Equilibria in Incomplete Stochastic Continuous-time Markets:

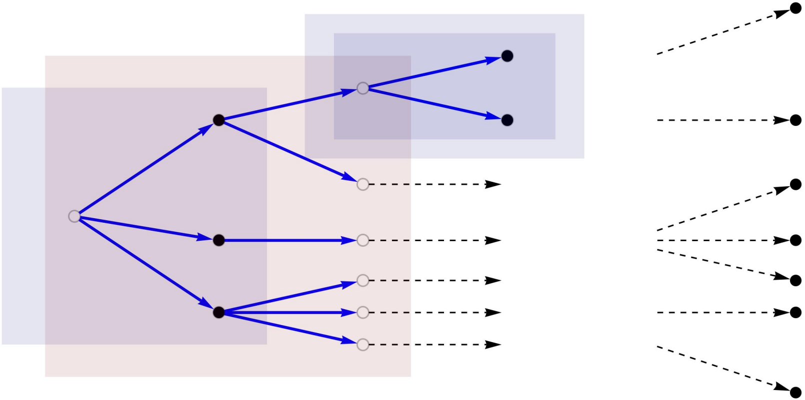
Existence and Uniqueness under "Smallness"

Constantinos Kardaras
Department of Statistics
London School of Economics

with **Hao Xing** (LSE) and **Gordan Žitković** (UT Austin)

7th General AMaMeF and Swissquote Conference,
September 10, 2015

Stochastic Finance Economies



Agents. Information. Preferences. Endowments. Assets.

Financial Equilibrium: Discrete Time

- Walras 1874,
- Arrow-Debreu '54, McKenzie '59,
- Radner '72 extends the classical Arrow-Debreu model.
- Hart '75 gives a non-existence example.
- Duffie-Shafer '85, '86 show that an equilibrium exists for *generic* endowments
- Cass, Drèze, Geanakoplos, Magill, Mas-Colell, Polemarchakis, Stieglitz, and others.

Financial Equilibrium: Continuous Time

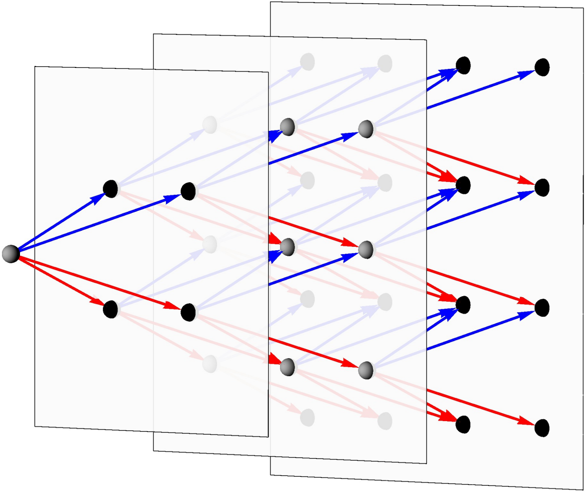
Complete Markets

- Merton '73
- Duffie-Zame '89, Araujo-Monteiro '89,
- Karatzas-Lakner-Lehoczky-Shreve '91.

Incomplete Markets

- Basak, Cheridito, Christensen, Choi, Cuoco, He, Hugonnier, Kupper, Larsen, Munk, Zhao, Žitković.

An Incomplete, Short-Lived-Asset Model



Our Problem

Setup: $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs B and W.

Our Problem

Setup: $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs B and W.

Price: $dS_t^\lambda = \lambda_t\, dt + 1 \cdot dB_t + 0 \cdot dW_t$.

Our Problem

Setup: $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs B and W.

Price: $dS_t^\lambda = \lambda_t \, dt + 1 \cdot dB_t + 0 \cdot dW_t$.

Agents: $U^i(x) = -\exp(-x/\delta^i), \;\; E^i \in \mathbb{L}^\infty(\mathcal{F}_T), \;\; i = 1, \dots, I$.

Our Problem

Setup: $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs B and W.

Price: $dS_t^\lambda = \lambda_t\,dt + 1\cdot dB_t + 0\cdot dW_t.$

Agents: $U^i(x) = -\exp(-x/\delta^i),\;\; E^i \in \mathbb{L}^\infty(\mathcal{F}_T),\;\; i = 1,\dots,I.$

Demand: $\widehat{\pi}^{\lambda,i} := \operatorname{argmax}_{\pi\in\mathcal{A}^\lambda} \mathbb{E}\left[U^i\Big(\int_0^T \pi_u\,dS_u^\lambda + E^i\Big)\right].$

Our Problem

Setup: $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs B and W.

Price: $dS_t^\lambda = \lambda_t \, dt + 1 \cdot dB_t + 0 \cdot dW_t$.

Agents: $U^i(x) = -\exp(-x/\delta^i), \;\; E^i \in \mathbb{L}^\infty(\mathcal{F}_T), \;\; i = 1, \dots, I$.

Demand: $\widehat{\pi}^{\lambda,i} := \operatorname{argmax}_{\pi\in\mathcal{A}^\lambda} \mathbb{E}\left[U^i\Big(\int_0^T \pi_u \, dS_u^\lambda + E^i\Big)\right]$.

Question: Is there an *equilibrium market price of risk* λ? That is, does there exist a process λ such that the *clearing condition* $\sum_{i=1}^I \widehat{\pi}^{\lambda,i} = 0$ holds?

Our Problem

Setup: $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs B and W.

Price: $dS^\lambda_t = \lambda_t\,dt + 1\cdot dB_t + 0\cdot dW_t$.

Agents: $U^i(x) = -\exp(-x/\delta^i),\ \ E^i \in \mathbb{L}^\infty(\mathcal{F}_T),\ \ i = 1,\dots,I$.

Demand: $\widehat{\pi}^{\lambda,i} := \operatorname{argmax}_{\pi\in\mathcal{A}^\lambda} \mathbb{E}\left[U^i\Big(\int_0^T \pi_u\,dS^\lambda_u + E^i\Big)\right]$.

Question: Is there an *equilibrium market price of risk* λ? That is, does there exist a process λ such that the *clearing condition* $\sum_{i=1}^I \widehat{\pi}^{\lambda,i} = 0$ holds?

Answer: Yes, when endowments are close to Pareto-optimality.

Risk-aware Parametrisation

Consider the risk-(tolerance-)denominated quantities

$$G^i = \frac{1}{\delta^i} E^i, \quad \text{and} \quad \widehat{\rho}^{\lambda,i} = \frac{1}{\delta^i} \widehat{\pi}^{\lambda,i}.$$

Then the market clearing condition becomes

$$\sum_i \alpha^i \widehat{\rho}^{\lambda,i} = 0, \quad \text{where} \quad \alpha^i = \frac{\delta^i}{\sum_j \delta^j}.$$

Risk-aware Parametrisation

Consider the risk-(tolerance-)denominated quantities

$$G^i = \frac{1}{\delta^i} E^i, \quad \text{and} \quad \widehat{\rho}^{\lambda,i} = \frac{1}{\delta^i} \widehat{\pi}^{\lambda,i}.$$

Then the market clearing condition becomes

$$\sum_i \alpha^i \widehat{\rho}^{\lambda,i} = 0, \quad \text{where} \quad \alpha^i = \frac{\delta^i}{\sum_j \delta^j}.$$

The risk-denominated **certainty equivalent** processes are

$$Y_t^{i,\lambda} = -\log \mathbb{E}_t \left[\exp \left(-\int_t^T \widehat{\rho}_u^{\lambda,i} dS_u^\lambda - G^i \right) \right], \quad t \in [0,T].$$

A BSDE Characterization

Define the **aggregator**

$$A[\boldsymbol{x}] = \textstyle\sum_i \alpha^i x^i, \text{ for } \boldsymbol{x} = (x^i)_i.$$

Theorem. A process $\lambda \in$ bmo is an equilibrium *if and only if*

$$\lambda = A[\boldsymbol{\mu}],$$

for some solution $(\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{Y}) \in \text{bmo} \times \text{bmo} \times \mathcal{S}^\infty$ of the following *nonlinear (quadratic)* and *fully-coupled* BSDE system:

$$\begin{cases} dY_t^i = \mu_t^i \, dB_t + \nu_t^i \, dW_t + \frac{1}{2}\Big((\nu_t^i)^2 - A[\boldsymbol{\mu}_t]^2 + 2A[\boldsymbol{\mu}_t]\mu_t^i\Big)\, dt, \\ Y_T^i = G^i, \qquad i = 1, \dots, I, \end{cases}$$

where $\boldsymbol{\mu} = (\mu^i)_i$, $\boldsymbol{\nu} = (\nu^i)_i$ and $\boldsymbol{Y} = (Y^i)_i$.

Nonlinear Systems of BSDEs

- [Darling 95], [Blache 05, 06]: Harmonic maps.
- [Tang 03]: Riccati systems,
- [Tevzadze 08]: existence when terminal condition is small.
- [Delarue 02], [Cheridito-Nam 14]: generator $f + z\,g$, where both f and g are Lipschitz.
- [Hu-Tang 14]: diagonally quadratic, small-time existence.

Applications:

- [Bensoussan-Frehse 90], [El Karoui-Hamadène 03]: stochastic differential games.
- [Frei-dos Reis 11], [Frei 14]: relative performance. Counterexample: bounded terminal condition, no solution.
- [Cheridito-Horst-Kupper-Pirvu 12]: equilibrium pricing.
- [Kramkov-Pulido 14]: price impact problem.

Existence and Uniqueness "with Cheating"

Theorem 0a. An equilibrium exists and is unique if $(G^i)_i$ is an (unconstrained) Pareto-optimal allocation. Then $\lambda \equiv 0$.

Note: $(G^i)_i$ is Pareto-optimal if and only if

$$G^i - G^j \in \mathbb{R}, \text{ for all } i, j.$$

Existence and Uniqueness "with Cheating"

Theorem 0a. An equilibrium exists and is unique if $(G^i)_i$ is an (unconstrained) Pareto-optimal allocation. Then $\lambda \equiv 0$.

Note: $(G^i)_i$ is Pareto-optimal if and only if

$$G^i - G^j \in \mathbb{R}, \text{ for all } i, j.$$

Definition. $(G^i)_i$ is **pre-Pareto** if there exists an equilibrium $\lambda \in \text{bmo}$ such that the allocation

$$\widetilde{G}^i = G^i + \int_0^T \widehat{\rho}_t^{i,\lambda} dS_t^{\lambda}, \; i = 1, \ldots, I, \text{ is Pareto optimal.}$$

Obviously ...

Existence and Uniqueness "with Cheating"

Theorem 0a. An equilibrium exists and is unique if $(G^i)_i$ is an (unconstrained) Pareto-optimal allocation. Then $\lambda \equiv 0$.

Note: $(G^i)_i$ is Pareto-optimal if and only if

$$G^i - G^j \in \mathbb{R}, \text{ for all } i, j.$$

Definition. $(G^i)_i$ is **pre-Pareto** if there exists an equilibrium $\lambda \in \text{bmo}$ such that the allocation

$$\widetilde{G}^i = G^i + \int_0^T \widehat{\rho}_t^{i,\lambda} dS_t^\lambda, \; i = 1, \dots, I, \text{ is Pareto optimal.}$$

Obviously ...

Theorem 0b. An equilibrium exists if $(G^i)_i$ is pre-Pareto.

However, ...

Existence and Uniqueness "with Cheating" II

Proposition. The following statements are equivalent:

1. $(G^i)_i$ is pre-Pareto.

Existence and Uniqueness "with Cheating" II

Proposition. The following statements are equivalent:

1. $(G^i)_i$ is pre-Pareto.
2. There exists an equilibrium $\lambda \in \mathrm{bmo}$ such that

$$\widehat{\mathbb{Q}}^{\lambda,i} = \widehat{\mathbb{Q}}^{\lambda,j}, \qquad \text{for all } i, j,$$

where $\widehat{\mathbb{Q}}^{\lambda,i}$, $i = 1, \dots, I$ denote the "dual optimizers".

Existence and Uniqueness "with Cheating" II

Proposition. The following statements are equivalent:

1. $(G^i)_i$ is pre-Pareto.
2. There exists an equilibrium $\lambda \in \text{bmo}$ such that

$$\widehat{\mathbb{Q}}^{\lambda,i} = \widehat{\mathbb{Q}}^{\lambda,j}, \qquad \text{for all } i, j,$$

   where $\widehat{\mathbb{Q}}^{\lambda,i}$, $i = 1, \dots, I$ denote the "dual optimizers".
3. For λ, ν defined by

$$\exp(-\textstyle\sum_i \alpha^i G^i) \propto \mathcal{E}\left(-\int_0^\cdot \lambda_t dB_t - \int_0^\cdot \nu_t dW_t\right)_T,$$

   there exist $(y^i)_i \in \mathbb{R}^I$ and $(\varphi^i)_i \in \text{bmo}^I$ such that

$$G^i - G^j = y^i - y^j + \int_0^T (\varphi^i_t - \varphi^j_t) dS^\lambda_t, \qquad \text{for all } i, j.$$

In each of these cases, λ as above is the unique equilibrium.

Certainty Equivalents and BMO

Let $G \in \mathbb{L}^\infty$. Define

$$X_t^G = -\log \mathbb{E}_t[\exp(-G)], \quad t \in [0, T],$$

and note the dynamics

$$dX_t^G = m_t^G \, dB_t + n_t^G \, dW_t + \frac{(m_t^G)^2 + (n_t^G)^2}{2} \, dt, \quad X_T^G = G.$$

Certainty Equivalents and BMO

Let $G \in \mathbb{L}^\infty$. Define

$$X_t^G = -\log \mathbb{E}_t[\exp(-G)], \quad t \in [0, T],$$

and note the dynamics

$$dX_t^G = m_t^G \, dB_t + n_t^G \, dW_t + \frac{(m_t^G)^2 + (n_t^G)^2}{2} \, dt, \quad X_T^G = G.$$

Define also the bmo^2-norm:

$$||(m,n)||_{\mathrm{bmo}^2(\widetilde{\mathbb{P}})} = \left\| \operatorname*{ess\,sup}_{\tau} \mathbb{E}_\tau^{\widetilde{\mathbb{P}}} \left[\int_\tau^T (m_t^2 + n_t^2) dt \right] \right\|_{\mathbb{L}^\infty}^{1/2}.$$

The General "Smallness" Result

For an allocation $(G^i)_i$, with $G^i \in \mathbb{L}^\infty$ for $i = 1, \ldots, I$, we define the **distance to Pareto optimality** $H((G^i)_i)$ by

$$H((G^i)_i) = \inf_{G \in \mathbb{L}^\infty} \max_i \left|\left| \left(m^{G^i} - m^G,\, n^{G^i} - n^G \right) \right|\right|_{\mathrm{bmo}^2(\mathbb{P}^G)},$$

where $d\mathbb{P}^G / d\mathbb{P} \propto \exp(-G)$.

The General "Smallness" Result

For an allocation $(G^i)_i$, with $G^i \in \mathbb{L}^\infty$ for $i = 1, \dots, I$, we define the **distance to Pareto optimality** $H((G^i)_i)$ by

$$H((G^i)_i) = \inf_{G \in \mathbb{L}^\infty} \max_i \left|\left| \left(m^{G^i} - m^G, \, n^{G^i} - n^G \right) \right|\right|_{\mathrm{bmo}^2(\mathbb{P}^G)},$$

where $d\mathbb{P}^G / d\mathbb{P} \propto \exp(-G)$.

Theorem. An equilibrium $\lambda \in \mathrm{bmo}$ exists and is unique if

$$H((G^i)_i) < \frac{3}{2} - \sqrt{2} \approx 0.0858.$$

NB: A similar result with "distance-to-Pareto" replaced by "distance-to-pre-Pareto" holds (mutadis mutandis), with a different proof technique.

Corollaries

Corollary 1. A unique equilibrium exists if

$$(1/\delta^i)||E^i||_{\mathbb{L}^\infty} \textit{ is sufficiently small for each } i.$$

Corollaries

Corollary 1. A unique equilibrium exists if

$$(1/\delta^i)||E^i||_{\mathbb{L}^\infty} \text{ is sufficiently small for each } i.$$

Corollary 2. A unique equilibrium exists if

there are sufficiently many sufficiently homogeneous agents,

i.e., if $I \geq I(||\sum_i E^i||_{\mathbb{L}^\infty}, \min_i \delta^i, \chi^E)$, where the **endowment heterogeneity index** $\chi^E \in [0,1]$ is defined via

$$\chi^E = \max_{i,j} \frac{||E^i - E^j||_{\mathbb{L}^\infty}}{||E^i||_{\mathbb{L}^\infty} + ||E^j||_{\mathbb{L}^\infty}}.$$

Corollaries, Continued

Corollary 3. (Small time existence and uniqueness.)
A unique equilibrium exists if

$$T < T^* = \frac{\left(3/2 - \sqrt{2}\right)^2}{\max_i \left(||D^b(G^i)||_{\mathcal{S}^\infty}^2 + ||D^w(G^i)||_{\mathcal{S}^\infty}^2 \right)},$$

provided all E^i have bounded Malliavin derivatives.

▸ Movie

Future Work

1. General global existence and uniqueness (?)
2. Sensitivity analysis around Pareto optimality.
3. Long-lived securities.
4. Endowments depending also on prices.

The End

Thanks for your attendance.

P.S. Preprint available on the `arXiv`.