Weak approximation of martingale representations

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joint work with Rama Cont

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Martingale representations

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space
- W d-dimensional Brownian motion, $(\mathcal{F}_t^W)$ its ($\mathbb{P}$-completed) natural filtration
- $T > 0$, $\forall H \in L^2(\mathcal{F}_T^W)$, $\exists \phi$ $(\mathcal{F}_t^W)$-predictable such that :

$$H = \mathbb{E}[H] + \int_0^T \phi \cdot dW$$

- Problem : ϕ non-explicit
- Motivations : ϕ represents hedging strategy in mathematical finance

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Notations

- $T > 0$, $D([0,T],\mathbb{R}^d)$ càdlàg paths with values in $\mathbb{R}^d$ equipped with the supremum norm :

$$\|\omega\|_\infty = \sup\{|\omega(t)|, t \in [0,T]\}$$

- For $\omega \in D([0,T],\mathbb{R}^d)$, we distinguish :
  - $\omega(t)$ value of ω at t
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Framework

- $dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t),\ X_0 = x_0 \in \mathbb{R}^d$
- $W \xrightarrow{\text{Ito map}} X \xrightarrow{g} H = g(X_T) \in L^2(\mathcal{F}_T^W)$
  - X non-Markovian
  - No differentiability assumptions on b or σ
  - No differentiability assumptions on g
- $H = \mathbb{E}[H] + \int_0^T \phi \cdot dW$
- Approximate ϕ by $(Z_n)_n$ explicit such that :

$$\left\| \int_0^T (Z_n(t) - \phi(t)) \cdot dW(t) \right\|_{\mathbb{L}^2} \leq C \sqrt{\frac{1 + \log n}{n}}$$

if g is Lipschitz continuous with respect to $\| \cdot \|_\infty$

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Non-anticipative functionals

- $F : (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d) \mapsto F(t, \omega) \in \mathbb{R}$ non-anticipative if :

$$F(t, \omega) = F(t, \omega_t)$$

- $F : (\Lambda_T, d_\infty) \longrightarrow \mathbb{R}$ where : $\Lambda_T = \{(t, \omega(t \wedge \cdot)), (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d)\}$ space of stopped path equipped with :

$$d_\infty((t, \omega), (t', \omega')) = \|\omega_t - \omega'_{t'}\|_\infty + |t - t'|$$

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Notions of differentiability

- A non-anticipative functional F is said to be :
  - horizontally differentiable at $(t, \omega) \in \Lambda_T$ if

$$\mathcal{D}F(t, \omega) = \lim_{h \to 0^+} \frac{F(t + h, \omega_t) - F(t, \omega_t)}{h}$$

  exists. $\mathcal{D}F$ is called the horizontal derivative of F.
  - vertically differentiable at $(t, \omega) \in \Lambda_T$ if the map :

$$\begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ e & \mapsto & F(t, \omega_t + e\mathbf{1}_{[t,T]}) \end{array}$$

  is differentiable at 0. Its gradient at 0 is called the vertical derivative of F at (t, ω) :

$$\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, \cdots, d) \in \mathbb{R}^d$$

  with

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Functional Ito formula

Theorem (Dupire(09) & Cont, Fournié(13))

Let S be a continuous semimartingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and F a non-anticipative functional which is :

- *horizontally differentiable*
- *twice vertically differentiable*
- *continuity properties of F, $\mathcal{D}F$, $\nabla_\omega F$ and $\nabla_\omega^2 F$*

we have, for any $t \in [0, T]$,

$$F(t, S_t) - F(0, S_0) = \int_0^t \mathcal{D}F(u, S_u)du + \int_0^t \nabla_\omega F(u, S_u) \cdot dS(u) + \frac{1}{2}\int_0^t \mathrm{tr}\left(\nabla_\omega^2 F(u, S_u) d[S](u)\right) \quad \mathbb{P}\text{−a.s.}$$

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- $b, \sigma : \Lambda_T \to \mathcal{M}_d(\mathbb{R})$ non-anticipative functionals

Assumption

$b, \sigma : (\Lambda_T, d_\infty) \to \mathcal{M}_d(\mathbb{R})$ are Lipschitz continuous

- $H = g(X_T)$ with g satisfying :

Assumption

g has polynomial growth with respect to $\|\cdot\|_\infty$

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Conditional expectation as non-anticipative functional

- $Y(t) = \mathbb{E}[g(X_T)|\mathcal{F}_t^W] = F(t, W_t)$
- If $\exists F$ smooth, then by functional Ito formula,

$$g(X_T) = \mathbb{E}[g(X_T)] + \int_0^T \underbrace{\nabla_\omega F(t, W_t)}_{\phi(t)} \cdot dW(t)$$

- Problem : F is in general not smooth !
- Idea : Approximate X by ${}_nX$ such that :

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Euler approximations

- $n \in \mathbb{N}$, $\delta = \frac{T}{n}$, grid $(t_j = j\delta, j = 0, \cdots, n)$
- For $\omega \in D([0,T], \mathbb{R}^d)$, we define the piecewise constant Euler approximation of X along ω as follows :

Definition (Euler scheme)

${}_nX(\omega)$ is constant in each interval $[t_j, t_{j+1})$ with ${}_nX(0,\omega) = x_0$ and for $0 \le j \le n-1$,

$$\begin{aligned} {}_nX(t_{j+1},\omega) &= {}_nX(t_j,\omega) + b(t_j, {}_nX_{t_j}(\omega))\delta \\ &\quad + \sigma(t_j, {}_nX_{t_j}(\omega))(\omega(t_{j+1}-) - \omega(t_j-)) \end{aligned}$$

- When $\omega = W$, ${}_nX(W_T)$ Euler scheme of X

Definition of the functional

- $F_n(t,\omega_t) = \mathbb{E}\left[g({}_nX(\omega_t \underset{t}{\oplus} B))\right]$ with B Brownian motion independent of W
- Concatenation of paths $\omega, \omega' \in D([0,T],\mathbb{R}^d)$ at $t \in [0,T]$:

$$\omega \underset{t}{\oplus} \omega' = \omega_t \underset{t}{\oplus} \omega' = \begin{cases} \omega(u) & u \in [0,t) \\ \omega(t) + \omega'(u) - \omega'(t) & u \in [t,T] \end{cases}$$

- By independence of increments of W, $F_n(t,W_t) = \mathbb{E}\left[g({}_nX(W_T))|\mathcal{F}_t^W\right]$

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Main results

Theorem (Cont, L' (14))

The functional F_n defined by $F_n(t, \omega_t) = \mathbb{E}\left[g({}_nX(\omega_t \underset{t}{\oplus} B))\right]$ satisfies all the properties to apply the functional Ito formula. And we have :

$$g({}_nX(W_T)) = \mathbb{E}[g({}_nX(W_T))] + \int_0^T \underbrace{\nabla_\omega F_n(t, W_t)}_{Z_n(t)} \cdot dW(t) \quad \mathbb{P} - a.s.$$

Main results(cont'd)

Corollary

If we assume in addition that g is Lipschitz continuous with respect to $\|\cdot\|_\infty$, we have :

$$\left\| \int_0^T (Z_n(t) - \phi(t)) \cdot dW(t) \right\|_{\mathbb{L}^2} \leq C\sqrt{\frac{1+\log n}{n}}$$

Proof : $\| \sup_t |{}_nX(W_T)(t) - X(t)| \|_{\mathbb{L}^2} \leq C\sqrt{\frac{1+\log n}{n}}$

Sketch of proof

- ${}_nX(\omega)$ depends only on $\omega(t_1-) - \omega(0), \cdots,$ $\omega(t_n-) - \omega(t_{n-1}-)$
- Consequently, for $t_k \leq t < t_{k+1}$, $g({}_nX(\omega_t \underset{t}{\oplus} B))$ depends only on $B(t_{k+1}) - B(t)$, $B(t_{k+2}) - B(t_{k+1}), \cdots,$ $B(t_n) - B(t_{n-1})$
- $g({}_nX(\omega_t \underset{t}{\oplus} B)) = f(B(t_{k+1}) - B(t), \cdots, B(t_n) - B(t_{n-1}))$

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Sketch of proof(cont'd)

- A jump of size z of ω at time t is equivalent to a "jump" of $B(t_{k+1}) - B(t)$:

$$g({}_nX((\omega_t + z\mathbf{1}_{[t,T]}) \underset{t}{\oplus} B)) = f(B(t_{k+1}) - B(t) + z, \cdots, B(t_n) - B(t_{n-1}))$$

- Vertical differentiability $\iff$ differentiability of $\mathbb{E}[f(\cdots)]$ with respect to z, ensured by the regularity of the Gaussian density function
- For horizontal differentiability, we have similarly :

$$g({}_nX(\omega_t \underset{t+h}{\oplus} B)) = f(B(t_{k+1}) - B(t+h), \cdots, B(t_n) - B(t_{n-1}))$$

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Explicit vertical derivative

- $\nabla_\omega F_n(t, \omega_t) = \mathbb{E}\left[g({}_nX(\omega_t \underset{t}{\oplus} B))\frac{B(t_{k+1})-B(t)}{t_{k+1}-t}\right]$
  for $t_k \leq t < t_{k+1}$, $\forall \omega \in D([0, T], \mathbb{R}^d)$
- $\nabla_\omega F_n(t, W_t) = \mathbb{E}\left[g({}_nX(W_T)\frac{W(t_{k+1})-W(t)}{t_{k+1}-t}|\mathcal{F}_t^W\right]$
  $\rightsquigarrow$ BSDE

Remarks : Malliavin calculus

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- When h smooth, Clark-Ocone formula provides the same result
- When h not smooth, $g({}_nX(W_T))$ might not be Malliavin differentiable or its Malliavin derivative might not be explicit
- We require regularity conditions only on :

$$F_n(t, W_t) = \mathbb{E}[g({}_nX(W_T)|\mathcal{F}_t^W]$$

which is more smooth than $g({}_nX(W_T))$

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- In BSDE, $Z_n(t_k) = \mathbb{E}\left[Y_n(t_{k+1}) \cdot \frac{W(t_{k+1})-W(t_k)}{t_{k+1}-t_k} | \mathcal{F}_{t_k}^W\right]$
- Our case is a particular case of BSDE with $f \equiv 0$, the results of BSDE apply : (see Zhang(04))
  - $Z_n(t) = \mathbb{E}\left[\frac{1}{t_{k+1}-t}\int_t^{t_{k+1}} Z_n(s)ds | \mathcal{F}_t^W\right]$
  - Z_n admits a càdlàg version
  - $\sum_{k=1}^n \mathbb{E}\int_{t_{k-1}}^{t_k} \left(|Z_t - Z_{t_{k-1}}|^2 + |Z_t - Z_{t_k}|^2\right) dt \leq \frac{C}{n}$

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Remarks : BSDE(cont'd)

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  $Y_n \rightsquigarrow$ conditional expectation
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Thank you for your attention !