A Framework for Analyzing Contagion in Banking Networks

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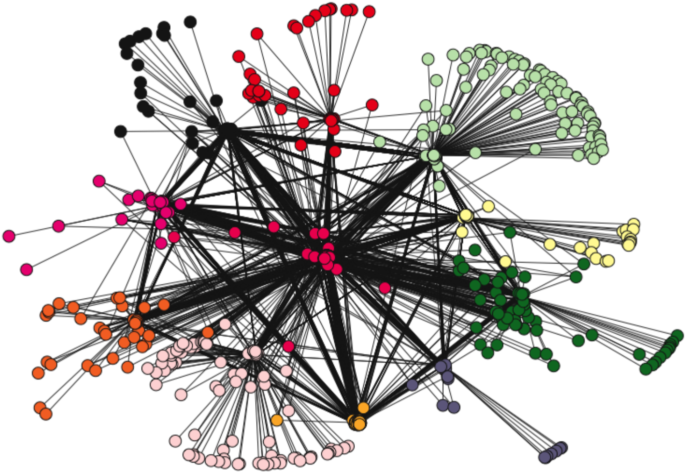
Systemic Network Risk: Overview

Andrew G Haldane's 2009 talk "Rethinking the Financial Network" is a brilliant summary of the nature of networks that compares the 2002 SARS epidemic to the 2008 collapse of Lehman Bros.

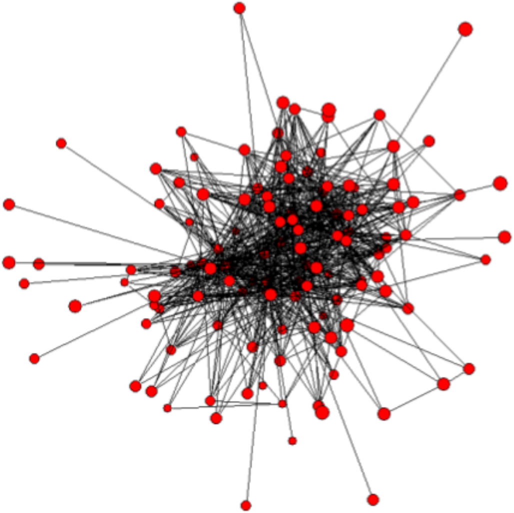
Quotation (Haldane 2009, p. 3)

Both events were manifestations of the behavior under stress of a complex, adaptive network. [...] Seizures in the electricity grid, degradation of ecosystems, the spread of epidemics and the disintegration of the financial system: each is essentially a different branch of the same network family tree.

Boss, Elsinger, Summer, Thurner: Austria 2002



Cont-Moussa-Bastos: Brazil 2007



Main Aims of this Research

- Create deliberately simplified models of systemic risk.
- Improve understanding of contagion in financial networks.
- To understand the most important determinants of financial stability, such as network connectivity, uncertainty.
- To provide analytical tools useful to regulators and policy makers.

Starting Assumptions

1. Nodes $v \in \mathcal{N}$ of the network consist of all financial institutions ("banks") in the system.
2. Edges $\ell \in \mathcal{E}$ denote the financial contracts banks exchange.
3. The system may be a single country's banks, or a larger jurisdiction (like EU).
4. Banks and their behaviour are characterized by their balance sheets.
5. No possibility of outside intervention (by governments or regulators).

Schematic Bank Balance Sheet

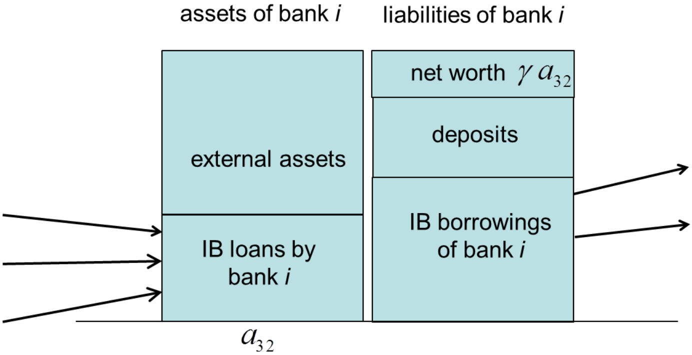


Figure: Schematic balance sheet of banks in the $(j, k) = (3, 2)$ class.

Further Assumptions

1. Banks have limited liability and become insolvent the first time their equity becomes non-positive.
2. Nonbank liabilities are senior to interbank liabilities.
3. Losses on interbank assets are shared equally across lenders.
4. Nonbank assets can be sold at their book value.
5. Contagion is only driven by domestic exposures.
6. No change in exogenous endowments during cascade.

Eisenberg-Noe 2001 Framework: Balance Sheets

Stylized financial system of N "banks":

- Assets A_v of bank v
  1. *external assets* Y_v
  2. *internal (Interbank) assets* Z_v
- Liabilities of bank v
  1. *external debts* D_v
  2. *internal (Interbank) debt* X_v
  3. *equity* or *net worth*, defined by $\gamma_v = \mathsf{Y}_v + \mathsf{Z}_v - \mathsf{D}_v - \mathsf{X}_v \geq 0$
- Interbank: W_ℓ, $\ell = (v, v')$ the amount bank v owes v'.
- Constraints

$$\mathsf{Z}_{v'} = \sum_v W_{vv'}, \quad \mathsf{X}_v = \sum_{v'} W_{vv'}, \quad \sum_{v'} \mathsf{Z}_{v'} = \sum_v \mathsf{X}_v$$

Schematic Bank Balance Sheet

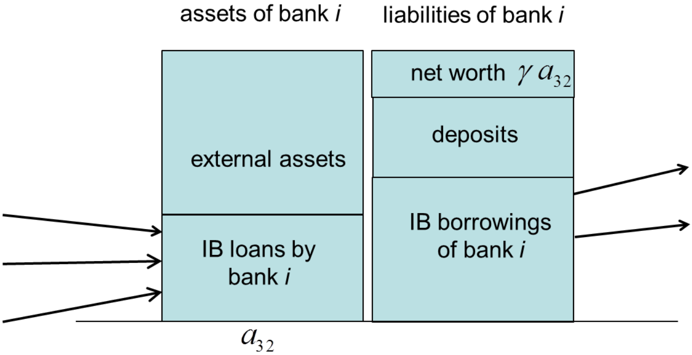


Figure: Schematic balance sheet of banks in the $(j, k) = (3, 2)$ class.

Default cascades

- Healthy banks maintain leverage ratio γ_v/A_v above a regulated value Λ_v.
- Following a bank specific catastrophic event, assets of a bank may suddenly contract by more than equity buffer γ, and bank becomes insolvent (defaulted).
- Assets of an insolvent bank must be quickly liquidated;
- Any proceeds go to pay off that bank's creditors, in order of seniority.
- Resultant shortfalls weaken creditors "downstream".
- Some further banks may default, creating a default cascade.

Two Simple Liquidation Mechanisms

p_v: amount available to pay v's internal debt at end of cascade/crisis.

- p_v is split fairly amongst creditor banks (in proportion to $\pi_{vv'} = W_{vv'}/\mathsf{X}_v$) .
- $\mathbf{p} = [p_1, \ldots, p_N]$ determined by Fixed Point Condition:

$$p_v = F_v(\mathbf{p}) := \min(\mathsf{X}_v, \max(\mathsf{Y}_v + \sum_{v'} \pi_{v'v} p_{v'} - \mathsf{D}_v, 0)), v = 1, \ldots, N$$

- Gai-Kapadia 2010 assume zero recovery at default leading to

$$p_v = F_v(\mathbf{p}) := \mathsf{X}_v \ \mathbf{1}(\mathsf{Y}_v + \sum_{v'} \pi_{v'v} p_{v'} - \mathsf{D}_v - \mathsf{X}_v > 0), v = 1, \ldots, N$$

Fixed Point Theorem

Proposition

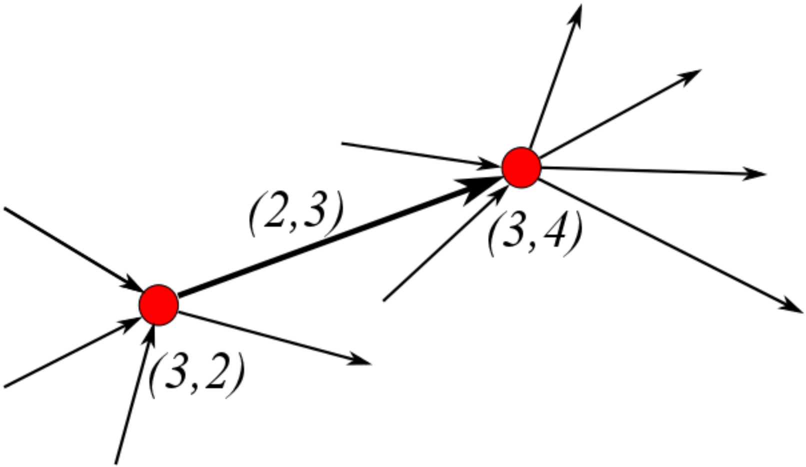
These vector mappings $F : \mathbb{R}_+^N \to \mathbb{R}_+^N$ *have at least one fixed point* $\mathbf{p}^*$.

Proof: Straightforward application of the Tarski Fixed Point Theorem. Fixed point may not be unique.

Directed Graph of size N

- Banks: set of nodes or vertices $\mathcal{N} = \{1, \dots, N\}$, numbered by integers.
- Interbank lending: set of possible directed edges or links $\mathcal{N} \times \mathcal{N}$.
- A graph $\mathcal{E}$ is an arbitrary subset $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$.
- We write v, v' etc for vertices, ℓ, ℓ' etc for links.
- $v' \in \mathcal{N}_v^+$ means "v' is exposed to v";
- $v' \in \mathcal{N}_v^-$ means "v' owes to v";

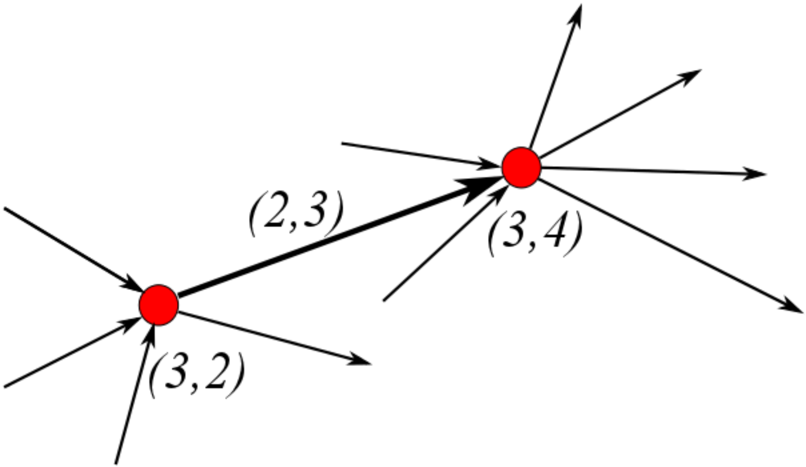
2 Nodes and 1 Edge



"+" means "out" and "−" means "in"

- $v \in \mathcal{N}_{jk}$ (i.e. has type (j, k)) if $\deg^-(v) = j$ and $\deg^+(v) = k$.
- $\ell \in \mathcal{E}_{kj}$ (i.e. has type (k, j)) if $\deg^-(\ell) = j$ and $\deg^+(\ell) = k$.

2 Nodes and 1 Edge



Random Graph Ensembles

The "skeleton" of the network is a random graph $\mathcal{E}$ characterized by:

- Node-type probability distribution: $P_{jk} = \mathbb{P}[v \in \mathcal{N}_{jk}]$.
- Edge-type distribution: $Q_{kj} = \mathbb{P}[\ell \in \mathcal{E}_{kj}]$.
- Dependence structure.

In particular we also have

- Marginals: $P^+_k = \sum_j P_{jk}$; also P^-_j, Q^+_k, Q^-_j.
- Mean degree: $z = \sum_{jk} k P_{jk} = \sum_{jk} j P_{jk}$.
- Edge-Assortativity: $Q - Q^+Q^- > 0$ means high degree nodes more likely to connect to high degree nodes.

Extended Gai-Kapadia 2010 Solvency Model (Hurd-Gleeson 2011 + recent work)

Financial network of IB exposures (similar to EN 2001).

- Random directed skeleton graph $\mathcal{E}$
- Random balance sheets: external assets Y_v and external liabilities D_v
- Random link weights: W_ℓ
- Solvency conditions:

$$\gamma_v = \mathsf{Y}_v + \sum_{v' \in \mathcal{N}_v^-} W_{v'v} - \mathsf{D}_v - \sum_{v' \in \mathcal{N}_v^+} W_{vv'} > 0$$

- Initial defaults: random set $\mathcal{M}_0 \subset \mathcal{N}$ of nodes have $\gamma_v \leq 0$.
- Assuming recovery fraction $R \leq 1$, a node will be insolvent after n steps of the cascade if

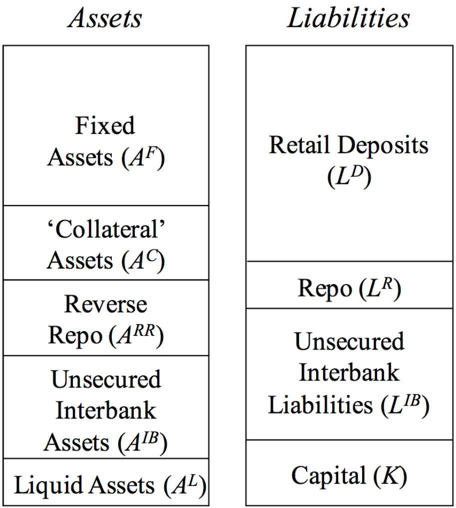
$$\gamma_v \leq (1 - R) \sum_{v' \in \mathcal{N}_v^- \cap \mathcal{M}_{n-1}} W_{v'v}$$

Liquidity Hoarding

Gai-Haldane-Kapadia 2011 introduce a model of illiquidity stress:

- Excessive illiquidity that hits a given bank creates "stress";
- The natural reaction of a stressed bank is to "delever", or shrink the balance sheet.
- This will shock the liability side of each of its debtor banks.
- Under some circumstances, such "upstream" shocks can cause further illiquidity stresses.
- These shocks may build up to create a global illiquidity cascade.

Schematic Balance Sheet (from GHK 2011)



Extended Gai-Haldane Kapadia 2011 Liquidity Model

Financial network of IB exposures

- Random directed skeleton graph $\mathcal{E}$
- Random balance sheets
- Random link weights: W_ℓ
- Stress conditions:

$$\beta_v = A_v^L + (1-h)[A_v^c + A_v^{RR}] - L_v^R - \sum_{v' \in \mathcal{N}_v^+} W_{vv'} > 0$$

- Initially stressed banks: random set $\mathcal{M}_0 \subset \mathcal{N}$ of nodes have $\beta_v \leq 0$.
- Assuming stress reaction fraction $\lambda \leq 1$, a node will be stressed after n steps of the cascade if

$$\beta_v \leq \lambda \sum_{v' \in \mathcal{N}_v^+ \cap \mathcal{M}_{n-1}} W_{vv'}$$

Random Financial Network (RFN)

...is a triple $(\mathcal{E}, \Gamma, W)$ where

- $\mathcal{E}$ is a directed random graph (the "skeleton");
- $\Gamma = (\Gamma_v)_{v \in \mathcal{N}}$ is the set of "random buffers";
- $W = (W_\ell)_{\ell \in \mathcal{E}}$ is the set of random interbank exposures.

Γ_v, W_ℓ may be multidimensional. Insolvent (or stressed) banks $v \in \mathcal{M}$ have $\Gamma_v \leq 0$.

LTIA: Locally Tree-like Independence Assumption

$N = \infty$ configuration graphs have the locally tree-like (LT) property: cycles of any fixed finite length occur only with zero probability. We extend this notion to RFNs:

Assumption

LT **independence assumption**

Generic Cascade Theorem (Schematic)

Let RFN $(\mathcal{E}, \Gamma, W)$ on $\mathcal{N}$ satisfy LTIA. For each pair $\ell = (v, v') \in \mathcal{E}$ and cascade step number $n \geq 0$ define Random Variables (RVs)

$$\begin{aligned} \tilde{W}^n_{v,v'} &= W_{v,v'} \, \mathbf{1}(v \in \mathcal{M}_n \text{ WOR } v') \\ \tilde{\Gamma}^n_{v,v'} &= \Gamma_v - \sum_{v'' \in \mathcal{N}_v \setminus v'} \tilde{W}^{n-1}_{v'',v} \end{aligned}$$

where $v \in \mathcal{M}_n$ WOR v' means $\tilde{\Gamma}^n_{v,v'} \leq 0$.

- Then the $n+1$st step of the cascade maps WOR RVs to WOR RVs.
- If $(\mathcal{E}, \tilde{\Gamma}^n, \tilde{W}^n)$ satisfies the LTIA, then so does $(\mathcal{E}, \tilde{\Gamma}^{n+1}, \tilde{W}^{n+1})$.

NB: Without Regarding (WOR) v' needs more explaining!

Consequences of Cascade Theorem

- The Cascade Theorem applies to Gai-Kapadia 2010, Gai-Haldane-Kapadia 2011 and more complex models.
- The cascade mapping is monotonic and bounded, hence converges to a fixed point as $n \to \infty$.
- The distributions of $\tilde{W}^n_{v,v'}, \tilde{\Gamma}^n_{v,v'}$ can be characterized inductively.
- Efficient "exact" numerical implementations are possible in case the RVs take values on a fixed grid $\{0, 1, \ldots, M\}$.
- Algorithm makes intensive use of the Fast Fourier Transform (FFT).

Observed Skeleton Graph Disassortativity: Is It Important?

- Edge-assortativity: Pearson correlation of matrix Q_{kj};
- Node-assortativity: Pearson correlation of matrix P_{jk};
- Graph-assortativity r: Pearson correlation of matrix

$$B_{jj'} = \sum_k \frac{P_{jk} Q_{kj'}}{P_k^+} = \mathbb{P}[j_v = j, j_{v'} = j' | v' \in \mathcal{N}_v^+]$$

Two Parameter GK2010 Model: Testing Disassortativity

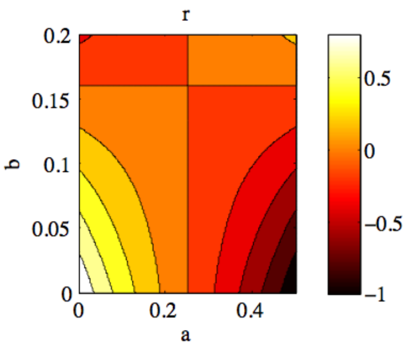


Figure: Graph Assortativity r and Frequency of Global Cascades f

Real World Networks

- Any finite size deterministic network fits into the stochastic framework.
- LTIA and hence Cascade Theorem may be approximately true.
- As global IB network data comes available, we can use these tools in the study of actual networks.
- It is important to know how well or badly the LTIA holds.

LTIA: Does it Work?

LTIA is exactly true in

1. $N = \infty$ configuration models;
2. $N < \infty$ tree graph models;
3. $N < \infty$ deterministic models.