Classification of affine processes with compact state spaces and counter examples for polynomial preserving processes

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This talk is based on joint work with Prof. Dr. Martin Larsson

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Definition

An affine process is a stochastically continuous strong Markov process X on a stochastic basis $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$ such that the state space $E \subseteq \mathbb{R}^d$ is a closed set and its characteristic function is exponential affine-linear in the state, i.e.

$$E_x(e^{i\langle u, X(t)\rangle}) = \Phi(t,u)\exp\left(\sum_{j=1}^{d} x_j \psi_j(t,u)\right)$$

for any $x \in E$, $u \in \mathbb{R}$, $t \geq 0$ where $\psi : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{C}^d$, $\Phi : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{C}$ are suitable continuous functions and $\mathbb{R}^d \subseteq \mathcal{U} \subseteq \mathbb{C}^d$ suitable.

See Duffie, Filipovic, Schachermayer [DFS 2003] for the original definition and Keller-Ressel, Schachermayer and Teichmann [KST 2013] for the given definition. The latter paper actually allows for measurable state spaces and for killing as well.

PPP

Definition

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An polynomial preserving process (PPP) is a stochastically continuous strong Markov process X on a stochastic basis $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$ such that the state space $E \subseteq \mathbb{R}^d$ is a closed set and where the extended generator maps polynomials to polynomials of at most the same degree, i.e. there is a function $\mathcal{G}$ from the set $\mathcal{P}$ of all polynomials in d variables into $\mathcal{P}$ such that

$$M_p(t) := p(X(t)) - \int_0^t \mathcal{G}p(X(s))ds, \quad t \geq 0$$

is a local martingale for any $p \in \mathcal{P}$ and such that the degree of $\mathcal{G}p$ is less or equal than the degree of p.

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is a local martingale for any $p \in \mathcal{P}$ and such that the degree of $\mathcal{G}p$ is less or equal than the degree of p.

Remark

An affine process is a PPP if and only if $E_x|X(1)|^n < \infty$ *for any* $x \in E$, $n \in \mathbb{N}$.

See Cuchiero, Keller-Ressel and Teichmann [CKT 2012].

Semimartingale property of PPP

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Let X be a PPP with state space E and $x \in E$. Then, X is a semimartingale under P_x and a version of its characteristics is given by

$$B(t) = \int_0^t b(X(s))ds,$$

$$C(t) = \int_0^t c(X(s))ds,$$

$$\nu(dx, dt) = F(X(t), dx)dt$$

for an affine function $b : \mathbb{R}^d \to \mathbb{R}^d$, a quadratic function $\tilde{c} : \mathbb{R}^d \to S_d$, $\tilde{c} := c + \int yy^\top F(\cdot, dy)$ and a signed transition kernel F from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $x \mapsto \int y^\alpha F(x, dy)$ is a polynomial of degree at most $|\alpha|$ for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 3$.

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If X is additionally an affine process, then c, F are affine as well.

See also [DFS 2003] and [KST 2013].

A natural conjecture(?)

Conjecture

Let X be an affine process with compact state space $E \subseteq \mathbb{R}^d$. Then, X is P_x-a.s. deterministic for any $x \in E$.

Contents

Limited jumps type affine processes

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A limited jump type (LJ) process X with parameters $(N, \lambda) \in \mathbb{N} \times (0, \infty)$ is a strong Markov process with state space $E := \{0, \ldots, N\}$ where its Markov triplet is given by $(b, 0, F)$ where

$$F(x, \cdot) = \lambda x \delta_{-1}$$
$$b(x) = -\lambda x$$

for $x \in E$ and generator

$$\mathcal{G}f(x) = \lambda x(f(x-1) - f(x)), \quad x \in E, f \in M_b(E, \mathbb{R}).$$

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Its characteristic function is given by $\mathrm{E}_x(e^{iuX(t)}) = \Psi(t, u)^x$ *where* $\Psi(t, u) = 1 - e^{-\lambda t} + e^{iu - \lambda t}$ *for any* $t \geq 0$, $u \in \mathbb{R}$.

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Remark

Its characteristic function is given by $\mathrm{E}_x(e^{iuX(t)}) = \Psi(t, u)^x$ *where* $\Psi(t, u) = 1 - e^{-\lambda t} + e^{iu - \lambda t}$ *for any* $t \geq 0$, $u \in \mathbb{R}$. *However, since* $\Psi(\frac{1}{\lambda} \log(2), \pi) = 0$ *it is not affine in the sense of [KST 2013].*

Example

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$E = \{0,1\}$, $F(x,\cdot) = x\delta_{-1} + (1-x)\delta_1$ for any $x \in E$. Define

$$\begin{aligned}\mathcal{G}f(x) &:= x(f(x-1) - f(x)) + (1-x)(f(x+1) - f(x)) \\ &= \int_{\mathbb{R}} (f(x+y) - f(x))F(x, dy)\end{aligned}$$

where $x \in E, f \in M_b(E, \mathbb{R})$.

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where $x \in E, f \in M_b(E, \mathbb{R})$.
The strong Markov process X with generator $\mathcal{G}$ has characteristic function given by

$$\mathrm{E}(e^{iuX(t)}) = \Phi(t,u)\exp(x\psi(t,u)), \quad t \geq 0, u \in \mathbb{R}$$

where $\Phi(t,u) = \cosh(t)e^{-t} + \sinh(t)e^{iu-t}$ and

$$\psi(t,u) = \log\left(\frac{\cosh(t)e^{iu} + \sinh(t)}{\cosh(t) + \sinh(t)e^{iu}}\right)$$

for any $t \geq 0$, $u \in \mathbb{R}$.

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for any $t \geq 0$, $u \in \mathbb{R}$.
In particular, the conjecture does not hold.

Grid type affine processes

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$E = \{(0,0),(1,0),(0,1)\}$,

$$F((x_1, x_2), dy) = (1 - x_1 - x_2)(\delta_{(1,0)} + \delta_{(0,1)}) + x_1\delta_{(-1,0)} + x_2\delta_{(1,-1)}$$

for any $(x_1, x_2) \in E$. Define

$$\mathcal{G}f(x) := \int_{\mathbb{R}} (f(x+y) - f(x))F(x, dy)$$

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where $x \in E, f \in M_b(E,\mathbb{R})$.
The strong Markov process X with generator $\mathcal{G}$ is GT with state space E.

A GT driven deterministic component

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Let A be the GT with state space $\{0,1\}$ and jump intensity 1, i.e. its jump measure is given by

$$F_A(x,dy) = x\delta_{-1} + (1-x)\delta_1, \quad x \in \{0,1\}$$

and its generator is given by

$$\mathcal{G}f(x) = x(f(x-1)+f(x)) + (1-x)(f(x+1)-f(x)), \quad x \in \{0,1\}, f \in M_b(\{0,1$$

Let $E := \{0,1\} \times [0,1]$, $b(x_1,x_2) = (1-2x_1, x_1-x_2)$ and B be the solution of

$$B(t) = \int_0^t (A(s) - B(s))ds, \quad t \geq 0$$

Then $X := (A,B)$ is affine with state space E and B is a CD with no jumps.

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The structure theorem for compact affine processes

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Theorem (K. and Larsson 2015)

Let X be a strong Markov process with compact state space $E \subseteq \mathbb{R}^d$ and affine Markov triplet (b, c, F). Then, it is a PPP and a semimartingale. Moreover, there is an affine-linear mapping $\pi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\pi(X) = (A, B, C)$ where A is $\mathbb{R}^k$ valued, B is $\mathbb{R}^l$ valued and C is $\mathbb{R}^{d-k-l}$-valued with $k, l \leq d$ and they have the following properties:

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1. *$A_1, \ldots, A_k$ are independent limited jump type processes.*

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1. *$A_1, \dots, A_k$ are independent limited jump type processes.*
2. *B is a grid type affine process conditioned on A being constant.*

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1. *$A_1, \ldots, A_k$ are independent limited jump type processes.*
2. *B is a grid type affine process conditioned on A being constant.*
3. *C is a deterministic affine process conditioned on (A, B) being constant.*
4. *C can only jump if A does.*

Contents

Modified characteristics

Definition and Lemma

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Definition and Lemma

Let X be a strong Markov process with closed state space $E \subseteq \mathbb{R}^d$ and Markov triplet (b, c, F) and assume that $\int |y|^2 F(x, dy) < \infty$ for any $x \in E$. The modified triplet *is defined by $(b, \tilde{c}, F)$ where*
$\tilde{c}(x) := c(x) + \int yy^\top F(x, dy)$.

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Definition and Lemma

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$\tilde{c}(x) := c(x) + \int yy^\top F(x, dy)$.
X is a PPP if and only if

- *b is linear,*
- *$\tilde{c}$ is a polynomial of order two or less,*
- *$\int |y|^n F(x, dy) < \infty$ for any $x \in E$, $n \geq 2$ and*
- *$x \mapsto \int y^\alpha F(x, dy)$ is a polynomial of order at most $n \geq 3$ where $\alpha \in \mathbb{N}^d$ with $|\alpha| = n$.*

Example I. PPP on $\mathbb{R}$

Proposition

Consider the modified triplet $(0, 1, F)$ *where*

$$F(x, \cdot) = 1_{\{x \neq 0\}} \frac{\delta_x + \delta_{-x}}{2x^2}, \quad x \in \mathbb{R}$$

and $E = \mathbb{R}$. *Define* $H_\alpha := \{\alpha 2^z, 0, -\alpha 2^z : z \in \mathbb{Z}\}$.

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and $E = \mathbb{R}$*. Define* $H_\alpha := \{\alpha 2^z, 0, -\alpha 2^z : z \in \mathbb{Z}\}$*.*
We have $\int y^n F(x, dy) = 1_{\{n \text{ even}\}} x^{n-2}$ *for any* $x \in E$*,* $n \geq 3$

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and $E = \mathbb{R}$. *Define* $H_\alpha := \{\alpha 2^z, 0, -\alpha 2^z : z \in \mathbb{Z}\}$.
We have $\int y^n F(x, dy) = 1_{\{n \text{ even}\}} x^{n-2}$ *for any* $x \in E$, $n \geq 3$
Then, there is a PPP X *with state space* E *and modified triplet* $(b, \tilde{c}, F)$ *such that* $P_0(\forall t \geq 0 : X(t) \in H_\alpha) = 1$.

Example II. PPP on $\mathbb{R}_+$ without diffusion component.

Proposition

Consider the Markov triplet $(1, 0, F)$ *where*

$$F(x, \cdot) = 1_{\{x \neq 0\}} \frac{\delta_x}{x}, \quad x \in \mathbb{R}$$

and $E = \mathbb{R}_+$. *Define* $H_\alpha := \{\alpha 2^z, 0 : z \in \mathbb{Z}\}$.

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Then, there is a PPP X *with state space* E *and modified triplet* $(b, \tilde{c}, F)$ *such that* $P_0(\forall t \geq 0 : X(t) \in H_\alpha) = 1$.

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Thank you for your attention!

An LJ driven deterministic component

Example

Let A be a LJ with parameters $(1,1)$, $b(x_1,x_2) = (-x_1, \frac{1}{2}x_1 - x_2)$, $E := \{(1,0)\} \cup (\{0\} \times [0,1])$, $b_0(x_1,x_2) = (0,-x_2)$, U be a uniform distributed random variable on $[0,1]$ which is independent of A,

$$\tau := \inf\{t \geq 0 : A(t) = 0\},$$
$$B(t) := 1_{\{t \geq \tau\}} e^{\tau - t} U,$$

and $X(t) := (A(t), B(t))$. Then B is CD and its jumps are controlled by A. Its Markov triplet are $(b, 0, F)$ where

$$F((x_1,x_2), dy) = x_1 \mu(dy)$$

where μ is the measure on $\mathbb{R}^2$ which is uniformly distributed on $\{-1\} \times [0,1]$.

Proposition (One dimensional Grid type processes)

Let $E := \{0, \ldots, N\}$ *for some* $N \geq 0$, $a, b > 0$ *and*

$$F(x, \cdot) = a\delta_1 + \frac{x}{N}(b\delta_{-1} - a\delta_1)$$

for $x \in E$. *Let* X *be the strong Markov process with generator*

$$\mathcal{G}f(x) = \int f(x+y) - f(x)F(x, dy), \quad x \in E, f \in M_b(E, \mathbb{R}).$$

Then, the characteristic function of X *is given by*

$$\mathrm{E}(e^{iuX(t)}) = \exp(\varphi(t,u) + x\psi(t,u)), \quad t \geq 0, u \in \mathbb{R}$$

where $\psi(t,u) = \log\left(1 + \frac{(a+b)(e^{iu}-1)}{\exp(-t(a+b)/N)(ae^{iu}+b) - a(e^{iu}-1)}\right)$ *and*
$\varphi(t,u) = \int_0^t \frac{(a+b)(e^{iu}-1)}{\exp(-s(a+b)/N)(ae^{iu}+b) - a(e^{iu}-1)} ds$