Affine Processes with stochastic discontinuities

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Swissquote Conference, EPFL Lausanne
September 9th, 2015

Part I

Motivation & Introduction

Economic Motivation I

Nature of 'discontinuities' in financial markets:

- **Type I discontinuities:** Events with substantial impact that come as a complete surprise (at 'unpredictable' times).
- **Type II discontinuities:** Events with substantial impact that occur at predictable or deterministic times (but with random outcome).

With the exception of firm-value-based credit risk models, most modeling frameworks have focused on type I events: e.g. intensity-based credit risk models; Lévy-based models of asset returns; ...

Economic Motivation II

Many examples of type II events (occurence at predictable or deterministic time) exist:

- Dividend payments
- Central bank decisions
- Political decisions (e.g. Greek debt crisis)
- Ex-post decisions on nature of default events
- ...

Mathematical formalism I

- A stopping time τ is called **predictable time** if

$$[\![0,\tau[\![:= \{(t,\omega) : 0 \le t < \tau(\omega)\}$$

  is a predictable set.
- Predictable times have an **announcing sequence**, i.e. there exist stopping times $\tau_n < \tau$ on $\{\tau > 0\}$ such that $\tau = \lim_{n\to\infty} \tau_n$. Conversely, stopping times with announcing sequences are predictable.
- A stopping time τ is **totally inaccessible** if $\mathbb{P}(\tau = \sigma) = 0$ for any finite predictable time σ.

Mathematical formalism II

- Type I events correspond to totally inaccessible stopping times, Type II events to predictable ones.
- Any stopping time can be split into a totally inaccessible part and an accessible part that can be exhausted by a sequence of predictable times.

Mathematical formalism III

Consider a cadlag adapted process X, its left limit process X_- and its pure-jump part $\Delta X = X - X_-$

We distinguish:

- **(pathwise) continuity:** $\Delta X \equiv 0$ (e.g. diffusion processes).
- **quasi-left-continuity:** $\Delta X_\tau = 0$ for all predicable times τ. This is the natural notion of 'stochastic continuity' in the semimartingale framework.
- **stochastic continuity:** X is continuous in probability.

Mathematical formalism III

- A cadlag adapted process is quasi-left-continuous if and only if its jumps can be exhausted by totally inaccessible stopping times.
- Hence, quasi-left-continuous processes represent economic models that allow only for Type-I discontinuities.
- Jump-processes typically used in mathematical finance, e.g. Lévy-processes are quasi-left-continuous (and hence also stochastically continuous).

Motto of this talk: Recognize importance of predictable jump-times (type II events) and combine them with the framework of affine processes.

Part II

Affine Processes

Affine Processes

Two main approaches to affine processes:

- Markovian approach (Duffie, Filipovic, Schachermayer, ...)

  An affine process is a Markov process, whose conditional characteristic function is exponentially-affine in the current state.

- Semimartingale approach (Kallsen, Muhle-Karbe, ...)

  An affine process is an Ito semimartingale, whose differential semimartingale characteristics are affine functions of the current state.

Up to details these approaches are largely equivalent, i.e. lead to the same class of processes.

Markovian approach

Definition (Affine Process)

An affine process is a Markov process X with state space $D \subset \mathbb{R}^d$, which satisfies

$$E\left[e^{\langle u, X_t \rangle} | \mathcal{F}_s\right] = \exp\left(\phi(s,t,u) + \langle \psi(s,t,u), X_s \rangle\right)$$

for all $u \in i\mathbb{R}^d$, $0 \le s \le t$ and $X_0 \in D$.

- Law of the process completely determined by ϕ, ψ
- Under additional assumptions ϕ, ψ are solutions of ordinary differential equations ('generalized Riccati equations')
- Most interesting properties of the process X can be formulated as analytical properties of ϕ, ψ.

Markovian approach II

- **Duffie, Filipovic & Schachermayer (2003):** Complete characterization of affine processes on $D = \mathbb{R}^m \times \mathbb{R}^n_{\geqslant 0}$ under assumption that X is stoch. continuous and time-homogeneous; ϕ and ψ are 'regular' (differentiable)
- **Filipovic (2005):** Complete characterization of affine processes on $D = \mathbb{R}^m \times \mathbb{R}^n_{\geqslant 0}$ under assumption that X is stoch. continuous (but not time-homogeneous) + regularity assumptions.
- **K.-R., Schachermayer & Teichmann (2011):** regularity assumption is implied by stoch. cont & time-homogeneity for $D = \mathbb{R}^m \times \mathbb{R}^n$
- Further improvements on regularity by K.-R., Schachermayer & Teichmann (2013), Cuchiero & Teichmann (2013).

Semimartingale approach

Following Kallsen (2006) an affine semimartingale is a semimartingale whose differential characteristics are affine functions of the current state:

$$B_t = \int_0^t \left(\beta_0(s) + \sum_{i=1}^d X^i_{s-}\beta_i(s)\right) ds$$

$$C_t = \int_0^t \left(\gamma_0(s) + \sum_{i=1}^d X^i_{s-}\gamma_i(s)\right) ds$$

$$\nu(ds, dx) = \left(\kappa_0(s, dx) + \sum_{i=1}^d X^i_{s-}\kappa_i(s, dx)\right) ds.$$

- Here, a regularity assumption is implicitly contained in the assumption that characteristics are absolutely continuous.
- In fact, continuity of the semimartingale char. already implies quasi-left-continuity of X.

State of the art

- No established theory for affine processes which are not quasi-left-continuous.
- Hence, no theory for affine models with type-II-events.
- In our approach for affine processes with stochastic discontinuity we use a mix of semimartingale approach and characteristic-function approach.

Part III

Affine Processes with stochastic discontinuities

Affine Processes with stochastic discontinuities

Definition (Affine Semimartingale – alternative definition)

A d-dimensional semimartingale X is called **affine** if there exist $\mathbb{C}$ and $\mathbb{C}^d$-valued (deterministic) functions $\phi(s,t,u)$ and $\psi(s,t,u)$, respectively, such that

$$E\left[e^{\langle u, X_t\rangle} | \mathcal{F}_s\right] = \exp\left(\phi(s,t,u) + \langle \psi(s,t,u), X_s\rangle\right) \qquad (\star)$$

holds for all $u \in i\mathbb{R}^d$, $0 \le s \le t$.

It is called **time-homogeneous**, if $\phi(s,t,u) = \phi(0, t-s, u)$ and $\psi(s,t,u) = \psi(0, t-s, u)$, again for all $u \in i\mathbb{R}^d$ and $0 \le s \le t$.

Proper affine families

Definition (Proper affine family)

A family $(X^x)_{x \in D}$ of affine processes with D a Borel set in $\mathbb{R}^d$ is called **proper**, if

1. the functions ϕ and ψ in $(\star)$ are the same for all X^x, and
2. the following **full-support condition** is fulfilled: For every $t > 0$ there exist $x_0, x_1, \dots, x_d \in D$, such that

$$\operatorname{aff}\left(\operatorname{supp} X_t^{x_0}, \dots, \operatorname{supp} X_t^{x_d}\right) = \mathbb{R}^d$$

$$\operatorname{aff}\left(\operatorname{supp} X_{t-}^{x_0}, \dots, \operatorname{supp} X_{t-}^{x_d}\right) = \mathbb{R}^d$$

This condition can be compared to assumptions on state space in the Markovian setting, but is weaker.

Proper affine families II

Some illustrative examples:

- The constant process $X = x_0 \in \mathbb{R}^d$ is an affine semimartingale, but (taken as a family with a single member) not a proper affine family, since the full support condition fails.

- The family $(X^{x_i} = x_i)_{i \in \{0,\dots,d\}}$ of constant processes is a proper affine family if and only if the points $x_0, \dots, x_d$ are affine independent.

- Even families with a single member can be proper affine families: Take d-dimensional Brownian motion started at a point $x_0 \in \mathbb{R}^d$. The support of Brownian motion is $\mathbb{R}^d$ for all $t > 0$, hence we may choose $x_0 = x_1 = \cdots = x_d$ in the full-support condition.

A regularity result

Lemma (Key regularity Lemma)

Let $(X^x)_{x\in D}$ *be a proper affine family. Then for all* $u \in i\mathbb{R}^d$ *and* $0 < s < t$ *the following holds:*

1. *the left limits*

$$\phi(s, t-, u) := \lim_{\epsilon\downarrow 0} \phi(s, t-\epsilon, u), \quad \psi(s, t-, u) := \lim_{\epsilon\downarrow 0} \psi(s, t-\epsilon, u)$$

   exist.

2. *The functions* $s \mapsto \phi(s, t, u)$*,* $s \mapsto \psi(s, t, u)$ *and* $s \mapsto \phi(s, t-u)$*,* $s \mapsto \psi(s, t-, u)$ *are càdlàg and of finite variation.*

...

A regularity result II

Lemma (Key regularity Lemma (cont.))

1. *ϕ and ψ satisfy the semi-flow property, i.e.*

$$\phi(s,t,u) = \phi(r,t,u) + \phi(s,r,\psi(r,t,u)), \quad \phi(t,t,u) = 0$$
$$\psi(s,t,u) = \psi(s,r,\psi(r,t,u)), \quad \psi(t,t,u) = u.$$

for all $0 \le s \le r \le t$.
Above eqs. still hold when any of r, s and t is be replaced by $r-$, $s-$ or $t-$ with the usual interpretation as a left limit.

This Lemma replaces the regularity assumptions of Duffie, Filipovic & Schachermayer (2003). Allows to apply Ito's formula to local martingale

$$M_s^{u,t} := E\left[e^{\langle u, X_t\rangle} | \mathcal{F}_s\right] = \exp\left(\phi(s,t,u) + \langle \psi(s,t,u), X_s\rangle\right).$$

Theorem (Main result)

Let $(X^x)_{x \in D}$ be a proper family of affine semimartingales. Then there exists a deterministic increasing process A, continuous functions $\beta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, $\gamma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{d \times d}$ and families of Borel measures $(\kappa_i(s, \cdot))_{s \geq o}$ on $D \backslash \{0\}$ such that the semimartingale characteristics (B, C, ν) of X satisfy

$$B_t = \int_0^t \left(\beta_0(s) + \sum_{i=1}^d X^i_{s-} \beta_i(s)\right) dA(s)$$

$$C_t = \int_0^t \left(\gamma_0(s) + \sum_{i=1}^d X^i_{s-} \gamma_i(s)\right) dA(s)$$

$$\nu(ds, dx) = \left(\kappa_0(s, dx) + \sum_{i=1}^d X^i_{s-} \kappa_i(s, dx)\right) dA(s).$$

Theorem (Main result (cont.))

The functions ϕ and ψ are continuous on the complement of $J := \{s > 0 : P(\Delta X_s \neq 0) > 0\}$ and their continuous parts ϕ^c and ψ^c solve the following generalized measure Riccati equations

$$\frac{d\phi^c(s,t,u)}{dA^c(s)} = -F(s, \psi(s-,t,u))$$
$$\frac{d\psi^c(s,t,u)}{dA^c(s)} = -G(s, \psi(s-,t,u))$$

for $s \in (J)^c \cap [0,t]$ with

$$F(s,u) = \langle \beta_0(s), u \rangle + \frac{1}{2} \langle u, \gamma_0(s) u \rangle + \int_D \left(e^{\langle x,u \rangle} - 1 \right) \kappa_0(s, dx)$$
$$G_i(s,u) = \langle \beta_i(s), u \rangle + \frac{1}{2} \langle u, \gamma_i(s) u \rangle + \int_D \left(e^{\langle x,u \rangle} - 1 \right) \kappa_i(s, dx).$$

. . .

Theorem (Main result (cont.))

Finally, let $z_s(\omega, u) := \int_D e^{\langle u,x\rangle}\nu(\omega, \{s\}, dx)$*. Then there exist functions* $\zeta_0^s, \ldots, \zeta_d^s$*, such that*

$$z_s(\omega, u) + 1 - z_s(\omega, 0) = \exp\Big(- \zeta_0^s(u) - \sum_{i=1}^d X_{s-}^i(\omega)\zeta_i^s(u)\Big)$$

P-almost surely. Moreover,

$$\begin{aligned} \Delta\phi(s,t,u) &= \zeta_0^s(\psi(s,t,u)), \\ \Delta\psi_i(s,t,u) &= \zeta_i^s(\psi(s,t,u)). \end{aligned} \tag{$\circ$}$$

for all $s \in J \cap [0,t]$*,* $u \in i\mathbb{R}^d$*.*

Summary of the main result

- The semimartingale characteristics of X are absolutely continuous with respect to a deterministic increasing process A and the 'differential characteristics' (wrt. A) are affine in the current state.
- A can be chosen such that the jumps of A coincide with the jumps of ϕ and ψ. These are exactly the predictable jump times of X.
- Between the jumps of A the functions ϕ and ψ satisfy generalized Riccati 'differential' equations. Again differentiation has to be interpreted as Radon-Nikodym derivatives wrt. A
- At the jump times of A the relation $(\circ)$ determines the discontinuities of ϕ and ψ.

Corollaries

Corollary

The following are equivalent:

- *A is continuous.*
- *$s \mapsto \phi(s,t,u)$ and $s \mapsto \psi(s,t,u)$ are continuous for all t,u.*
- *X is quasi-left-continuous*

Corollary

The following are equivalent:

- *A is absolutely continuous.*
- *$s \mapsto \phi(s,t,u)$ and $s \mapsto \psi(s,t,u)$ are absolutely continuous for all t,u.*
- *X is an Ito semimartingale (a semimartingale with absolutely continuous characteristics).*

Summary and Outlook

- Our results give a characterization of affine semimartingales with stochastic discontinuities under a 'full support' condition;
- The dynamics of such processes are still described by 'generalized Riccati equations' (now with discontinuities);
- Also several time-series models, such as $AR(p)$ models fit into the framework of stochastically discontinuous affine processes;
- Existence results and sufficient admissibility conditions for affine semimartingales with stochastic discontinuities are work in progress.

Thank you for your attention!