e.g.* Bouchard and Pham (2004) and Blanchet-Scalliet et al. (2008)

Utility maximization problem with random horizon

Let τ be a default time. The problem becomes

$$(\mathcal{P}^\tau) \quad V^\tau(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X^{x,\Pi}_{T \wedge \tau} - \xi)].$$

- The case "τ is an $\mathbb{F}$ stopping time" was studied by Karatzas and Wang (2000) (among others), the general case was studied in *e.g.* Bouchard and Pham (2004) and Blanchet-Scalliet et al. (2008)

$\hookrightarrow$ Using the convex duality theory (Bouchard, Pham, Touzi, ... among others) to prove the existence of an optimal strategy.

Let τ be a default time. The problem becomes

$$(\mathcal{P}^\tau) \quad V^\tau(x) := \sup_{\Pi\in\mathcal{A}} \mathbb{E}[U(X^{x,\Pi}_{T\wedge\tau} - \xi)].$$

- The case "τ is an $\mathbb{F}$ stopping time" was studied by Karatzas and Wang (2000) (among others), the general case was studied in *e.g.* Bouchard and Pham (2004) and Blanchet-Scalliet et al. (2008)

$\hookrightarrow$ Using the convex duality theory (Bouchard, Pham, Touzi, ... among others) to prove the existence of an optimal strategy.

$\hookrightarrow$ This approach does not provide a characterization of either the optimal strategy or of the value function.

Utility maximization problem with random horizon

Let τ be a default time. The problem becomes

$$(\mathcal{P}^\tau) \quad V^\tau(x) := \sup_{\Pi \in \mathcal{A}} \mathbb{E}[U(X^{x,\Pi}_{T\wedge\tau} - \xi)].$$

- The case "τ is an $\mathbb{F}$ stopping time" was studied by Karatzas and Wang (2000) (among others), the general case was studied in *e.g.* Bouchard and Pham (2004) and Blanchet-Scalliet et al. (2008)

$\hookrightarrow$ Using the convex duality theory (Bouchard, Pham, Touzi, ... among others) to prove the existence of an optimal strategy.

$\hookrightarrow$ This approach does not provide a characterization of either the optimal strategy or of the value function.

$\hookrightarrow$ Use the BSDE approach in this talk, as in Kharroubi, Lim and Ngoupeyou (13), by assuming that τ is not an $\mathbb{F}$ stopping time.

In this talk: "no constraints on the set of admissible strategies $\mathcal{A}$" to simplify. We assume that

$$\mathcal{A} := \left\{ (\pi_t)_{t\in[0,T]} \in \mathcal{P}(\mathbb{G}),\ \pi_t \in \mathbb{R},\ dt \otimes \mathbb{P} - a.e.,\ \pi \mathbf{1}_{(\tau\wedge T, T]} = 0 \right\}.$$

See the paper for the general case.

Let $H_t := \mathbf{1}_{\tau \leqslant t},\ t \geqslant 0.$ (the right-continuous default indicator process).

Let $H_t := \mathbf{1}_{\tau \leqslant t}, \ t \geqslant 0.$ (the right-continuous default indicator process).

Let $\mathbb{G}$ be "the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time."

Enlargment of filtration and Immersion Hypothesis

Let $H_t := \mathbf{1}_{\tau \leqslant t}, \; t \geqslant 0.$ (the right-continuous default indicator process).

Let $\mathbb{G}$ be "the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time."

We make a density assumption on τ denoted by (H1) which implies "Immersion Hypothesis": any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

Enlargment of filtration and Immersion Hypothesis

Let $H_t := \mathbf{1}_{\tau \leqslant t},\ t \geqslant 0.$ (the right-continuous default indicator process).

Let $\mathbb{G}$ be "the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time."

We make a density assumption on τ denoted by (H1) which implies "Immersion Hypothesis": any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

$\Longrightarrow$ There exists a non-negative $\mathbb{G}$-predictable process $\lambda^{\mathbb{G}}$ (called the $\mathbb{G}$ intensity) such that

$$M_t := H_t - \int_0^t \lambda_s^{\mathbb{G}} ds,$$

is a $\mathbb{G}$-martingale, with $\lambda_t^{\mathbb{G}} = \lambda_t \mathbf{1}_{t \leqslant \tau}$, where λ is an $\mathbb{F}$-predictable process.

Enlargment of filtration: Assumptions on λ

In Kharoubbi Lim and Ngoupeyou (13), λ is bounded. Here we make two assumptions on λ

(H2) $\mathbb{E}\left[\left(\int_0^T \lambda_s ds\right)^2\right] < +\infty.$

(H2') $\mathbb{E}\left[\left(\int_0^t \lambda_s ds\right)^2\right] < +\infty, \ \forall t < T$

and $\mathbb{E}\left[\int_0^T \lambda_s ds\right] = +\infty.$

Enlargment of filtration: Assumptions on λ

In Kharoubbi Lim and Ngoupeyou (13), λ is bounded. Here we make two assumptions on λ

(H2) $\mathbb{E}\left[\left(\int_0^T \lambda_s ds\right)^2\right] < +\infty.$

$\Downarrow$

$\text{Supp}(\tau) \supsetneq [0, T]$

(H2') $\mathbb{E}\left[\left(\int_0^t \lambda_s ds\right)^2\right] < +\infty, \ \forall t < T$ and $\mathbb{E}\left[\int_0^T \lambda_s ds\right] = +\infty.$

$\Downarrow$

$\text{Supp}(\tau) = [0, T]$

Enlargment of filtration: Assumptions on λ

In Kharoubbi Lim and Ngoupeyou (13), λ is bounded. Here we make two assumptions on λ

(H2) $\mathbb{E}\left[\left(\int_0^T \lambda_s ds\right)^2\right] < +\infty.$	**(H2')** $\mathbb{E}\left[\left(\int_0^t \lambda_s ds\right)^2\right] < +\infty, \ \forall t < T$ and $\mathbb{E}\left[\int_0^T \lambda_s ds\right] = +\infty.$
$\Downarrow$	$\Downarrow$
$\mathrm{Supp}(\tau) \supsetneq [0, T]$	$\mathrm{Supp}(\tau) = [0, T]$

Using $\mathbb{P}[\tau > t | \mathcal{F}_t] = e^{-\int_0^t \lambda_s ds}$, see El Karoui, Jeanblanc, Jiao (2010).

Enlargment of filtration: Assumptions on λ

- (**H2**): with positive probability, Problem $(\mathcal{P}^\tau)$ is the same as the classical maximization problem with terminal time T.
  - $\hookrightarrow$ Generalizes the case λ bounded in Kharroubi, Lim and Ngoupeyou (2013).

Enlargment of filtration: Assumptions on λ

- (**H2**): with positive probability, Problem $(\mathcal{P}^\tau)$ is the same as the classical maximization problem with terminal time T.
  - $\hookrightarrow$ Generalizes the case λ bounded in Kharroubi, Lim and Ngoupeyou (2013).

- (**H2'**): with probability 1 the final horizon is less than T.

Enlargment of filtration: Assumptions on λ

- (**H2**): with positive probability, Problem $(\mathcal{P}^\tau)$ is the same as the classical maximization problem with terminal time T.
  - $\hookrightarrow$ Generalizes the case λ bounded in Kharroubi, Lim and Ngoupeyou (2013).

- (**H2'**): with probability 1 the final horizon is less than T.
  - $\hookrightarrow$ Example 1: **Life-insurance type markets.** Products with very long maturities (up to 95 years for universal life policies and to 120 years for whole life maturity).

    Example 2: Markets whose maximal lifetime is finite and known at the beginning of the investment period (like for instance **carbon emission markets in the United States.**)

Problem $(\mathcal{P}^\tau)$ and BSDE

Theorem (Jeanblanc, M., Possamaï, Réveillac (2015))

Assume that $(H1)$ *and* $(H2)$ *or* $(H2')$ *hold and* ξ *is bounded and* $\mathbb{G}_\tau$ *measurable. Assume that the BSDE*

$$Y_t = \xi - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \cdot dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s - \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, U_s) ds, \ t \in [0, T], \tag{1}$$

with

$$f(s, \omega, z, u) := z \cdot \theta_s + \frac{\|\theta_s\|^2}{2\alpha} - \lambda_s \frac{e^{\alpha u} - 1}{\alpha},$$

admits a unique solution such that Y *and* U *are uniformly bounded and such that* $\mathbb{E}\left[\int_0^T Z_s^2 ds\right] < +\infty$. *Then,*

$$V(x) = -\exp(-\alpha(x - Y_0)),$$

and an optimal strategy $p^* \in \mathcal{A}$ *for Problem* $(\mathcal{P}^\tau)$ *is given by*

$$p_t^* = Z_t + \frac{\theta_t}{\alpha}, \ t \in [0, T], \ \mathbb{P} - a.s.$$

Uniqueness of the solution of BSDE (1)

Lemma

Assume that $(H1)$ *and* $(H2)$ *or* $(H2')$ *hold. Then, there exists at most a solution* $(Y, Z, U) \in \mathbb{S}^2_{\mathbb{G}} \times \mathbb{H}^2_{\mathbb{G}} \times \mathbb{L}^2_{\mathbb{G}}$ *to BSDE* (1).

Decomposition Lemma

For f. According to a classical decomposition result (see *e.g.* Jeulin (1980))

$$f(t,.)\mathbf{1}_{t<\tau} = f^b(t,.)\mathbf{1}_{t<\tau},$$

where $f^b : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathbb{F}$-progressively measurable.

Decomposition Lemma

For f. According to a classical decomposition result (see *e.g.* Jeulin (1980))

$$f(t,.)\mathbf{1}_{t<\tau} = f^b(t,.)\mathbf{1}_{t<\tau},$$

where $f^b : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R} \longrightarrow \mathbb{R}$ is $\mathbb{F}$-progressively measurable.

For ξ. We have

Lemma (Jeanblanc, M., Possamaï, Réveillac (2015))

Let ξ be a bounded $\mathbb{G}_{T\wedge\tau}$-measurable random variable. Then, there exist a bounded $\mathcal{F}_T$-measurable random variable ξ^b and a bounded $\mathbb{F}$-predictable process ξ^a such that

$$\xi = \xi^b\mathbf{1}_{T<\tau} + \xi^a_\tau\mathbf{1}_{\tau\leqslant T}.$$

The proof is mainly based on a result of Song (2014).

Decomposition Lemma

For f. According to a classical decomposition result (see *e.g.* Jeulin (1980))

$$f(t,.)\mathbf{1}_{t<\tau} = f^b(t,.)\mathbf{1}_{t<\tau},$$

where $f^b : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathbb{F}$-progressively measurable.

For ξ. We have

Lemma (Jeanblanc, M., Possamaï, Réveillac (2015))

Let ξ be a bounded $\mathbb{G}_{T\wedge\tau}$-measurable random variable. Then, there exist a bounded $\mathcal{F}_T$-measurable random variable ξ^b and a bounded $\mathbb{F}$-predictable process ξ^a such that

$$\xi = \xi^b \mathbf{1}_{T<\tau} + \xi^a_\tau \mathbf{1}_{\tau\leqslant T}.$$

The proof is mainly based on a result of Song (2014).

- **In the following: we focus on (H2′).** Same results holds under $(H2)$ following the proofs in Kharroubi Lim and Ngoupeyou.

BSDE with random horizon under (H2')

Proposition (Jeanblanc, M., Possamaï, Réveillac (2015))

Assume (H1)-(H2'). Let A be a real-valued, $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|A|^2] < +\infty$. Assume that the BSDE

$$Y_t^b = A - \int_t^T f^b(s, Y_s^b, Z_s^b, \xi_s^a - Y_s^b)ds - \int_t^T Z_s^b \cdot dW_s, \quad t \in [0,T], \quad (2)$$

admits a solution (Y^b, Z^b) in $\mathbb{S}_{\mathbb{F}}^2 \times \mathbb{H}_{\mathbb{F}}^2$. Then (Y, Z, U) given by

$$\begin{aligned} Y_t &= Y_t^b \mathbf{1}_{t<\tau} + \xi_\tau^a \mathbf{1}_{t\geqslant\tau}, \\ Z_t &= Z_t^b \mathbf{1}_{t\leqslant\tau}, \\ U_t &= (\xi_t^a - Y_t^b)\mathbf{1}_{t\leqslant\tau}, \end{aligned}$$

is a solution of BSDE (1) and (Y, Z, U) belongs to $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{S}_{\mathbb{G}}^2$.

Definition of a solution of a Bownian BSDE with exploding coefficient

ξ an $\mathcal{F}_T$-measurable random variable, $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ an $\mathbb{F}$-progressively measurable mapping.

$$Y_t^b = \xi - \int_t^T f(s, Y_s^b, Z_s^b)ds - \int_t^T Z_s^b \cdot dW_s, \; t \in [0,T],$$

A pair of $\mathbb{F}$-adapted processes (Y^b, Z^b) where Z^b is predictable is a solution of the Brownian BSDE if:

- The previous relation is satisfied
- $$\mathbb{E}\left[\int_0^T |f(t, Y_t, Z_t)|dt + \left(\int_0^T \|Z_t\|^2 dt\right)^{1/2}\right] < +\infty. \quad (3)$$

BSDE with random horizon under (H2')

- Under $(H2')$, $Y_T = \xi_\tau^a$ which does not depend on A.

BSDE with random horizon under (H2')

- Under $(H2')$, $Y_T = \xi^a_\tau$ which does not depend on A.

$\hookrightarrow$ two different value functions for the problem... It contradicts the well-posedness of the probem (see for instance Bouchard & Pham (2004)).

BSDE with random horizon under (H2')

- Under $(H2')$, $Y_T = \xi^a_\tau$ which does not depend on A.

$\hookrightarrow$ two different value functions for the problem... It contradicts the well-posedness of the probem (see for instance Bouchard & Pham (2004)).

- Consider the following ODE

$$y'_t = \lambda_t(e^{\xi^a_t - y_t} - 1), \; y_T = A.$$

It admits a solution if and only if $A = \xi^a_T$.

BSDE with random horizon under (H2')

- Under $(H2')$, $Y_T = \xi^a_\tau$ which does not depend on A.

$\hookrightarrow$ two different value functions for the problem... It contradicts the well-posedness of the probem (see for instance Bouchard & Pham (2004)).

- Consider the following ODE

$$y'_t = \lambda_t(e^{\xi^a_t - y_t} - 1),\ y_T = A.$$

It admits a solution if and only if $A = \xi^a_T$.

- The definition (3) of a solution of a Bownian BSDE with exploding coefficient suggests that the Brownian BSDE has a solution iff $A = \xi^a_T$.

Existence of a solution under (H2')

Proposition

Under $(H1) - (H2')$, there exists a solution to the Brownian BSDE (2) iff $A = \xi_T^a$.

Idea of the proof:

- consider $(Y^{b,n}, Z^{b,n})$ solution of the Brownian BSDE (2) with $\lambda^n := \lambda \wedge n$.
- Lower and upper bound for $Y^{b,n}$ uniform in n.
- Comparison Theorem implies that $(Y^{b,n})_n$ is non decreasing.
- Study the continuity of the solution when $t \to T$.

Numerical simulations

It is just an empirical study. We do not provide a numerical analysis and we do not study the speed of convergence with respect to the truncation level n (leave this aspect for future researches).

- We take $\lambda_t := \frac{1}{T-t}$. Let $\lambda^n := \lambda \wedge n$. $(\lambda^n)_n$ is associated with a sequence $(\tau^n)_n$ which converges to τ.

Numerical simulations

It is just an empirical study. We do not provide a numerical analysis and we do not study the speed of convergence with respect to the truncation level n (leave this aspect for future researches).

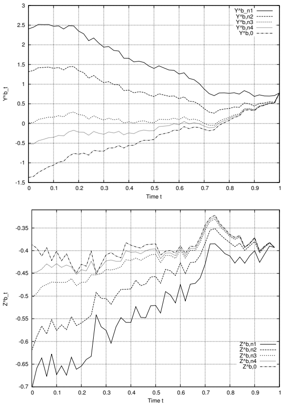
- We take $\lambda_t := \frac{1}{T-t}$. Let $\lambda^n := \lambda \wedge n$. $(\lambda^n)_n$ is associated with a sequence $(\tau^n)_n$ which converges to τ.
- Hypothesis (H1) holds for every τ_n (see Filipovic (2009)).

Numerical simulations

It is just an empirical study. We do not provide a numerical analysis and we do not study the speed of convergence with respect to the truncation level n (leave this aspect for future researches).

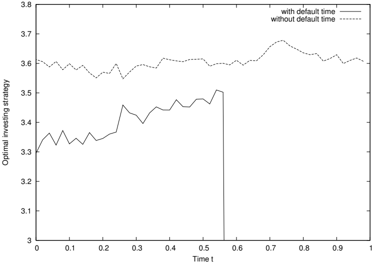
- We take $\lambda_t := \frac{1}{T-t}$. Let $\lambda^n := \lambda \wedge n$. $(\lambda^n)_n$ is associated with a sequence $(\tau^n)_n$ which converges to τ.
- Hypothesis (H1) holds for every τ_n (see Filipovic (2009)).
- We take $\xi_T^a := \left(K - S_0 e^{\sigma W_T + \left(\mu - \frac{\sigma^2}{2}\right) T}\right)^+$.
- We use an implicit scheme (see Bouchard & Touzi (04), Bender & Denk (07)... among others).

The same path of the solutions of Brownian BSDE (2) for a truncation levels n_i



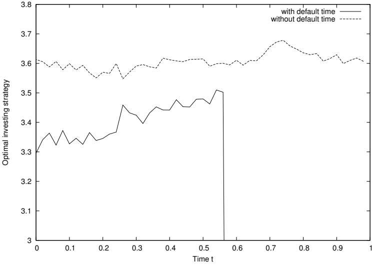
Numerical simulations: optimal strategy

An optimal strategy associated to the exponential utility maximization problem with ω such that $\tau(\omega) = 0.562075$ and without default time.



Numerical simulations: optimal strategy

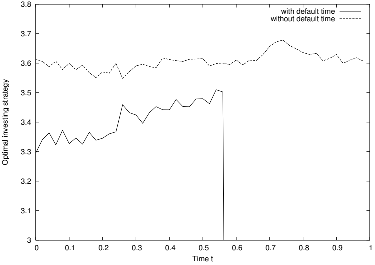
An optimal strategy associated to the exponential utility maximization problem with ω such that $\tau(\omega) = 0.562075$ and without default time.



- The investor tends to be more cautious by investing less in the risky asset.

Numerical simulations: optimal strategy

An optimal strategy associated to the exponential utility maximization problem with ω such that $\tau(\omega) = 0.562075$ and without default time.



- The investor tends to be more cautious by investing less in the risky asset.
- For small times: the trading strategies are merely mirrors of each other.
  When you approach the default: the strategy becomes more and more similar to the one in the non-default case and the former tends to coalesce with the latter.