Dirichlet Forms and Finite Element Methods for the SABR Model

Blanka Horvath

7th General AMaMeF and Swissquote Conference
September 10, 2015

Based on joint work with Oleg Reichmann

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Outline

- Aim: Suggest an analytic framework and a numerical pricing method for vanilla options the SABR model which is
  - valid in the critical ultra low rate regimes prevalent today
  - applicable with mild restrictions on the model parameters
  - allows for an error analysis
  - easily extendible to more complex contracts
- Approaches to the SABR model considered so far
- Challenges arising at interest rates near zero

  exemplified in option prices via heat kernel methods
- A functional analytic viewpoint and a finite element method for the SABR model

The SABR Model

The benchmark SABR (stochastic α,β,ρ) model was designed by Hagan, Kumar, Lesniewski and Woodward at Bear Stearns and BNP Paribas in the early 2000's.

The model with parameters $\alpha > 0$, $\beta \in [0, 1]$, and $\rho \in [-1, 1]$ is defined on a space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ by the SDE

$$\begin{aligned} \mathrm{d}X_t &= Y_t X_t^{\beta} \mathrm{d}W_t, & X_0 &= x > 0, \\ \mathrm{d}Y_t &= \alpha Y_t \mathrm{d}Z_t, & Y_0 &= y > 0, \\ \mathrm{d}\langle Z, W\rangle_t &= \rho \mathrm{d}t, & 0 &\leq t \leq T < \infty, \end{aligned}$$

where W, Z are ρ-correlated Brownian motions.

The asset follows a CEV process with a lognormal volatility.

Approaches to the SABR model

$$\begin{aligned} \mathrm{d}X_t &= Y_t X_t^{\beta}\mathrm{d}W_t \\ \mathrm{d}Y_t &= \alpha Y_t \mathrm{d}Z_t \\ \mathrm{d}\langle Z, W\rangle_t &= \rho \mathrm{d}t \end{aligned} \qquad \begin{aligned} \dot{u} &= y^2\left(x^{2\beta}\partial^2_{xx} + 2\rho\alpha x^{\beta}\partial^2_{xy} + \alpha^2\partial^2_{yy}\right)u \\ u(0;x,y) &= u_0(x,y) \end{aligned}$$

- Heat kernel methods: approximation of the fundamental solution of the Kolmogorov equation via (short time) asymptotic expansions [Hagan, Lesniewski, Woodward '03], [Henry-Labordère '08], [Paulot '07], . . .
- Monte Carlo methods: path simulation of the process combined with suitable Monte Carlo approximation [Chen, Oosterlee, van der Weide '11]
- Splitting methods: the infinitesimal generator of the process is decomposed into suitable operators for which the pricing equations can be computed more efficiently [Bayer, Friz, Loeffen '13]
- PDE methods: finite differences [Hagan, Kumar, Lesniewski, Woodward '14], [Le Floc'h, Kennedy '14]

SABR's popularity is largely due to the parsimonious formula for the Black implied volatility $\sigma_{K,T}^{imp}$, derived by Hagan, Kumar, Lesniewski, Woodward in the early 2000 and justified via heat kernel methods in [Hagan, Lesniewski, Woodward, '03]:

$$\sigma_{K,T}^{imp}(x,y,\alpha,\beta,\rho) =$$

$$= \frac{y\log(x/K)}{\left(\frac{x^{1-\beta}-K^{1-\beta}}{1-\beta}\right)}\left(\frac{\zeta}{\hat{\chi}(\zeta)}\right)\left\{1+\left[\frac{2\frac{\beta(\beta-1)}{x_{av}^2}-\left(\frac{\beta}{x_{av}}\right)^2+\frac{1}{x_{av}^2}}{24}y^2x_{av}^{2\beta}\right.\right.$$

$$\left.\left.+\frac{1}{4}\rho\alpha y\frac{\beta}{x_{av}}x_{av}^{\beta}+\frac{2-3\rho^2}{24}\alpha^2\right]T+\dots\right\}$$

with the abbreviations

$$\hat{\chi}(\zeta) := \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right)$$

such as $x_{av} := \sqrt{xK}$, $\gamma_1 := \frac{\beta}{x_{av}}$, $\gamma_2 := \frac{\beta(\beta-1)}{x_{av}^2}$ and $\zeta := \frac{\alpha}{\alpha}\frac{x-K}{x^\beta}$.

For European contracts on a forward with exercise date T and settlement date T_{set} the Black implied volatility $\sigma_{K,T}^{imp}$ solves

$$C_B(T, T_{set}, x, K, \sigma) = D(T_{set})C(K, T)$$

- $D(T_{set})$ a discount factor, $C(K, T)$ is the price of a call option given by the market at strike K and maturity T
- C_B stands for the Black formula for a call option

$$C_B(T, T_{set}, x, K, \sigma) = D(T_{set})\left(x\Phi(d_1) - K\Phi(d_2)\right),$$

with $d_{1,2} = \frac{\log(x/K) \pm \sigma^2 T}{\sigma\sqrt{T}}$.

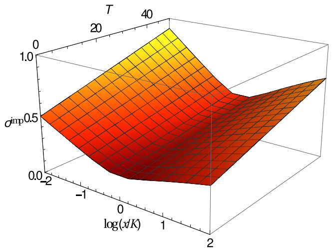


Figure: SABR implied volatility surface $(y, \alpha, \beta, \rho) = (0.3, 0.45, 0.5, -0.25)$

The SABR model quickly became ubiquitous in fixed income markets in the early 2000s

- the implied volatility formula allowed for close fits to to the observed market prices for the interest rates around 5% base rates which were prevalent at the time.
- As markets started moving towards low interest rates, the formula exhibited irregularities:

  in the ultra-low rate regime we are facing today, the formula turns out to be erroneous

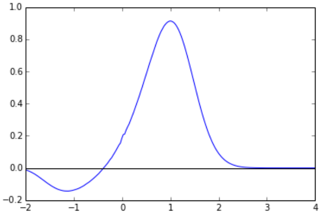


Figure: SABR densities obtained from the implied volatility surface

- The SABR formula was derived by heat kernel techniques: The short time asymptotic behavior of the SABR probability density was characterized in the leading order by one of the iconic results in the theory of second-order elliptic operators

Varadhan's Formula '67

$$\lim_{t \to 0} t \log p_t(x_1, y_1; x_2, y_2) = -\frac{d(x_1, y_1; x_2, y_2)^2}{2}$$

- For $u \in C^{1,2}(J; \mathbb{R}_{\geq 0} \times \mathbb{R}) \cap C^0(\bar{J}; \mathbb{R}_{\geq 0} \times \mathbb{R})$ the Kolmogorov pricing equation to SABR is

$$\begin{aligned} \dot{u}(t; x, y) &= Au(t; x, y) \quad \text{in } J \times \mathbb{R}_{\geq 0} \times \mathbb{R}, \\ u(0; x, y) &= u_0(x, y) \qquad \text{in } \mathbb{R}_{\geq 0} \times \mathbb{R} \end{aligned}$$

where the infinitesimal generator A of reads

$$Au(t; x, y) = y^2 \left(x^{2\beta} \partial^2_{xx} + 2\rho\alpha x^\beta \partial^2_{xy} + \alpha^2 \partial^2_{yy} \right) u(t; x, y).$$

Why Dirichlet forms?

- Extensions of asymptotic results in this spirit to subelliptic and hypoelliptic operators are available: corresponding results were obtained by [Deuschel, Friz, Jacquier and Violante '14], [De Marco, Friz '13], [Gulisashvili, Laurence '14], ...
- Extension to CEV-type degeneracies with accessible boundary is not obvious: Symmetric Dirichlet forms extend this framework to allow for such degeneracies in the generator. Results so far in [Hino, Ramirez '03], and for a related univariate diffusions in [ter Elst, Robinson, Sikora '07]
- Symmetric Dirichlet forms for SABR and the related Dirichlet geometry are studied in [Döring, H., Teichmann '15]: $\Rightarrow$ Symmetric Dirichlet forms can be derived for specific parameter regimes, in general non-symmetric.

Approaches to the SABR model

$$\begin{aligned} \mathrm{d}X_t &= Y_t X_t^{\beta} \mathrm{d}W_t \\ \mathrm{d}Y_t &= \alpha Y_t \mathrm{d}Z_t \\ \mathrm{d}\langle Z, W\rangle_t &= \rho \mathrm{d}t \end{aligned} \qquad \begin{aligned} \dot{u} &= y^2 \left(x^{2\beta}\partial_{xx}^2 + 2\rho\alpha x^{\beta}\partial_{xy}^2 + \alpha^2\partial_{yy}^2\right) u \\ u(0; x, y) &= u_0(x, y) \end{aligned}$$

Heat kernel methods: approximation of the fundamental solution of the Kolmogorov equation via (short time) asymptotic expansions [Hagan, Lesniewski, Woodward '03], [Henry-Labordére '08], [Paulot '07], ...
Monte Carlo methods: path simulation of the process combined with suitable Monte Carlo approximation [Chen, Oosterlee, Weide '11]
Splitting methods: the infinitesimal generator of the process is decomposed into suitable operators for which the pricing equations can be computed more efficiently [Bayer, Friz, Loeffen. '13]
PDE methods: finite differences [Hagan, Kumar, Lesniewski, Woodward '14], [Le Floc'h, Kennedy '14]
finite element methods?

Finite Element Method for SABR

- FEM provide a robust and flexible framework in the context of non-symmetric Dirichlet forms.
- They are well adapted to degeneracies, but in spite of this they did not appear in the context of SABR so far.
- FEM in the context of mathematical finance: [Wilmott, Howison, Dewynne '95]
  applied to a large class of financial models in a recent textbook [Hilber, Reichmann, Schwab, Winter '13],
  see also [Reichmann, Schwab '13], [Matache, Schwab '04], [von Petersdorff, Schwab '03]...

Finite Element Method for SABR

Baseline: instead of discretizing the Kolmogorov PDE

$$\dot{u} = Au. \tag{1}$$

directly, pass to a variational formulation: For this, specify an appropriate functional analytic framework i.e. a (dense) triplet

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$$

for a vector space $D(A) \subset \mathcal{V}$ and a Hilbert space $\mathcal{H}$.
Extend the scalar product $(\cdot,\cdot)_{\mathcal{H}}$ to a dual pairing $(\cdot,\cdot)_{\mathcal{V}^*\times\mathcal{V}}$.
Variational (re-)formulation of (1):

$$(\dot{u}, v)_{\mathcal{V}^*\times\mathcal{V}} = (Au, v)_{\mathcal{V}^*\times\mathcal{V}}, \quad v \in \mathcal{V}.$$

Then localize (area of interest), and discretize in space and time.

Advantage: Allows for solutions with less regularity than (1), one can choose $\mathcal{V}$, $\mathcal{H}$ *suitable* to the degeneracies of A.

Weak formulation

Define a bilinearform on $\mathcal{V}$: $\quad a(u,v) := -(Au,v)_{\mathcal{V}^* \times \mathcal{V}}, \quad u,v \in \mathcal{V}$
The weak fomulation then reads: Find $u \in L^2(J;\mathcal{V}) \cap H^1(J;\mathcal{V}^*)$ such that $u(0) = u_0$, and $(\dot{u},v)_{\mathcal{V}^* \times \mathcal{V}} = -a(u,v)$ for all $v \in \mathcal{V}$.

Well-posedness: If there exist $C_1, C_2 > 0$, $C_3 \geq 0$

$$\forall u,v \in \mathcal{V}: \quad |a(u,v)| \leq C_1 ||u||_{\mathcal{V}} ||v||_{\mathcal{V}}$$
$$\forall u \in \mathcal{V}: \quad |a(u,u)| \geq C_2 ||u||^2_{\mathcal{V}} - C_3 ||u||^2_{\mathcal{H}}.$$

Then: there exists a unique solution of the (weak) PDE in $\mathcal{V}$, and A is the infinitesimal generator of a C^0 semigroup $(P_t)_{t\geq 0}$ in $\mathcal{H}$, the unique solution of the weak PDE can be represented as

$$u(t) = P_t(u_0), \quad t \geq 0.$$

A priori $\mathcal{V}$-norm estimates for u are possible $\Rightarrow$ Error analysis.

Analytic Setting for SABR

For convenience, pass to logarithmic volatility

$$
\begin{aligned}
dX_t &= X_t^\beta Y_t dW_t = X_t^\beta \exp(\tilde{Y}_t) dW_t \qquad & X_0 = x > 0, \\
d\tilde{Y}_t &= \alpha dZ_t - \alpha^2 \tfrac{1}{2} dt & \tilde{Y}_0 = \log y,\ y_0 > 0 \\
d\langle W, Z\rangle_t &= \rho dt.
\end{aligned}
$$

Localize the state space to a bounded domain $G \subset \mathbb{R}_{\geq 0} \times \mathbb{R}$.

The area of interest is $G = [0, R_x] \times [-R_y, R_y]$, $R_x, R_y > 0$

Localization: Similar to approximating vanilla options with barrier options. A probabilistic argument to estimate the localization error was given in [Cont, Voltchkova '05].

Origin necessarily should be included in the forward dimension. Mass at zero in the SABR disribution: computations of this [Gulisashvili, H. Jacquier '15].

Gelfand triplet $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ for SABR

- Consider $G = [0, R_x) \times (-R_y, R_y) \subset \mathbb{R}_+ \times \mathbb{R}, \quad R_x, R_y > 0$ and for any $\beta \in [0,1]$ $\mu \in [\max\{-1, -2\beta\}, 1-2\beta]$, the space

$$\begin{aligned} \mathcal{H} := \mathcal{L}^2(G, x^{\mu/2}) \quad &= \{u : G \to \mathbb{R} \text{ mb} \mid ||u||_{\mathcal{L}^2(G,x^{\mu/2})} < \infty\} \text{ where} \\ ||u||_{\mathcal{L}^2(G,x^{\mu/2})} \quad &:= \left(\int_G |u(x,y)|^2 x^\mu dx\, dy\right)^{1/2} \end{aligned}$$

- Consider on G the functions $\mathcal{C}_0^\infty(G)$ and define

$$||u||_{\mathcal{V}}^2 := ||x^{\beta+\mu/2}\partial_x u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}u||^2_{\mathcal{L}^2(G)},$$

$$\mathcal{V} := \overline{\mathcal{C}_0^\infty(G)}^{||\cdot||_{\mathcal{V}}}$$

- SABR bilinear form $(a(\cdot,\cdot), \mathcal{V})$ on $\mathcal{H}$

$$\begin{aligned} a(u,v) = &\tfrac{1}{2}\int_G x^{2\beta+\mu} e^{2y}\partial_x u \partial_x v dxdy + \tfrac{2\beta+\mu}{2}\int_G x^{2\beta+\mu-1} e^{2y}\partial_x u v dxdy \\ &+ \rho\alpha \int_G x^{\beta+\mu} e^y \partial_x u \partial_y v dxdy + \rho\alpha \int_G x^{\beta+\mu} e^y \partial_x u v dxdy \\ &+ \tfrac{\alpha^2}{2}\int\int_G x^\mu \partial_y u \partial_y v dxdy - \tfrac{\alpha^2}{2}\int_G x^\mu \partial_y u v dxdy \end{aligned}$$

The spaces $\mathcal{H}$, $\mathcal{V}$ and the SABR bilinear form $a(\cdot, \cdot)$ consistently extend the analytic setting for the CEV model ($\alpha = 0$), presented in [Hilber, Reichmann, Schwab, Winter '13]

Advantage of FEM in contrast to probabilistic methods:
Passing from the univariate (CEV) to the bivariate (SABR) case

- in probabilistic methods typically relies on the assumption $\rho = 0$
- in the finite element method this can be relaxed to $|\rho|\alpha^2 < 2$.

Well-posedness of the SABR pricing equation

For every configuration $(\beta, |\rho|, \alpha) \in [0,1] \times [0,1] \times \mathbb{R}_+$ of the SABR parameters, which satisfy the condition $|\rho|\alpha^2 < 2$

the weak formulation of the SABR pricing equation admits a unique solution $u \in L^2(J, \mathcal{V}) \cap H^1(J, \mathcal{V}^*)$, $J = [0, T]$.

This unique solution can be represented for any u_0 in $\mathcal{H}$ as

$$u(t,z) = P_t u_0(z), \qquad t \geq 0, z \in G$$

for a C^0 semigroup $(P_t)_{t \geq 0}$ on $\mathcal{H}$ with generator A.

Dirichlet form for the full SABR model

The pair $(a(\cdot,\cdot), \mathcal{V})$ is a (non-symmetric) Dirichlet form on the Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ for every $(\beta, |\rho|, \alpha) \in [0,1] \times [0,1] \times \mathbb{R}_+$ with $|\rho|\alpha^2 < 2$, and for any $\mu \in [\max\{-1, -2\beta\}, 1 - 2\beta]$.

Discussion of the parameter restrictions

- The parameter restriction

$$|\rho|\alpha^2 < 2 \qquad (2)$$

induced by well-posedness (Gårding inequality)

$$a(u,u) \quad \geq C_2||u||^2_{\mathcal{V}} - C_3||u||^2_{\mathcal{H}}$$

Quick reasoning: for some $\delta > 0$ and $\varepsilon > 0$

$$\begin{aligned} a(u,u) \quad &\geq \left(\frac{1}{2} - \frac{|\rho|\alpha^3}{4\delta} - \frac{|\rho|\alpha^3\varepsilon}{4}\right)||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2} \\ &+ \left(\frac{\alpha^2}{2} - \frac{|\rho|\alpha^3\delta}{4}\right)||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2} - \frac{|\rho|\alpha^3}{4\varepsilon}||x^{\mu/2}u||^2_{\mathcal{L}^2} \end{aligned}$$

$$\begin{aligned} &= C_2\left(||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2} + ||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2} + ||x^{\mu/2}u||^2_{\mathcal{L}^2}\right) \\ &= \quad -C_3||x^{\mu/2}u||^2_{\mathcal{L}^2} \end{aligned}$$

- Mild restriction: α typically calibrates to values $\alpha < 1$.
  In this case (2) is uniformly fulfilled for all $\rho \in [-1,1]$.

Space discretization: The semidiscrete problem

Let $u_{(0,L)} = P_L u_0$, were $P_L : \mathcal{V} \to \mathcal{V}_L$, $u \longmapsto u_L$ is a projection to a finite dimensional approximation space $\mathcal{V}_L$. Find $u_L \in H^1(J; \mathcal{V}_L)$, such that

$$u_L(0) = u_{(0,L)}, \qquad (\tfrac{d}{dt} u_L, v_L)_{\mathcal{V}^* \times \mathcal{V}} = -a(u_L, v_L), \quad \forall v_L \in \mathcal{V}_L.$$

The fully discrete scheme

Given the initial data $u_L^0 := u_{(0,L)} = P_L u^0$, and for $m = 0, \ldots, M-1$ the following uniform time mesh

$$k := \tfrac{T}{M}, \qquad \text{and} \qquad t^m = mk,\ m = 0, \ldots, M.$$

find $u_L^{m+1} \in \mathcal{V}_L$ such that for all $v_L \in \mathcal{V}_L$:

$$\tfrac{1}{k}(u_L^{m+1} - u_L^m, v_L)_{\mathcal{V}^* \times \mathcal{V}} = -a(\theta u_L^{m+1} + (1-\theta) u_L^m, v_L).$$

The full discretization space is constructed as the tensor product of univariate discretization spaces

$$\mathcal{V}_L := V_{L_x} \otimes V_{L_y}.$$

In each dimension we have a multiresolution analysis on $I \subset \mathbb{R}$ of a nested family of spaces

$$V^0 \subset V^1 \subset \ldots \subset V^{N^L+1} = V_L \subset \ldots \subset L^2(I, \omega), \qquad (3)$$

L denotes the discretization level and $\overline{\bigcup_{l \in \mathbb{N}} V^l} = L^2(I, \omega)$.

$L^2(I, \omega)$ is a space of square integrable functions with weight ω.
$\Rightarrow$ We use and the weighted norm equivalences via wavelets described in [Beuchler, Schneider, Schwab '04].

The fully discrete finite element scheme for the SABR model reads

$$(\tfrac{1}{k}\mathbf{M} + \theta\mathbf{A})\underline{u}^{m+1} = \tfrac{1}{k}\mathbf{M}\underline{u}^{m} - (1-\theta)\mathbf{A}\underline{u}^{m}, \quad m = 0, 1, \ldots, M-1,$$

where $\mathbf{M}$ denotes the mass matrix, $\mathbf{A}$ the stiffness matrix, and $\underline{u}^m$ is the coefficient matrix of u_L^m with respect to the basis of $\mathcal{V}_L$.

The mass matrix reads

$$\mathbf{M} = \mathbf{M}^x_{x^\mu} \otimes \mathbf{M}^y_1, \tag{4}$$

and the stiffness matrix $\mathbf{A}$ takes the form

$$\begin{aligned}\mathbf{A} = \frac{1}{2} & \left(\mathcal{Q}_{xx}\mathbf{S}^x_{x^{2\beta+\mu}} \otimes \mathbf{M}^y_{e^{2y}} + \mathcal{Q}_{yy}\mathbf{M}^x_{x^\mu} \otimes \mathbf{S}^y\right) \\ & + \left(\mathcal{Q}_{xy}\mathbf{B}^x_{x^{\beta+\mu}} \otimes \mathbf{B}^y_{e^y}\right) \\ & + \left(c_{x_1}\mathbf{B}^x_{x^{2\beta+\mu-1}} \otimes \mathbf{M}^y_{e^{2y}} + c_{x_2}\mathbf{B}^x_{x^{\beta+\mu}} \otimes \mathbf{M}^y_{e^y} + c_y\mathbf{M}^x_{x^\mu} \otimes \mathbf{B}^y\right),\end{aligned} \tag{5}$$

where the coefficients are $(\mathcal{Q}_{xx}, \mathcal{Q}_{yy}, \mathcal{Q}_{xy}) = (1, \alpha^2, 2\rho\alpha)$ and $(c_{x_1}, c_{x_2}, c_y) = (2\beta+\mu, 2\rho\alpha, -\alpha^2)$.

Recall: Gelfand triplet $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ for SABR

Consider $G = [0, R_x) \times (-R_y, R_y) \subset \mathbb{R}_+ \times \mathbb{R}$, $\quad R_x, R_y > 0$ and for any $\beta \in [0, 1]$ $\mu \in [\max\{-1, -2\beta\}, 1 - 2\beta]$, the space

$$\mathcal{H} := \mathcal{L}^2(G, x^{\mu/2}) \quad = \{u : G \to \mathbb{R} \text{ mb} \mid ||u||_{\mathcal{L}^2(G,x^{\mu/2})} < \infty\} \text{ where}$$
$$||u||_{\mathcal{L}^2(G,x^{\mu/2})} \quad := \left(\int_G |u(x,y)|^2 x^\mu dx\, dy\right)^{1/2}$$

Consider on G the functions $\mathcal{C}_0^\infty(G)$ and define

$$||u||_{\mathcal{V}}^2 := ||x^{\beta+\mu/2}\partial_x u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}u||^2_{\mathcal{L}^2(G)},$$

$$\mathcal{V} := \overline{\mathcal{C}_0^\infty(G)}^{||\cdot||_{\mathcal{V}}}$$

SABR bilinear form $(a(\cdot,\cdot), \mathcal{V})$ on $\mathcal{H}$

$$a(u,v) = \frac{1}{2}\int_G x^{2\beta+\mu} e^{2y} \partial_x u \partial_x v dxdy + \frac{2\beta+\mu}{2}\int_G x^{2\beta+\mu-1} e^{2y}\partial_x u v dxdy$$
$$+ \rho\alpha \int_G x^{\beta+\mu} e^y \partial_x u \partial_y v dxdy + \rho\alpha \int_G x^{\beta+\mu} e^y \partial_x u v dxdy$$
$$+ \frac{\alpha^2}{2}\int\int_G x^\mu \partial_y u \partial_y v dxdy - \frac{\alpha^2}{2}\int_G x^\mu \partial_y u v dxdy$$

Remark on the error analysis

- Consider $u^m(x) := u(t^m, x)$, with t^m, $m = 0, \ldots, M$ and u_L^m as in the fully discrete scheme. Decomposition of total FEM error:

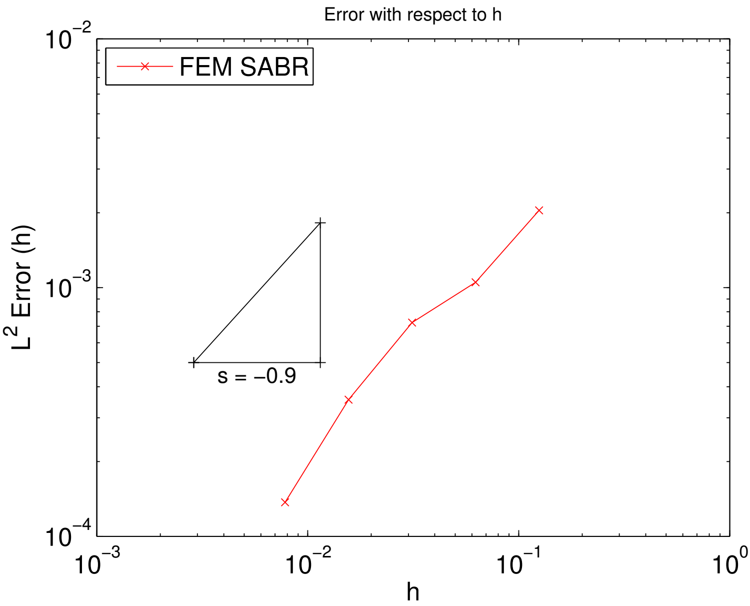
$$e_L^m := u^m(x) - u_L^m(x) = (u^m - P_L u^m) + (P_L u^m - u_L^m) =: \eta^m + \xi_L^m,$$

  for the the time-points t^m, $m = 0, \ldots, M$, where $P_L : \mathcal{V} \to \mathcal{V}_L$ denotes the projection on the finite dimensional approximation space $\mathcal{V}_L$.

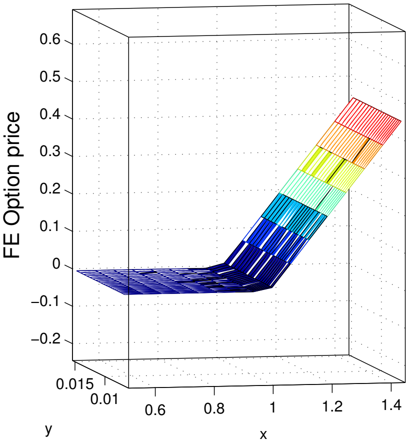
- Approximation error η^m satisfies for $m = 0, \ldots, M$

$$||u - P_L u||^2_{L^2(I, x^{\mu/2})} \leq Ch||u||^2_{H^1(I, x^{\mu/2})},$$

  in each dimension for some $C > 0$, where we have set $h = 2^{-2L}$.



Error with respect to h
FEM SABR
s = −0.9
L^2 Error (h)
h
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}


FE Option price
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
0.015
0.01
y
0.6
0.8
1
1.2
1.4
x

Conclusions

- The presented finite element method provides a means for numerical option pricing for under SABR model, and allows for error an analysis.
- This method is applicable in the critical ultra low rate regimes we are facing today.
- Only mild assumptions on the model parameters are needed and these are easily met in all pracical pricing scenarios.

Thank you for your attention!