Polynomial Preserving Jump-Diffusions on the Unit Interval

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D MATH

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Outline

- **Polynomial preserving processes**

- **Polynomial preserving processes on $[0, 1]$**

Polynomial preserving processes

Polynomial preserving processes

- Markov semimartingale X with state space $E \subseteq \mathbb{R}^d$
- (Extended) generator $\mathscr{G}$ given by

$$\mathscr{G}f(x) = b(x)^\top \nabla f(x) + \frac{1}{2}\operatorname{Tr}\left(a(x)\nabla^2 f(x)\right) + \int \Big(f(x+\xi) - f(x) - \xi^\top \nabla f(x)\Big)\nu(x, d\xi)$$

- **Assumption:** $\int \|\xi\|^{2n}\nu(x, d\xi) \leq K_n(1 + \|x\|^{2n})$ for some $K_n \in \mathbb{R}$, all $x \in E$, all $n \in \mathbb{N}$.

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Definition. $\mathscr{G}$ is called **polynomial preserving (PP)** if

$$\mathscr{G}\,\mathrm{Pol}_n(E) \subseteq \mathrm{Pol}_n(E) \qquad \text{for all} \qquad n \in \mathbb{N},$$

where $\mathrm{Pol}_n(E) = \{\text{polynomials on } E \text{ of degree} \leq n\}$. In this case, X is called a polynomial preserving process.

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- Coordinate representations:

$$\begin{aligned} p(x) &= H(x)^\top \vec{p} & \qquad \vec{p} &\in \mathbb{R}^N \\ \mathscr{G} p(x) &= H(x)^\top G \, \vec{p} & \qquad G &\in \mathbb{R}^{N \times N} \end{aligned}$$

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- **Key consequence:**

$$\begin{aligned} \mathbb{E}\left[p(X_T) \mid \mathscr{F}_t\right] &= e^{(T-t)\mathscr{G}} p\,(X_t) \qquad \text{(formally)} \\ &= H(X_t)^\top e^{(T-t)G}\, \vec{p} \end{aligned}$$

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- This only involves a matrix exponential as opposed to solving a PDE which leads to tractable pricing models

Polynomial preserving processes

Lemma (Cuchiero, Keller-Ressel, Teichmann, 2012):

$\mathscr{G}$ is (PP) if and only if

$$b_i(x) \in \mathrm{Pol}_1(E)$$

$$a_{ij}(x) + \int_{\mathbb{R}^d} \xi_i \xi_j \, \nu(x, d\xi) \in \mathrm{Pol}_2(E)$$

$$\int_{\mathbb{R}^d} \xi_1^{k_1} \cdots \xi_d^{k_d} \, \nu(x, d\xi) \in \mathrm{Pol}_{k_1+\cdots+k_d}(E) \quad \text{if } k_1 + \cdots + k_d \geq 3$$

Corollary. If X is affine, then it is (PP).

Polynomial preserving processes on $[0, 1]$

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Why do we care about [0,1]?

- Stochastically evolving probabilities (e.g. default probabilities)
- Stochastically evolving correlations
- Stochastic recovery rates
- Electricity modeling
- Stepping stone toward simplex $E = \{x \in \mathbb{R}^d_+ : x_1 + \cdots + x_d = 1\}$
- ... and other compact state spaces

Polynomial preserving processes on $[0, 1]$

Let $E = [0, 1]$ and consider an operator of the form

$$\mathscr{G} f = \frac{1}{2} a f'' + b f' + \int \Big(f(\cdot + \xi) - f - \xi f' \Big) \nu(\cdot, d\xi)$$

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Remarks:

- Without jumps the solution is the well-known Jacobi process:

$$dX_t = (b + BX_t)dt + \sigma\sqrt{X_t(1 - X_t)}dW_t$$

  with $b \geq 0$, $b + B \leq 0$.

- With jumps, much richer behavior is possible.
- We restrict ν to have simple polynomial jump sizes.

Polynomial jump sizes

Definition. We say that $\nu(x, d\xi)$ has **simple polynomial jump sizes** if

$$\nu(A, d\xi) = \lambda(x) \int \mathbf{1}_A(\gamma(x, y))\mu(dy)$$

for some measurable $\lambda \geq 0$, some measure μ on $\mathbb{R}^{N+1}$ for some N, and

$$\gamma(x, y) = y_0 + y_1 x + \cdots + y_N x^N$$

- **Meaning:** Jump size is polynomial in the current state,

$$\Delta X_t = \gamma(X_{t-}, Y), \qquad Y \sim \mu,$$

  and arrive with intensity $\lambda(X_t)$ (if μ is a probability measure)
- **Remark:** Affine processes do not admit state-dependent jump sizes

Example 1

Jacobi process with constant intensity jumps:

$$dX_t = (b + BX_t)dt + \sigma\sqrt{X_t(1 - X_t)}dW_t + \sum_{i=0}^{N_t}\Big(Y_{i,1}(-X_{t-}) + Y_{i,2}(1 - X_{t-})\Big)$$

where N_t is a Poisson process with constant intensity λ, and

$$\{Y_{i,1}, Y_{i,2} \colon i = 1, 2, \ldots\} \quad \overset{\text{iid}}{\sim} \quad \mu, \qquad \operatorname{supp}\mu \subseteq [0,1] \times [0,1]$$

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Coefficients of $\mathscr{G}$:

$$a(x) = \sigma^2 x(1 - x) \qquad \gamma(x, y) = y_1(-x) + y_2(1 - x)$$
$$b(x) = b + Bx \qquad \lambda(x) \equiv \lambda$$

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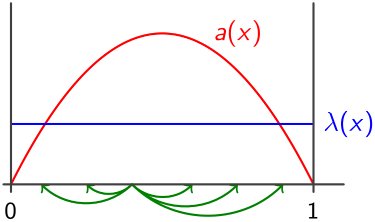

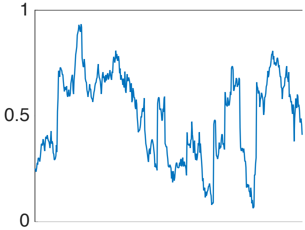

Example 2

Jacobi process with unbounded intensity jumps:

$$a(x) = \sigma^2 x(1-x) \qquad \gamma(x,y) = -yx$$

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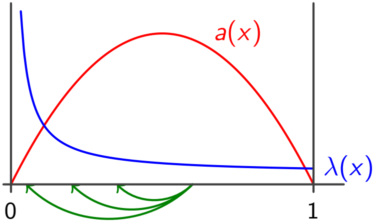

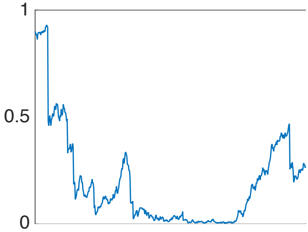

Example 3

Dunkl-like process: Fix $x^* \in (0,1)$. For $x \neq x^*$,

$$a(x) = \sigma^2 x(1-x) \qquad \gamma(x,y) = -y(x-x^*)$$

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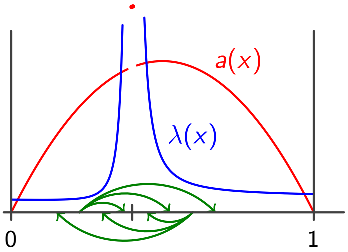
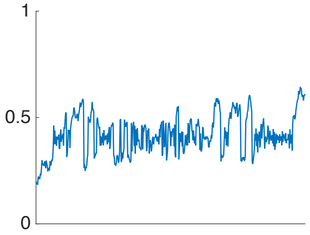




Characterization theorem

Theorem. Assume $\nu(x, d\xi)$ has simple polynomial jump sizes. Then $\mathscr{G}$ is the (extended) generator of a unique polynomial preserving process on $[0,1]$ if and only if

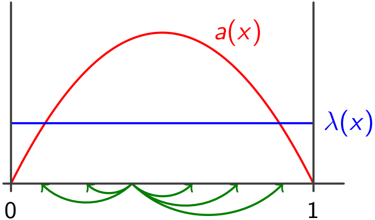
- **Affine drift:** $b(x) = b + Bx$ for some $b, B \in \mathbb{R}$
- **Affine jump sizes:** Can take $\mu(dy)$ on $[0,1] \times [0,1]$ such that $\int \|y\|^2 \mu(dy) < \infty$, and

$$\gamma(x, y) = y_1(-x) + y_2(1 - x)$$

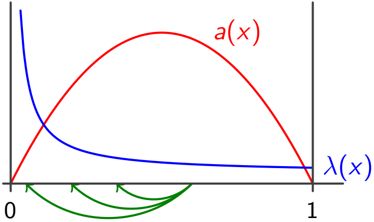
- **Boundary conditions:**
  - $a(0) = a(1) = 0$
  - $b(0) - \lambda(0) \int y_2 \mu(dy) \geq 0$
  - $b(1) + \lambda(1) \int y_1 \mu(dy) \leq 0$
- a, λ and $\operatorname{supp} \mu$ are of one of four types (up to reflection in $\frac{1}{2}$):

Characterization theorem

Case 1



Case 2



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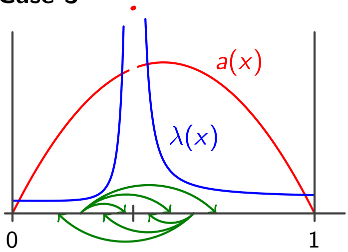
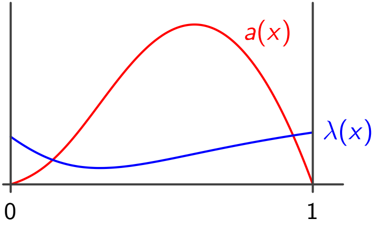


Case 4



Case 4

Intensity and diffusion are of the form

$$\lambda(x) = \frac{c_0 + c_1 x + c_2 x^2}{x^2 + |\alpha|^2} \qquad \text{and} \qquad a(x) = x(1-x)\frac{a_0 + a_1 x + a_2 x^2}{x^2 + |\alpha|^2}$$

for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$ such that

$$\int \Big(y_1(-\alpha) + y_2(1-\alpha) \Big)^n \mu(dy) = 0 \quad \text{for all} \quad n \geq 3$$

where $\operatorname{supp} \mu \subset [0,1] \times [0,1]$.

- We do not know whether such α and μ exist!
- If none exist, then $a(x)$ is necessarily essentially quadratic.

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Observation: The set

$$\left\{\mathscr{G} : \mathscr{G} \text{ is the generator of a (PP) process on } [0,1]\right\}$$

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Thus: Can combine Cases 1–3 to get (PP) generators with (for a.e. x)

$$a(x) = \sigma^2 x(1-x) \qquad b(x) = b + Bx \qquad \nu(x, A) = \int \mathbf{1}_A(\gamma(x,y))K(x,dy)$$

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where

$$K(x,dy) = m(dy) + \frac{\mu_0^{(0)}(dy) + x\mu_1^{(0)}(dy)}{x} + \frac{\mu_0^{(1)}(dy) + x\mu_1^{(1)}(dy)}{1-x} + \sum_{k=2}^{K} \frac{\mu_0^{(k)}(dy) + x\mu_1^{(k)}(dy) + x^2\mu_2^{(k)}(dy)}{(x-x_k)^2}$$

with $x_k \in (0,1)$ and signed measures $\mu_i^{(k)}$ concentrated on

$$\{y \in [0,1]\times[0,1] : (1-x_k)y_2 = x_k y_1\}$$

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Example. A (PP) process is obtained by $a(x) \equiv 0$, $b(x) = 1 - 2x$, and

$$\nu(x, A) = \int \mathbf{1}_A(\gamma(x,y)) K(x, dy)$$

where

$$K(x, dy) = \frac{1}{x(1+x)} \delta_{(1,0)}(dy) + \frac{1}{(1-x)(1+x)} \delta_{(0,1/2)}(dy)$$

However, neither [green] nor [blue] individually gives rise to a (PP) generator.

Conclusion

- (PP) processes can be used to build flexible and tractable models
- Beyond the affine case, little is known about (PP) jump-diffusions
- Characterization of (PP) processes on the unit interval with simple polynomial jump sizes
- Outlook:
  - What about Case 4?
  - "Composite" polynomial jump sizes
  - Boundary attainment?
  - (PP) processes on the unit simplex

Thank you!