Nonlinear Price Impact and Portfolio Choice

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Outline

- Motivation:
  Optimal Rebalancing and Execution.
- Model:
  Nonlinear Price Impact.
  Constant investment opportunities and risk aversion.
- Results:
  Optimal policy and welfare. Implications.

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Price Impact and Market Frictions

- Classical theory: no price impact.
  Same price for any quantity bought or sold.
  Merton (1969) and many others.
- Bid-ask spread: constant (proportional) “impact”.
  Price depends only on sign of trade.
  Constantinides (1985), Davis and Norman (1990), and extensions.
- Price linear in trading rate.
  Asymmetric information equilibria (Kyle, 1985), (Back, 1992).
  Quadratic transaction costs (Garleanu and Pedersen, 2013)
- Price nonlinear in trading rate.
  Square-root rule: Loeb (1983), BARRA (1997), Grinold and Kahn (2000).
  Empirical evidence: Hasbrouck and Seppi (2001), Plerou et al. (2002), Lillo et al. (2003), Almgren et al. (2005).
- Literature on nonlinear impact focuses on optimal execution.
  Portfolio choice?

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Portfolio Choice with Frictions

- With constant investment opportunities and constant relative risk aversion:
- Classical theory: hold portfolio weights constant at Merton target.
- Proportional bid-ask spreads:
  hold portfolio weight within buy and sell boundaries (no-trade region).
- Linear impact:
  trading rate proportional to distance from target.
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- Inputs
  - Price exogenous. Geometric Brownian Motion.
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  - Nonlinear price impact:
    trading rate one-percent higher means impact α-percent higher.
- Outputs
  - Optimal trading policy and welfare.
  - High liquidity asymptotics.
  - Linear impact and bid-ask spreads as extreme cases.
- Focus is on temporary price impact:
  - No permanent impact as in Huberman and Stanzl (2004)
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Market

- Brownian Motion $(W_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$.
- *Best quoted* price of risky asset. Price for an infinitesimal trade.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

- Trade $\Delta\theta$ shares over time interval Δt. Order filled at price

$$\tilde{S}_t(\Delta\theta) := S_t \left(1 + \lambda \left|\frac{S_t \Delta\theta_t}{X_t \Delta t}\right|^{\alpha} \operatorname{sgn}(\dot{\theta})\right)$$

  where X_t is investor's wealth. Proxies total market's wealth.
- λ measures illiquidity. $1/\lambda$ market depth. Like Kyle's (1985) lambda.
- Price worse for larger quantity $|\Delta\theta|$ or shorter execution time Δt.
  Price linear in quantity, inversely proportional to execution time.
- Impact of dollar trade $S_t\Delta\theta$ declines as large investor's wealth increases.
- Makes model scale-invariant.
  Doubling wealth, and all subsequent trades, doubles final payoff exactly.

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Alternatives?

- Alternatives: quantities $\Delta\theta$, or share turnover $\Delta\theta/\theta$. Consequences?
- Quantities ($\Delta\theta$):
  Bertsimas and Lo (1998), Almgren and Chriss (2000), Schied and Shoneborn (2009), Garleanu and Pedersen (2011)

$$\tilde{S}_t(\Delta\theta) := S_t + \lambda \frac{\Delta\theta}{\Delta t}$$

- Price impact independent of price. Not invariant to stock splits!
- Suitable for short horizons (liquidation) or mean-variance criteria.
- Share turnover:
  Stationary measure of trading volume (Lo and Wang, 2000). Observable.

$$\tilde{S}_t(\Delta\theta) := S_t \left(1 + \lambda \frac{\Delta\theta}{\theta_t \Delta t}\right)$$

- Problematic. Infinite price impact with cash position.

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Wealth and Portfolio

- Continuous time: cash position

$$dC_t = -S_t\left(1+\lambda\left|\frac{\dot\theta_t S_t}{X_t}\right|^{\alpha}\operatorname{sgn}(\dot\theta)\right)d\theta_t = -\left(\frac{S_t\dot\theta_t}{X_t}+\lambda\left|\frac{\dot\theta_t S_t}{X_t}\right|^{1+\alpha}\right)X_t dt$$

- Trading volume as wealth turnover $u_t := \frac{\dot\theta_t S_t}{X_t}$.
  Amount traded in unit of time, as fraction of wealth.
- Dynamics for wealth $X_t := \theta_t S_t + C_t$ and risky portfolio weight $Y_t := \frac{\theta_t S_t}{X_t}$

$$\frac{dX_t}{X_t} = Y_t(\mu dt + \sigma dW_t) - \lambda|u_t|^{1+\alpha}dt$$

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- Illiquidity...
- ...reduces portfolio return ($-\lambda u_t^{1+\alpha}$).
  Turnover effect quadratic: quantities times price impact.
- ...increases risky weight ($\lambda Y_t u_t^{1+\alpha}$). Buy: pay more cash. Sell: get less.
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  Turnover effect linear in risky weight Y_t. Vanishes for cash position.

Wealth and Portfolio

- Continuous time: cash position

$$dC_t = -S_t \left(1 + \lambda \left|\frac{\dot\theta_t S_t}{X_t}\right|^{\alpha} \operatorname{sgn}(\dot\theta)\right) d\theta_t = -\left(\frac{S_t\dot\theta_t}{X_t} + \lambda \left|\frac{\dot\theta_t S_t}{X_t}\right|^{1+\alpha}\right) X_t dt$$

- Trading volume as wealth turnover $u_t := \frac{\dot\theta_t S_t}{X_t}$.
  Amount traded in unit of time, as fraction of wealth.
- Dynamics for wealth $X_t := \theta_t S_t + C_t$ and risky portfolio weight $Y_t := \frac{\theta_t S_t}{X_t}$

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Admissible strategy: process $(u_t)_{t\geq 0}$, adapted to $\mathcal{F}_t$, such that system

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- Investor with relative risk aversion γ.
- Maximize equivalent safe rate, i.e., power utility over long horizon:

$$\max_u \lim_{T\to\infty} \frac{1}{T} \log E\left[X_T^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

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Verification

Theorem

If $\frac{\mu}{\gamma\sigma^2} \in (0,1)$*, then the optimal wealth turnover and equivalent safe rate are:*

$$\hat{u}(y) = \left|\frac{q(y)}{(\alpha+1)\lambda(1-yq(y))}\right|^{1/\alpha} \operatorname{sgn}(q(y)) \qquad \mathrm{EsR}_\gamma(\hat{u}) = \beta$$

where $\beta \in (0, \frac{\mu^2}{2\gamma\sigma^2})$ *and* $q : [0,1] \mapsto \mathbb{R}$ *are the unique pair that solves the ODE*

$$-\hat{\beta} + \mu y - \gamma\frac{\sigma^2}{2}y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q$$
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Asymptotics

Theorem

c_α and s_α unique pair that solves

$$s'(z) = z^2 - c - \alpha(\alpha+1)^{-(1+1/\alpha)}|s(z)|^{1+1/\alpha} \qquad \lim_{z\to\pm\infty} \frac{|s_\alpha(z)|}{|z|^{\frac{2\alpha}{\alpha+1}}} = (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}}$$

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Asymptotic optimal strategy and welfare:

$$\hat{u}(y) = -\left|\frac{s_\alpha(\lambda^{-\frac{1}{\alpha+3}}(y-\bar{Y})/A_\alpha)}{B_\alpha(\alpha+1)}\right|^{1/\alpha} \operatorname{sgn}(y-\bar{Y})$$

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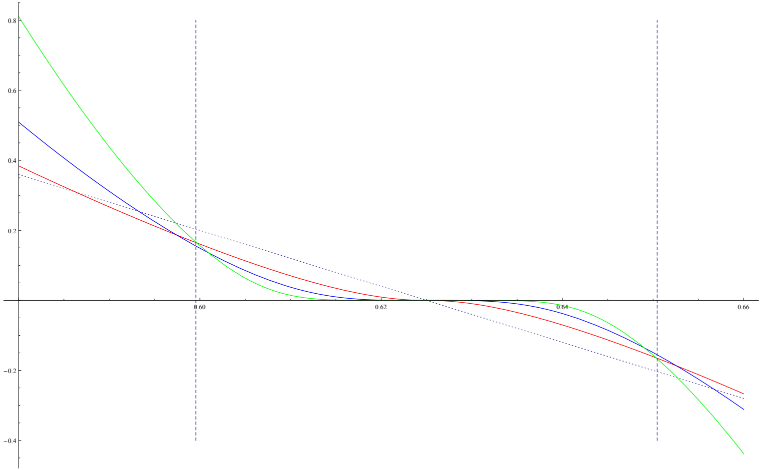
Trading Rate ($\mu = 8\%$, $\sigma = 16\%$, $\lambda = 0.1\%$, $\gamma = 5$)



Trading rate (vertical) against current risky weight (horizontal) for $\alpha = 1/8, 1/4, 1/2, 1$. Dashed lines are no-trade boundaries ($\alpha = 0$).

Trading Policy

- Trade towards $\bar{Y}$. Buy for $y < \bar{Y}$, sell for $y > \bar{Y}$.
- Trade faster if market deeper. Higher volume in more liquid markets.
- Trade slower than with linear impact near target. Faster away from target. With linear impact trading rate proportional to displacement $|y - \bar{Y}|$.
- As $\alpha \downarrow 0$, trading rate:
  vanishes inside no-trade region
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Welfare

- Welfare cost of friction:

$$c_\alpha \left[\left(\frac{\sigma^2}{2} \right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4 \right]^{\frac{\alpha+1}{\alpha+3}} \lambda^{\frac{2}{\alpha+3}}$$

- Last factor accounts for effect of illiquidity parameter.
- Middle factor reflects volatility of portfolio weight.
- Constant c_α depends on α alone. No explicit expression for general α.
- Exponents $2/(\alpha+3)$ and $(\alpha+1)/(\alpha+3)$ sum to one. Geometric average.

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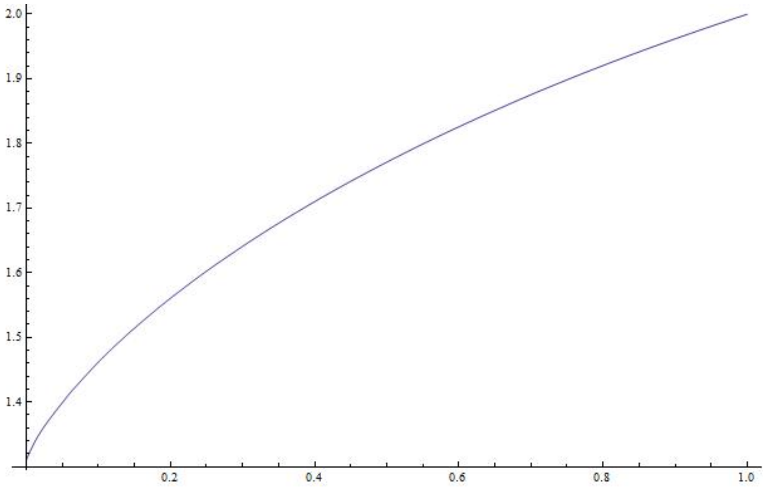
Universal Constant c_α



c_α (vertical) against α (horizontal).

Portfolio Dynamics

Proposition

Rescaled portfolio weight $Z_s^\lambda := \lambda^{-\frac{1}{\alpha+3}}(Y_{\lambda^{2/(\alpha+3)}s} - \bar{Y})$ *converges weakly to the process* Z_s^0*, defined by*

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Proposition

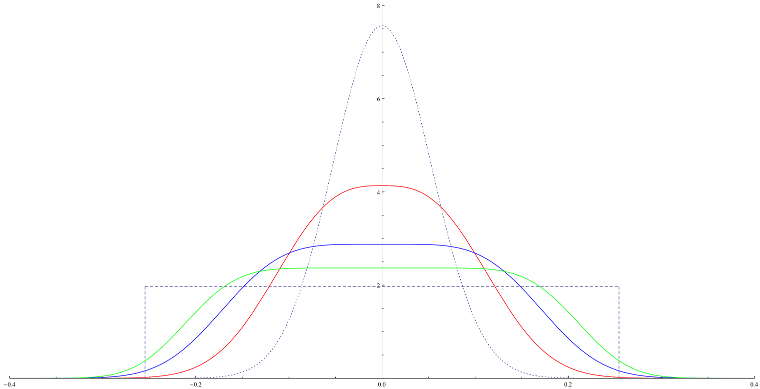
Rescaled portfolio weight $Z_s^\lambda := \lambda^{-\frac{1}{\alpha+3}}(Y_{\lambda^{2/(\alpha+3)}s} - \bar{Y})$ *converges weakly to the process* Z_s^0*, defined by*

$$dZ_s^0 = v_\alpha(Z_s^0)ds + \bar{Y}(1-\bar{Y})\sigma dW_s$$

$$v_\alpha(z) := -\left|\frac{B_\alpha s_\alpha(z/A_\alpha)}{(\alpha+1)}\right|^{1/\alpha} \operatorname{sgn}(z)$$

- "Nonlinear" stationary process. Ornstein-Uhlenbeck for linear impact.
- No explicit expression for drift – even asymptotically.
- Long-term distribution?

Long-term weight ($\mu = 8\%$, $\sigma = 16\%$, $\gamma = 5$)



Density (vertical) of the long-term density of rescaled risky weight Z^0 (horizontal) for $\alpha = 1/8, 1/4, 1/2, 1$. Dashed line is uniform density ($\alpha \to 0$).

Linear Impact ($\alpha = 1$)

- Solution to

$$s'(z) = z^2 - c - \alpha(\alpha+1)^{-(1+1/\alpha)}|s(z)|^{1+1/\alpha}$$

is $c_1 = 2$ and $s_1(z) = -2z$.

- Optimal policy and welfare:

$$\hat{u}(y) = \sigma\sqrt{\frac{\gamma}{2\lambda}}(\bar{Y} - y) + O(1)$$

$$\mathrm{EsR}_\gamma(\hat{u}) = \frac{\mu^2}{2\gamma\sigma^2} - \sigma^3\sqrt{\frac{\gamma}{2}}\bar{Y}^2(1-\bar{Y})^2\lambda^{1/2} + O(\lambda)$$

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Transaction Costs ($\alpha \downarrow 0$)

- Solution to

$$s'(z) = z^2 - c - \alpha(\alpha+1)^{-(1+1/\alpha)}|s(z)|^{1+1/\alpha}$$

converges to $c_0 = (3/2)^{2/3}$ and

$$s_0(z) := \lim_{\alpha\to 0} s_\alpha(z) = \begin{cases} 1, & z \in (-\infty, -\sqrt{c_0}], \\ z^3/3 - c_0 z, & z \in (-\sqrt{c_0}, \sqrt{c_0}), \\ -1, & z \in [\sqrt{c_0}, +\infty). \end{cases}$$

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$$Y_\pm = \frac{\mu}{\gamma\sigma^2} \pm \left(\frac{3}{4\gamma}\bar{Y}^2(1-\bar{Y})^2\right)^{1/3} \varepsilon^{1/3}$$

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- Compare to transaction cost model (Gerhold et al., 2014).

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Trading Volume and Welfare

- Expected Trading Volume

$$|ET| := \lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{u}_\lambda(Y_t)|dt = K_\alpha \left[\left(\frac{\sigma^2}{2}\right)^3 \gamma \bar{Y}^4(1-\bar{Y})^4\right]^{\frac{1}{\alpha+3}} \lambda^{-\frac{1}{\alpha+3}} + o(\lambda^{-\frac{1}{\alpha+}}$$

- Define welfare loss as decrease in equivalent safe rate due to friction:

$$\mathrm{LoS} = \frac{\mu^2}{2\gamma\sigma^2} - \mathrm{EsR}_\gamma(\hat{u})$$

- Zero loss if no trading necessary, i.e. $\bar{Y} \in \{0, 1\}$.
- Universal relation:

$$\mathrm{LoS} = N_\alpha \lambda \,|\mathrm{ET}|^{1+\alpha}$$

  where constant N_α depends only on α.
- Linear effect with transaction costs (price, not quantity). Superlinear effect with liquidity (price *times* quantity).

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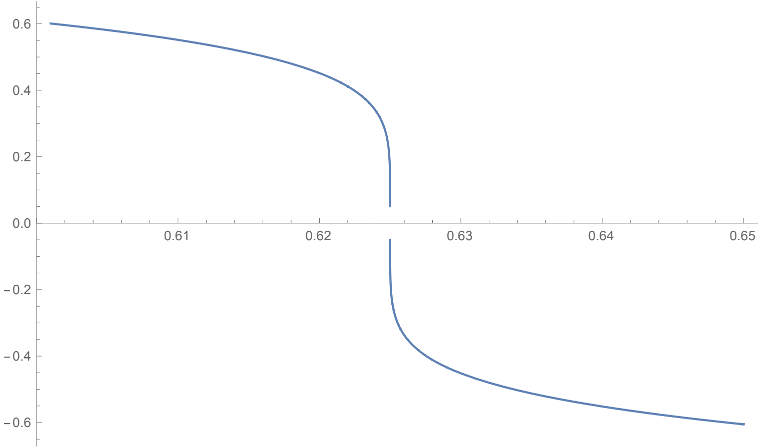
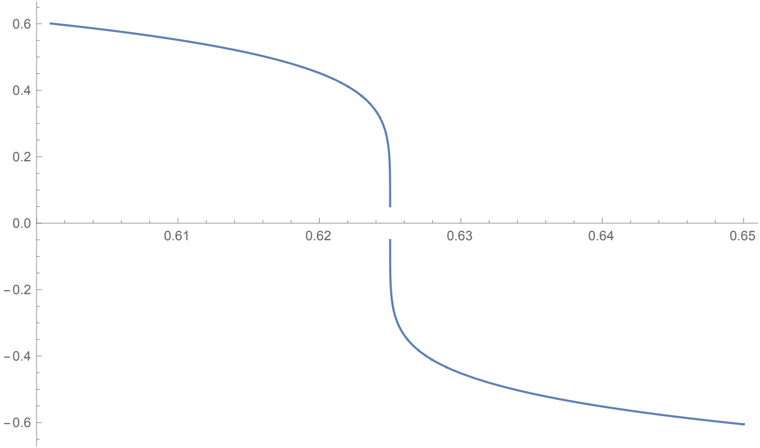
Hacking the Model ($\alpha > 1$)



- Empirically improbable. Theoretically possible.
- Trading rates below one cheap. Above one expensive.
- As $\alpha \uparrow \infty$, trade at rate close to one. Compare to Longstaff (2001).

Hacking the Model ($\alpha > 1$)



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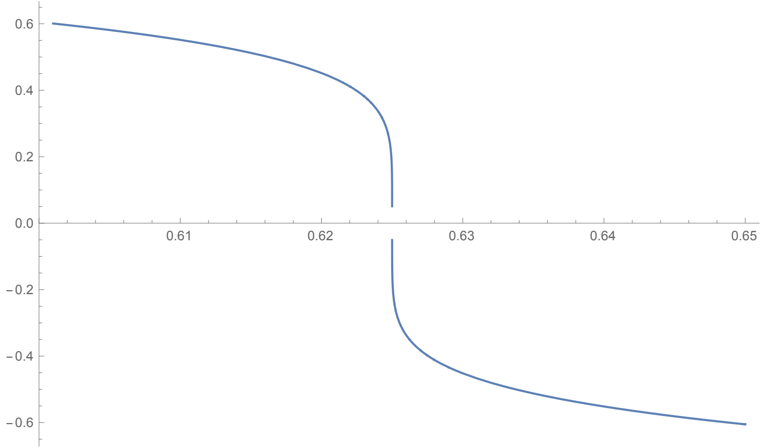

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Neither a Borrower nor a Shorter Be

Theorem

If $\frac{\mu}{\gamma\sigma^2} \leq 0$, then $Y_t = 0$ and $\hat{u} = 0$ for all t optimal. Equivalent safe rate zero.
If $\frac{\mu}{\gamma\sigma^2} \geq 1$, then $Y_t = 1$ and $\hat{u} = 0$ for all t optimal. Equivalent safe rate $\mu - \frac{\gamma}{2}\sigma^2$.

- If Merton investor shorts, keep all wealth in safe asset, but do not short.
- If Merton investor levers, keep all wealth in risky asset, but do not lever.
- Portfolio choice for a risk-neutral investor!
- Corner solutions. But without constraints?
- Intuition: the constraint is that wealth must stay positive.
- Positive wealth does not preclude borrowing with block trading, as in frictionless models and with transaction costs.
- Block trading unfeasible with price impact proportional to turnover. Even in the limit.
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Control Argument

- Value function v depends on (1) current wealth X_t, (2) current risky weight Y_t, and (3) calendar time t.

$$dv(t, X_t, Y_t) = v_t dt + v_x dX_t + v_y dY_t + \frac{v_{xx}}{2} d\langle X\rangle_t + \frac{v_{yy}}{2} d\langle Y\rangle_t + v_{xy} d\langle X, Y\rangle_t$$
$$= v_t dt + v_x(\mu X_t Y_t - \lambda X_t |u_t|^{\alpha+1})dt + v_x X_t Y_t \sigma dW_t$$
$$+ v_y(Y_t(1 - Y_t)(\mu - Y_t\sigma^2) + u_t + \lambda Y_t |u_t|^{\alpha+1})dt + v_y Y_t(1 - Y_t)\sigma dW_t$$
$$+ \left(\frac{\sigma^2}{2} v_{xx} X_t^2 Y_t^2 + \frac{\sigma^2}{2} v_{yy} Y_t^2(1 - Y_t)^2 + \sigma^2 v_{xy} X_t Y_t^2(1 - Y_t)\right) dt,$$

- Maximize drift over u, and set result equal to zero:

$$v_t + y(1-y)(\mu-\sigma^2 y)v_y + \mu xyv_x + \frac{\sigma^2 y^2}{2}\left(x^2 v_{xx} + (1-y)^2 v_{yy} + 2x(1-y)v_{xy}\right)$$
$$+ \max_u \left(-\lambda x|u|^{\alpha+1} v_x + v_y\left(u + \lambda y|u|^{\alpha+1}\right)\right) = 0.$$

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- Homogeneity in wealth $v(t,x,y) = x^{1-\gamma} v(t,1,y)$.
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- Substitution $v(t,x,y) = \frac{x^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(\beta(T-t)+\int^y q(z)dz)}$ reduces HJB equation

$$-\beta + \mu y - \gamma \frac{\sigma^2}{2} y^2 + qy(1-y)(\mu - \gamma\sigma^2 y) + \frac{\sigma^2}{2} y^2 (1-y)^2 (q' + (1-\gamma)q^2)$$
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- Guess that $q(y) \to 0$ as $\lambda \downarrow 0$. Limit equation:

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Lemma

Let q solve the HJB equation, and define $Q(y) = \int^y q(z)dz$. There exists a probability $\hat{P}$, equivalent to P, such that the terminal wealth X_T of any admissible strategy satisfies:

$$E[X_T^{1-\gamma}]^{\frac{1}{1-\gamma}} \leq e^{\beta T + Q(y)} E_{\hat{P}}[e^{-(1-\gamma)Q(Y_T)}]^{\frac{1}{1-\gamma}} ,$$

and equality holds for the optimal strategy.

- Solution of HJB equation yields asymptotic upper bound for any strategy.
- Upper bound reached for optimal strategy.
- Valid for any β, for corresponding Q.
- Idea: pick largest β^* to make Q disappear in the long run.
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$$\beta^* < \frac{\mu^2}{2\gamma\sigma^2} \qquad \text{(frictionless solution)}$$

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$$E[X_T^{1-\gamma}]^{\frac{1}{1-\gamma}} \leq e^{\beta T + Q(y)} E_{\hat{P}}[e^{-(1-\gamma)Q(Y_T)}]^{\frac{1}{1-\gamma}} ,$$

and equality holds for the optimal strategy.

- Solution of HJB equation yields asymptotic upper bound for any strategy.
- Upper bound reached for optimal strategy.
- Valid for any β, for corresponding Q.
- Idea: pick largest β^* to make Q disappear in the long run.
- A priori bounds:

$$\beta^* < \frac{\mu^2}{2\gamma\sigma^2} \qquad \text{(frictionless solution)}$$

$$\max\left(0, \mu - \frac{\gamma}{2}\sigma^2\right) < \beta^* \qquad \text{(all in safe or risky asset)}$$

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Existence

Theorem

Assume $0 < \frac{\mu}{\gamma\sigma^2} < 1$. *There exists* β^* *such that HJB equation has solution* $q(y)$ *with positive finite limit in* 0 *and negative finite limit in* 1.

- for $\beta > 0$, there exists a unique solution $q_{0,\beta}(y)$ to HJB equation with positive finite limit in 0.
- for $\beta > \mu - \frac{\gamma\sigma^2}{2}$, there exists a unique solution $q_{1,\beta}(y)$ to HJB equation with negative finite limit in 1.
- there exists β_u such that $q_{0,\beta_u}(y) > q_{1,\beta_u}(y)$ for some y;
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Explosion with Leverage

Lemma

If Y_t that satisfies $Y_0 \in (1, +\infty)$ and

$$dY_t = Y_t(1 - Y_t)(\mu dt - Y_t\sigma^2 dt + \sigma dW_t) + u_t dt + \lambda Y_t |u_t|^{1+\alpha} dt$$

explodes in finite time with positive probability.

Lemma

Let τ be the exploding time of Y_t. Then wealth $X_\tau = 0$ a.s on $\{\tau < +\infty\}$.

- Feller's criterion for explosions.
- No strategy admissible if it begins with levered or negative position.

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- Finite market depth. Execution price power of wealth turnover.
- Large investor with constant relative risk aversion.
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- Halfway between linear impact and bid-ask spreads.
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