No arbitrage conditions in HJM multiple curve term structure models

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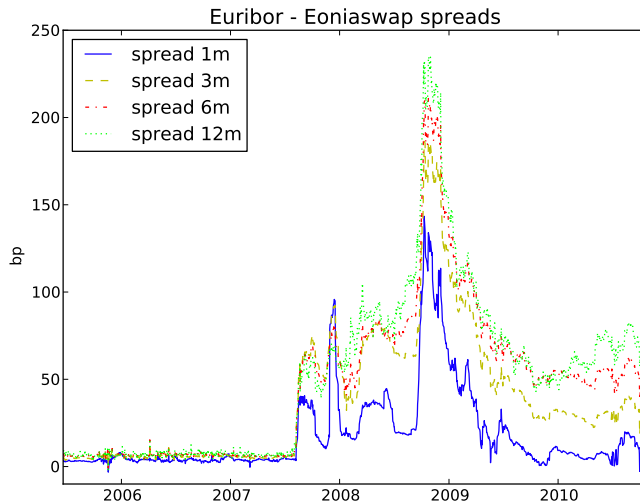
7th General AMaMeF and Swissquote Conference
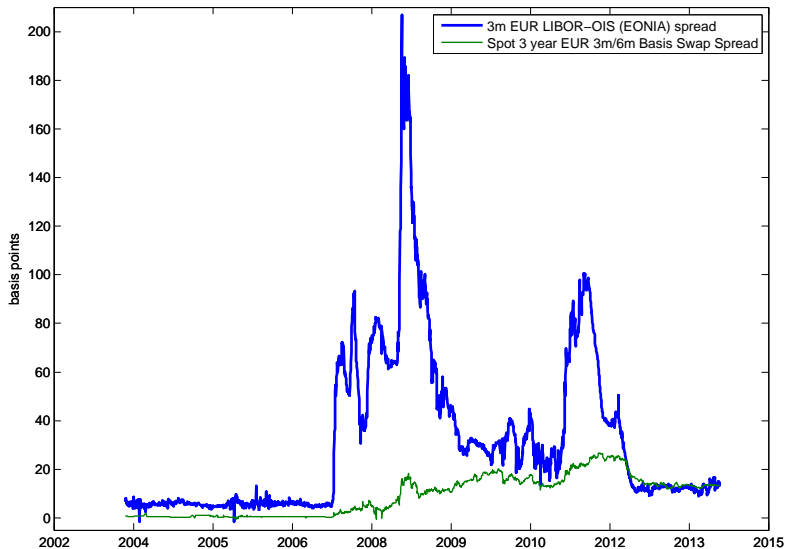Lausanne, 7-10 September 2015

Introduction and motivation

A number of anomalies has appeared in the financial markets due to the recent financial crisis. In particular, in interest rate markets the following issues have emerged:

- Counterparty risk – risk of non-payment of promised cash-flows due to default of the counterparty in a bilateral OTC interest rate derivative transaction and funding (liquidity) issues – cost of borrowing and lending for funding of a position in a contract: CVA, DVA, FVA... adjustments

- Libor/Euribor rate cannot be considered risk-free any longer: it reflects credit and liquidity risk of the interbank sector; a large gap between the Libor and the OIS rate and the Libor rates for different tenors:
  multiple curve issue

Euribor - Eoniaswap spreads

Multiple curve modeling and pricing:

- future cash flows in interest rate derivatives are related to the underlying tenor-dependent interest rates (Libor/Euribor)

- another curve is used for discounting and and is common for all tenors and all derivatives (OIS curve)

- In the following we shall consider just a two-curve model for simplicity, corresponding to one given tenor, hence we shall have one Libor curve and one discounting curve

Multiple curve modeling and absence of arbitrage

When developing an arbitrage-free model for the term structure, a first issue is to identify the basic traded assets in the market:

- they are FRA contracts, which are basic linear interest rate derivatives and building blocks for interest rate swaps

- they can be based on the discretely compounded overnight rate or on the Libor rates

- after the crisis, the prices of FRA contracts based on these two rates became disconnected and have to be modeled separately

- the issue of absence of arbitrage translates then into the question of existence of a numéraire and a martingale measure related to it, under which the discounted prices of the OIS- and the Libor-FRA contracts are local martingales

Multiple curve modeling and absence of arbitrage

- we choose the OIS curve that results from OIS swaps and OIS FRAs as the discount curve

- this choice is motivated the fact that, being based on overnight rates, the OIS bear little risk and moreover, these rates are used as remuneration rates in collateralization, which by now has become standard for all typical interest rate derivative OTC contracts

- having bootstrapped the OIS discount curve $(T \mapsto p(t, T))$, we consider the corresponding instantaneous forward rate $f(t, T) := -\frac{\partial}{\partial T} \log p(t, T)$ and the short rate $r_t := f(t, t)$

- we define a corresponding bank account

$$B_t := \exp \left(\int_0^t r_s ds \right)$$

and a related standard martingale measure Q having B as a numéraire

Multiple curve modeling and absence of arbitrage

A model for a market consisting of OIS bonds and Libor-FRA contracts:

1. model the OIS bond prices $p(t, T)$

2. consider an FRA for the time interval $[T, T + \Delta]$: the underlying rate is the Libor rate $L(T; T, T + \Delta)$, which is an actual rate determined at time T by the LIBOR panel

   The FRA value at time t is given by

   $$P^{FRA}(t; T, T + \Delta, K) = \Delta p(t, T + \Delta)\mathbb{E}^{Q^{T+\Delta}}[L(T; T, T + \Delta) - K|\mathcal{F}_t]$$

   where $Q^{T+\Delta}$ denotes the forward martingale measure associated to the date $T + \Delta$, with bond price $p(t, T + \Delta)$ as a numeraire

   The implied FRA rate is hence

   $$K_t = \mathbb{E}^{\mathbb{Q}^{T+\delta}}[L(T; T, T + \Delta)|\mathcal{F}_t] \neq \frac{1}{\Delta}\left(\frac{p(t, T)}{p(t, T + \Delta)} - 1\right)$$

   We call this rate the Libor FRA rate or simply the forward Libor rate and denote it by $L(t; T, T + \Delta)$

The modeling is done in two steps:

1. Firstly we model the OIS bond prices directly under a martingale measure, denoted Q.

2. This gives us the explicit expressions for the forward measures, namely e.g. the forward measure $Q^{T+\Delta}$ is given by the following Radon-Nikodym density

$$\frac{dQ^{T+\Delta}}{dQ}\Big|_{\mathcal{F}_t} = \frac{p(t, T+\Delta)}{B_t p(0, T+\Delta)},$$

3. In a second step we shall consider suitable quantities connected to the FRA rates and develop for them a model in an HJM fashion so that the complete model is free of arbitrage.

HJM framework for multiple curves

- For simplicity we consider in the sequel models driven by Brownian motion only; however the reasoning remains analogous for more general drivers, only the drift conditions become more complicated

Modeling of the OIS prices: since we intend to follow a HJM methodology, we start from the Q-dynamics of the instantaneous forward rates:

$$df(t, T) = a(t, T)dt + \sigma(t, T)dW_t$$

It follows, recalling that $p(t, T) = \exp\left(-\int_t^T f(t, u)du\right)$,

$$dp(t, T) = p(t, T)\left((r_t - A(t, T) + \frac{1}{2} \mid \Sigma(t, T)\mid^2)dt - \Sigma(t, T)dW_t\right)$$

where W is a multidimensional Q-Brownian motion and

$$r_t = f(t, t), \quad A(t, T) := \int_t^T a(t, u)du, \quad \Sigma(t, T) := \int_t^T \sigma(t, u)du$$

HJM framework for multiple curves

The discounted prices of the OIS bonds have to be Q-local martingales, which implies the standard HJM drift condition

$$A(t, T) = \frac{1}{2} \mid \Sigma(t, T) \mid^2$$

What about the forward Libor rates (FRA rates):

$$L(t; T, T + \Delta) = E^{T+\Delta} \{L(T; T, T + \Delta) \mid \mathcal{F}_t\}?$$

Various approaches are possible as we shall see in the sequel...

I. Hybrid LMM-HJM approach

One idea is to mimic the Libor market model (top-down approach) and model directly the forward Libor rates in a suitable fashion: we refer to such models as hybrid LMM-HJM models

In Crépey, G., Ngor and Skovmand (2014) (see also Moreni and Pallavicini (2014)) the following quantities are modeled

$$
\begin{aligned}
G(t; T, T + \Delta) &= \Delta E^{T+\Delta} \{ L(T; T, T + \Delta) \mid \mathcal{F}_t \} \\
&= \Delta L(t; T, T + \Delta)
\end{aligned}
\tag{1}
$$

- by definition, $G(t; T, T + \Delta)$ must be a martingale under the forward measure $Q^{T+\Delta}$, hence no arbitrage conditions translate into a drift condition on the process $G(t; T, T + \Delta)$

- the quantities $G(t; T, T + \Delta)$ are observable in the market (for short maturities FRA rates are directly observable; for longer maturities they can be bootstrapped from observed swap rates of Libor-indexed swaps)

I. Hybrid LMM-HJM approach

More precisely, we model directly the dynamics of the process $G(t; T, T + \Delta)$, however not under the forward measure $Q^{T+\Delta}$ as in the LMM setup, but under the martingale measure Q as in the HJM setup

$$G(t; T, T + \Delta) = G(0; T, T + \Delta)$$
$$= \exp\left(\int_0^t \alpha(s, T, T + \Delta)ds + \int_0^t \varsigma(s, T, T + \Delta)dW_s \right)$$

The drift $\alpha(s, T, T + \Delta)$ has to be such that $G(t; T, T + \Delta)$ is a martingale under $Q^{T+\Delta}$:

$$\alpha(s, T, T + \Delta) = -\frac{1}{2} \mid \varsigma(s, T, T + \Delta) \mid^2 + \langle \varsigma(s, T, T + \Delta), \Sigma(s, T + \Delta) \rangle$$

which follows by the fact that the measure change $\frac{dQ^{T+\Delta}}{dQ} \mid_{\mathcal{F}_t} = \frac{p(t, T+\Delta)}{B_t p(0, T+\Delta)}$ yields that $dW_t^{T+\Delta} := dW_t + \Sigma(t, T + \Delta)dt$ is a $Q^{T+\Delta}$-Brownian motion

II. Approach based on fictitious bond prices

A second idea is to introduce fictitious bonds $\bar{p}(t, T)$ in order to reproduce the classical relationship between the Libor rates and the bond prices:

$$L(T; T, T + \Delta) = \frac{1}{\Delta} \left(\frac{1}{\bar{p}(T, T+\Delta)} - 1 \right) \tag{2}$$

Then

$$\begin{aligned} L(t; T, T + \Delta) &= E^{T+\Delta} \left\{ L(T; T, T + \Delta) \mid \mathcal{F}_t \right\} \\ &= \frac{1}{\Delta} E^{T+\Delta} \left\{ \left(\frac{1}{\bar{p}(T, T+\Delta)} - 1 \right) \mid \mathcal{F}_t \right\} \end{aligned} \tag{3}$$

- we assume $\bar{p}(t, T)$ are defined for all t and that $\bar{p}(T, T) = 1$

- the $\bar{p}(t, T)$-bonds are not observable in the market

- these bonds are referred to as Libor bonds or interbank bonds by some authors

II. Approach based on fictitious bond prices

We shall consider two alternative ways to obtain an arbitrage-free model for the $\bar{p}(t, T)$- prices:

(a) The first alternative, in which we denote $\bar{p}(t, T)$ more specifically as $p^{\Delta}(t, T)$, is to impose the above relationship not only at the level of the spot Libor rates, but also for the forward Libor rates by assuming

$$
\begin{aligned}
L(t; T, T + \Delta) &= E^{T+\Delta} \left\{ L(T; T, T + \Delta) \mid \mathcal{F}_t \right\} \\
&= \frac{1}{\Delta} E^{T+\Delta} \left\{ \left(\frac{1}{p^{\Delta}(T, T+\Delta)} - 1 \right) \mid \mathcal{F}_t \right\} \\
&= \frac{1}{\Delta} \left(\frac{p^{\Delta}(t, T)}{p^{\Delta}(t, T+\Delta)} - 1 \right)
\end{aligned}
\tag{4}
$$

Thus, the no-arbitrage condition here translates into a martingale condition on the ratio $\frac{p^{\Delta}(t, T)}{p^{\Delta}(t, T+\Delta)}$ under $Q^{T+\Delta}$

This approach bears some similarity with the hybrid HJM-LMM approach since the condition is imposed on the ratio of the bond prices, rather than on a discounted bond price itself. Moreover, the ratios $\frac{p^{\Delta}(t, T)}{p^{\Delta}(t, T+\Delta)}$ are observable quantities

II. Approach based on fictitious bond prices

(b) The second alternative is to interpret the fictitious bonds $\bar{p}(t, T)$ on the basis of either a credit risk analogy or a foreign exchange analogy.

- This will lead to pseudo no-arbitrage conditions, which are not implied by the absence of arbitrage in the market for FRAs and OIS bonds, but rather by each specific analogy

- On the basis of the relationship

$$L(T; T, T + \Delta) = \frac{1}{\Delta}\left(\frac{1}{\bar{p}(T, T+\Delta)} - 1\right)$$

$$\begin{aligned}L(t; T, T + \Delta) &= E^{T+\Delta}\{L(T; T, T + \Delta) \mid \mathcal{F}_t\} \\ &= \frac{1}{\Delta}E^{T+\Delta}\left\{\left(\frac{1}{\bar{p}(T, T+\Delta)} - 1\right) \mid \mathcal{F}_t\right\}\end{aligned}$$

(5)

a crucial quantity to compute in these models is

$$\bar{\nu}_{t,T} = E^{T+\Delta}\left\{\frac{1}{\bar{p}(T, T + \Delta)} \mid \mathcal{F}_t\right\}$$

Relationship between various modeling quantities

Summarizing the definitions we introduced, we get the following equivalent representations for the forward Libor rate

$$L(t; T, T+\Delta) = \begin{cases} \frac{1}{\Delta} G(t; T, T+\Delta) & \text{(used in Approach I.)} \\[2mm] \frac{1}{\Delta}\left(\frac{p^\Delta(t,T)}{p^\Delta(t,T+\Delta)} - 1\right) & \text{(used in Approach II.a)} \\[2mm] \frac{1}{\Delta}\left(\bar{\nu}_{t,T} - 1\right) & \text{(used in Approach II.b)} \end{cases} \quad (6)$$

from which it also follows that

$$\bar{\nu}_{t,T} = \frac{p^\Delta(t,T)}{p^\Delta(t,T+\Delta)} = G(t; T, T+\Delta) + 1 \qquad (7)$$

Now we extend the model for the OIS prices by introducing the instantaneous forward rates $\bar{f}(t, T)$ (resp. $f^{\Delta}(t, T)$) such that

$$\bar{f}(t, T) = f(t, T) + g(t, T),$$

where $f(t, T)$ are the forward rates corresponding to the OIS bonds and $g(t, T)$ are the forward rate spreads.

Remark: this is a common practice to model quantities related to the Libor curves via spreads over the OIS quantities

HJM model for the fictitious bond prices

Under the martingale measure Q we thus assume the following dynamics for the spreads:

$$dg(t, T) = a^*(t, T)dt + \sigma^*(t, T)dW_t$$

and for $\bar{f}(t, T) = f(t, T) + g(t, T)$ we get

$$d\bar{f}(t, T) = \bar{a}(t, T)dt + \bar{\sigma}(t, T)dW_t$$

with $\bar{a}(t, T) = a(t, T) + a^*(t, T)$ and similarly for $\bar{\sigma}$.

From $\bar{p}(t, T) = \exp\left(-\int_t^T \bar{f}(t, u)du\right)$, we thus get

$$d\bar{p}(t, T) = \bar{p}(t, T)\left((\bar{r}_t - \bar{A}(t, T) + \frac{1}{2}\mid\bar{\Sigma}(t, T)\mid^2)dt - \bar{\Sigma}(t, T)dW_t\right)$$

where W is the same Q-Brownian motion as before and

$$\bar{r}_t = \bar{f}(t, t), \quad \bar{A}(t, T) := \int_t^T \bar{a}(t, u)du, \quad \bar{\Sigma}(t, T) := \int_t^T \bar{\sigma}(t, u)du$$

HJM model for the fictitious bond prices

In the first alternative **II.a**, treated in G. and Runggaldier (2015) and Miglietta (2015), we have thus

$$dp^\Delta(t, T) = p^\Delta(t, T)\left((r_t^\Delta - A^\Delta(t, T) + \frac{1}{2}\mid\Sigma^\Delta(t, T)\mid^2)dt - \Sigma^\Delta(t, T)dW_t\right)$$

and the drift $A^\Delta(t, T)$ has to be chosen in such a way that the ratio $\frac{p^\Delta(t, T)}{p^\Delta(t, T+\Delta)}$ becomes a $Q^{T+\Delta}$-martingale

This yields the following drift condition

$$\begin{aligned}A^\Delta(t, T + \Delta) - A^\Delta(t, T) = &-\frac{1}{2}\mid\Sigma^\Delta(t, T + \Delta) - \Sigma^\Delta(t, T)\mid^2\\ &+\langle\Sigma(t, T + \Delta), \Sigma^\Delta(t, T + \Delta) - \Sigma^\Delta(t, T)\rangle\end{aligned}$$

Remark: Note that the drift condition is expressed only on the difference $A^\Delta(t, T + \Delta) - A^\Delta(t, T)$ and does not uniquely define the coefficient $A^\Delta(t, T)$; this is a consequence of a non-unique definition of $p^\Delta(t, T)$ via

$$L(t; T, T + \Delta) = \frac{1}{\Delta}\left(\frac{p^\Delta(t, T)}{p^\Delta(t, T + \Delta)} - 1\right)$$

Coming to the second alternative **II.b**, we emphasize the following:

- the drift conditions on $\bar{p}(t, T)$ are derived on the basis of different interpretations of the fictitious bonds and they are not directly implied by the no-arbitrage conditions for the FRA contracts, as it was the case before

- we call these conditions pseudo no-arbitrage conditions as they are not strictly necessary to ensure the absence of arbitrage in the model for FRAs and OIS bonds

- exploiting the existing HJM approaches with more than one curve, one may consider a credit risk (cf. for example Bielecki and Rutkowski (2002)) and a foreign exchange analogy (cf. Amin and Jarrow (1991))

Credit risk analogy

We concentrate here on the approach developed in Crépey, G. and Nguyen (2012) (see also Morini (2009), Filipović and Trolle (2013) and Ametrano and Bianchetti (2013))):

- $\bar{p}(t, T)$ are interpreted here as pre-default values of bonds issued by an average representative of the Libor panel, which may possibly default at a hypothetical random time τ^*

- the defaultable bonds are now traded assets and hence their discounted values have to be Q-martingales, which yields a drift condition

- using a reduced form approach where the default is modeled via a default intensity process and for a specific choice of the recovery scheme this leads to a particularly nice form of the drift condition on $\bar{p}(t, T)$ given by

$$\bar{A}(t, T) = \frac{1}{2} \mid \bar{\Sigma}(t, T) \mid^2$$

This is a classical HJM condition (as for the OIS bonds), where all the risk-free quantities $A(\cdot)$ and $\Sigma(\cdot)$ are replaced by the corresponding "risky" ones $\bar{A}(\cdot)$ and $\bar{\Sigma}(\cdot)$

Foreign exchange analogy

This analogy has been first suggested in Bianchetti (2010) and exploited in Cuchiero, Fontana and Gnoatto (2015) and Miglietta (2015). The specific HJM setup presented here is from G. and Runggaldier (2015).

- we denote the fictitious bonds more specifically by $p^f(t, T)$ to distinguish them from the previous case and interpret them as bonds denominated in a different, foreign currency

- the OIS bonds $p(t, T)$ are seen as domestic bonds

- thus, the fictitious bonds $p^f(t, T)$ are now traded assets in the foreign economy and the drift conditions on $p^f(t, T)$ stem from the absence of arbitrage in the foreign market

- the two markets are linked via an exchange rate, which allows to derive corresponding conditions in the domestic market ensuring the absence of arbitrage in the foreign market

This yields the following drift condition

$$A^f(t, T + \Delta) - A^f(t, T) = -\frac{1}{2} \mid \Sigma^f(t, T + \Delta) - \Sigma^f(t, T) \mid^2$$
$$+ \langle \Sigma^f(t, T + \Delta) - \beta_t, \Sigma^f(t, T + \Delta) - \Sigma^f(t, T) \rangle$$

- as for $p^\Delta(t, T)$, the drift condition is again only on the difference $A^f(t, T + \Delta) - A^f(t, T)$, due to the fact that the ratio $\frac{p^f(t,T)}{p(t,T+\Delta)}$ has to be a martingale under a certain foreign forward measure

- the drift conditions on $p^\Delta(t, T)$ and $p^f(t, T)$ look similar, but are not the same: besides the appearance of the volatility β_t of the spot exchange rate, the second condition results from a further measure transformation (the difference being justified by the different interpretations of the fictitious bonds).

Overview of the HJM multiple curve drift conditions

- hybrid LMM-HJM approach via rates

$$\alpha(t, T, T + \Delta) = -\frac{1}{2} \mid \varsigma(t, T, T + \Delta) \mid^2 + \langle \varsigma(t, T, T + \Delta), \Sigma(t, T + \Delta) \rangle$$

- hybrid LMM-HJM approach via fictitious bonds

$$A^\Delta(t, T + \Delta) - A^\Delta(t, T) = -\frac{1}{2} \mid \Sigma^\Delta(t, T + \Delta) - \Sigma^\Delta(t, T) \mid^2$$
$$+ \langle \Sigma^\Delta(t, T + \Delta) - \Sigma^\Delta(t, T), \Sigma(t, T + \Delta) \rangle$$

- credit risk analogy

$$\bar{A}(t, T) = \frac{1}{2} \mid \bar{\Sigma}(t, T) \mid^2$$

- foreign exchange analogy

$$A^f(t, T + \Delta) - A^f(t, T) = -\frac{1}{2} \mid \Sigma^f(t, T + \Delta) - \Sigma^f(t, T) \mid^2$$
$$+ \langle \Sigma^f(t, T + \Delta) - \beta_t, \Sigma^f(t, T + \Delta) - \Sigma^f(t, T) \rangle$$

Concluding remarks

- we have studied various possible modeling choices in the HJM setup for multiple curves

- different choices lead to different types of martingale (drift) conditions, which are not interconnected in general

- we distinguish between "true" no-arbitrage conditions implied by absence of arbitrage between basic quantities (OIS bonds and FRA contracts) in the multiple curve markets and pseudo no-arbitrage conditions implied by specific interpretations of the fictitious Libor bonds

- apart from the forward Libor rates, one possibility is to model various forward discretely compounded spreads in addition to the OIS bond prices (cf. Cuchiero et al. (2015)) – this is again a hybrid HJM-LMM approach with "true" no-arbitrage conditions

- regarding implementation and calibration of these models, the choice might depend on a specific task at hand, however we note that in some of the mentioned alternatives the initial term structure of the modeling quantities may not be observable

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Thank you for your attention