# Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting

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based on joint work with Stefan Ankirchner, Monique Jeanblanc and Alexandre Popier


7th General AMaMeF and Swissquote Conference
September 8, 2015
Lausanne

Financial support from the French Banking Federation through the Chaire Markets in Transition is gratefully acknowledged.

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Case study: Sell $x$ shares of Adidas within $T$ minutes using market orders.

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | A1EWWW | adidas AG | | | | | | | 83,680 | 133 | 12:33:29 | CO | 83,140 |
| | Bid/Ask Orders | | | | | | | | | | | | |
| | | | 2 | 505 | 83,650 | 83,680 | 162 | 2 | | | | | |
| | | | 5 | 586 | 83,640 | 83,690 | 275 | 2 | | | | | |
| | | | 9 | 925 | 83,630 | 83,700 | 670 | 7 | | | | | |
| | | | 7 | 869 | 83,620 | 83,710 | 1.125 | 10 | | | | | |
| | | | 5 | 566 | 83,610 | 83,720 | 1.062 | 8 | | | | | |
| | | | 6 | 676 | 83,600 | 83,730 | 1.085 | 8 | | | | | |
| | | | 7 | 583 | 83,590 | 83,740 | 405 | 4 | | | | | |
| | | | 5 | 790 | 83,580 | 83,750 | 952 | 9 | | | | | |
| | | | 7 | 776 | 83,570 | 83,760 | 246 | 4 | | | | | |
| | | | 2 | 117 | 83,560 | 83,770 | 888 | 6 | | | | | |

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Assumption (Almgren&Chriss):

$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta z$$

$z$: amount sold at time $t$
$\eta$: price impact factor

# Stochastic Liquidity

| Symb | WKN | Name | Bid Anz | Bid Vol in Stck | Bid | Ask | Ask Vol in Stck | Ask Anz | Preis | Letzter Umsatz | Zeit Preis | Ph | Vortag |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | A1EWWW | adidas AG | 1 | 397 | 84,840 | 84,880 | 312 | 2 | 84,890 | 89 | 12:38:40 | CO | 85,920 |

Bid/Ask Orders

| Bid Anz | Bid Vol in Stck | Bid | Ask | Ask Vol in Stck | Ask Anz |
|---|---|---|---|---|---|
| 1 | 876 | 84,870 | 84,900 | 281 | 2 |
| 3 | 455 | 84,860 | 84,910 | 392 | 3 |
| 5 | 494 | 84,850 | 84,920 | 275 | 2 |
| 9 | 1.187 | 84,840 | 84,930 | 1.040 | 9 |
| 9 | 1.408 | 84,830 | 84,940 | 889 | 5 |
| 7 | 602 | 84,820 | 84,950 | 994 | 7 |
| 7 | 760 | 84,810 | 84,960 | 358 | 4 |
| 3 | 400 | 84,800 | 84,970 | 631 | 6 |
| 5 | 929 | 84,790 | 84,980 | 922 | 6 |
| 3 | 639 | 84,780 | 84,990 | 974 | 7 |

Bid/Ask Orders

| Bid Anz | Bid Vol in Stck | Bid | Ask | Ask Vol in Stck | Ask Anz |
|---|---|---|---|---|---|
| 4 | 276 | 84,850 | 84,900 | 484 | 5 |
| 2 | 275 | 84,840 | 84,910 | 631 | 5 |
| 7 | 843 | 84,830 | 84,920 | 808 | 8 |
| 9 | 829 | 84,820 | 84,930 | 976 | 9 |
| 9 | 1.696 | 84,810 | 84,940 | 937 | 6 |
| 4 | 522 | 84,800 | 84,950 | 1.171 | 7 |
| 6 | 921 | 84,790 | 84,960 | 358 | 4 |
| 4 | 717 | 84,780 | 84,970 | 471 | 5 |
| 2 | 134 | 84,770 | 84,980 | 438 | 3 |
| 4 | 274 | 84,760 | 84,990 | 723 | 3 |

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$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta_t z$$

$z$: amount sold at time $t$

$(\eta_t)$: price impact process

# The model: Trading rates determine remaining position

- $T < \infty$: time horizon
- $x \in \mathbb{R}$: initial position
- $X_t$: position size at time $t \in [0, T]$:

$$dX_t = x + \int_0^t \alpha_s ds + \int_0^t \beta_s dN_s$$

  - $\alpha_t$: trading rate at time $t$.
  - $\beta_t$: amount placed as a passive order at time $t$
  - $N$: Poisson process with intensity $\mu > 0$

- **Constraint:** $X_T = 0$ on a set $\mathcal{S} \in \mathcal{F}_T$.

# A reduced form model à la Almgren & Chriss

$$E\left[\int_0^T \left( \underbrace{\eta_t|\alpha_t|^p}_{\text{"execution costs"}} + \underbrace{\lambda_t|\beta_t|^p}_{\text{"slippage costs"}} + \underbrace{\gamma_t|X_t|^p}_{\text{"risk"}} \right) dt + \underbrace{\xi 1_{\mathcal{S}^c}|X_T|^p}_{\text{terminal costs}} \right] \longrightarrow \min$$

- $p > 1$ ($q$ its Hölder conjugate)
- $(\eta_t), (\lambda_t), (\gamma_t)$: nonnegative, progressively measurable
- $\xi$: nonnegative, $\mathcal{F}_T$-measurable random variable
- stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ satisfying usual conditions

# Related literature

- Schied 2013: Solves a variant of this problem in a Markovian framework using superprocesses
- Graewe, Horst, Séré 2015: Show smoothness of the value function in a Markovian framework
- Graewe, Horst, Qiu 2014: Analyze both Markovian and non-Markovian dependence of the coefficients by means of BSPDEs

# A maximum principle

Let $(Y, \psi, M)$ satisfy

- on $[0, T)$

$$dY_t = \left( (p-1)\frac{Y_t^q}{\eta_t^{q-1}} + \Theta(t, Y_t, \psi_t) - \gamma_t \right) dt + \psi_t d\tilde{N}_t + dM_t$$

  with $\Theta$ Lipschitz continuous in $y$ and $\psi$
- $M$ is a local martingale orthogonal to $\tilde{N}$
- $\lim_{t \to T} Y_t = \xi 1_{\mathcal{S}^c} + \infty 1_{\mathcal{S}}$.

Then the process given by

$$X_t^* = x \exp\left[ -\int_0^t \left( \frac{Y_u}{\eta_u} \right)^{q-1} du \right] \exp\left[ (q-1) \int_0^t \ln\left( \frac{\lambda_u}{Y_{u-} + \psi_u} \right) dN_u \right].$$

is optimal and the value function is given by $v(t, x) = Y_t x^p$.

# BSDEs with singular terminal condition

Consider the BSDE

$$dY_t = -f(t, Y_t, \psi_t)dt + \int_{\mathcal{Z}} \psi_t(z)\widetilde{\pi}(dz, dt) + dM_t$$
$$Y_T = \xi$$

where $\tilde{\pi}$ is a compensated Poisson measure and $P[\xi = \infty] > 0$.
Central assumptions on $f$:

- monotonicity in $y$:

$$(f(t, y, \psi) - f(t, y', \psi))(y - y') \leq \chi(y - y')^2.$$

- Lipschitz continuity in $\psi$:

$$|f(t, y, \psi) - f(t, y, \varphi)| \leq K\|\psi - \varphi\|_{L^2_\mu}.$$

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- Lipschitz continuity in $\psi$:

$$|f(t, y, \psi) - f(t, y, \varphi)| \leq K\|\psi - \varphi\|_{L^2_\mu}.$$

- at least polynomial growth in $y$:

$$-f(t, y, \psi) \geq \frac{1}{\eta_t}|y|^q - f(t, 0, \psi), \quad y \geq 0, \quad q > 1.$$

# Approximation from below

Consider the BSDE

$$
\begin{aligned}
dY_t^L &= -f(t, Y_t^L, \psi_t)dt + \int_{\mathcal{Z}} \psi_t^L(z)\widetilde{\pi}(dz, dt) + dM_t^L \\
Y_T^L &= \xi \wedge L
\end{aligned}
\tag{1}
$$

## Theorem

*For every $L > 0$ there exists a solution $(Y^L, \psi^L, M^L)$ to (1) satisfying the estimate*

$$
Y_t^L \leq \frac{K}{(T-t)^p} \left[ E\left( \int_t^T \left(\eta_s^{p-1} + (T-s)^p f(s,0,0)^+\right)^l ds \middle| \mathbb{F}_t \right) \right]^{1/l}.
$$

# Existence and Minimality

**Theorem**

*There exists a process* $(Y, \psi, M)$ *such that for every* $t < T$ *and as* $L \nearrow \infty$

- $Y_t^L \nearrow Y_t$ *a.s.*
- $\psi^L \to \psi$ *in* $L_\pi([0, t])$
- $M^L \to M$ *in* $\mathcal{M}^I([0, t])$.

*The process* $(Y, \psi, M)$ *satisfies*

$$dY_t = -f(t, Y_t, \psi_t)dt + \int_{\mathcal{Z}} \psi_t(z)\widetilde{\pi}(dz, dt) + dM_t$$

*on* $[0, t)$ *and* $\liminf_{t \to T} Y_t \geq \xi$. *Moreover,* $Y$ *is minimal.*

# Back to the control problem

Assume that

$$E\left[\int_0^T \gamma_t^2 dt\right] < \infty, \quad E\left[\int_0^T \eta_t^2 dt\right] < \infty \quad \text{and} \quad E\left[\int_0^T \frac{1}{\eta_t^{q-1}} dt\right] < \infty.$$

## Corollary

*There exists a minimal supersolution* $(Y, \psi, M)$ *to*

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} + \Theta(t, Y_t, \psi_t) - \gamma_t\right) dt + \psi_t d\tilde{N}_t + dM_t \qquad (2)$$

*with* $\liminf_{t\to T} Y_t \geq \xi 1_{\mathcal{S}^c} + \infty 1_{\mathcal{S}}$.

# Optimal controls

$$E\left[\int_0^T \left(\underbrace{\eta_t|\alpha_t|^p}_{\text{"execution costs"}} + \underbrace{\lambda_t|\beta_t|^p}_{\text{"slippage costs"}} + \underbrace{\gamma_t|X_t|^p}_{\text{"risk"}}\right) dt + \underbrace{\xi 1_{\mathcal{S}^c}|X_T|^p}_{\text{terminal costs}}\right] \longrightarrow \min$$

## Theorem

*The process given by*

$$X_t^* = x \exp\left[-\int_0^t \left(\frac{Y_u}{\eta_u}\right)^{q-1} du\right] \exp\left[(q-1)\int_0^t \ln\left(\frac{\lambda_u}{Y_{u-}+\psi_u}\right) dN_u\right].$$

*is optimal and the value function is given by* $v(t,x) = Y_t x^p$.

The proof is based on a penalization argument

# Random execution period

Replace the deterministic time horizon $T$ by a stopping time $\tau$.

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## Example

*Consider the BSDE*

$$dY_t = Y_t^2 dt + dM_t$$
$$Y_\tau = \infty$$

*with* $E\left[\frac{1}{\tau}\right] = \infty$.
*Consider first the terminal condition* $Y_\tau^L = L$. *Then one can show that*

$$Y_0^L \geq E\left[\frac{1}{\tau + 1/L}\right].$$

*In particular*

$$\liminf_{L\to\infty} Y_0^L \geq E\left[\frac{1}{\tau}\right] = \infty.$$

# Random execution period

Let $\Gamma$ be a diffusion in $\mathbb{R}^d$

$$d\Gamma_t = b(\Gamma_t)dt + \sigma(\Gamma_t)dW_t$$

with $\sigma$ being uniformly elliptic. Let $D \subset \mathbb{R}^d$ be open and bounded with $C^2$-boundary. Define

$$\tau = \tau_D = \inf\{t \geq 0, \quad \Gamma_t \notin D\}.$$

Consider the BSDE

$$dY_t = -f(t, Y_t, \psi_t)dt + \int_{\mathcal{Z}} \psi_t(z)\widetilde{\pi}(dz, dt) + dM_t$$
$$Y_\tau = \xi$$

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Consider first the terminal condition $Y_\tau^L = \xi \wedge L$.

**Theorem**

*For every $L > 0$ there exists a solution $(Y^L, \psi^L, M^L)$ satisfying the estimate*

$$Y_t^L \leq \frac{C}{\mathrm{dist}(\Gamma_{t\wedge\tau})^{p-1}}.$$

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Then we obtain existence of a minimal supersolution to the singular BSDE and optimal controls as before.

# Processes with uncorrelated multiplicative increments

## Definition

*$\eta$ has uncorrelated multiplicative increments (umi) if*

$$E\left[\frac{\eta_t}{\eta_s}\Big|\mathcal{F}_s\right] = E\left[\frac{\eta_t}{\eta_s}\right]$$

*for all $s \leq t < T$.*

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Examples

- $\eta$ is deterministic
- $\eta$ is a martingale
- $d\eta_t = \mu(t)\eta_t dt + \sigma(t, \eta_t) dW_t$
- $\eta_t = e^{Z_t}$ where $Z$ is a Lévy process

# umi processes $\leftrightarrow$ deterministic controls

Assume $\gamma = 0$ and $\mu = 0$.

## Proposition

*Suppose that $\eta$ has umi, then*

$$Y_t = \frac{1}{\left(\int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds\right)^{p-1}}$$

*is the minimal solution to* (2) *with singular terminal condition. The deterministic control*

$$X_t = x \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}} ds} \int_t^T \frac{1}{E[\eta_s]^{q-1}} ds$$

*is optimal. In particular, if $p = 2$, then $\dot{X}_t = -c \frac{1}{E[\eta_t]}$.*

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*is optimal. In particular, if $p = 2$, then $\dot{X}_t = -c \frac{1}{E[\eta_t]}$.*

*Vice versa, assume that the optimal control $X_t = xe^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$ is deterministic. Then $\eta$ has umi.*

# Extensions

- Include directional views for the price process
- Incorporate volume uncertainty

# Thank you!