Hybrid FX-Interest rate models: a tale of two risks

Based on a joint work with M. Grasselli and E. Platen

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Abstract

The valuation of long-dated foreign-exchange products can be reliably performed by means of frameworks where exchange and interest rates are jointly modelled. In a multi-currency setting, analytical tractability can be granted if the functional form of the FX dynamics is invariant with respect to the inversion of the spot and the construction of FX triangles via suitable products/ratios of FX rates. We provide examples of models driven by multifactor affine stochastic volatility models in a standard risk-neutral setting which fulfill the aforementioned requirement. In the second part, we consider a new stochastic volatility model that includes as special cases the square root Heston model and the 3/2 model. The new model is analytically tractable and the computation of the characteristic function of asset prices involves a non trivial extension of known results in probability. We will use this model as a building block in order to develop a multifactor version that considerably extends the model investigated in De Col et al. (2013) and Baldeaux et al. (2015), including stochastic interest rates. The new methodology does not require the existence of a risk neutral probability measure.

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- The pricing of FX options written on a single FX rate is a well-understood problem, the complexity of which is comparable to the case of equity options, so many different possibilities have been investigated

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- An FX desk may be required to enter and manage position in **long dated FX products**

Stylized facts: products of exchange rates

Denote by $S^{i,j}(t)$ the exchange rate between currency i and j (FORDOM convention, i.e. for a EURUSD spot of 1.20, one EUR costs 1.20 USD)

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So, from a modelling perspective, a desireable feature of an FX model is:

$$dS^{i,j}(t) \overset{f}{=} d(S^{i,l}(t)S^{l,j}(t)).$$

in the sense that the functional form of the model should be invariant w.r.t. the construction of suitable products/ratios of exchange rates.

Triangles of smiles

Market implied volatility surfaces for currency options are quite complex
$\rightarrow$ non trivial implied correlation structure among FX rates
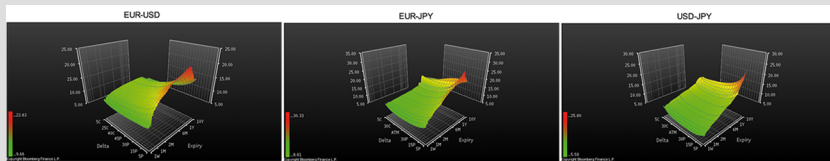
A model able to simultaneously fit three volatility surfaces is able to adequately manage correlations in a currency triangle.

Long dated FX products

Following [Clark, 2011], the risk involved in long-dated products may be intuitively understood in terms of a simple example.

- The value of an ATM call option, with maturity T, in a Black-Scholes setting, is usually approximated by means of the formula $0.4\sigma\sqrt{T}$.
- The vega of such a position is simply given by $0.4\sqrt{T}$.
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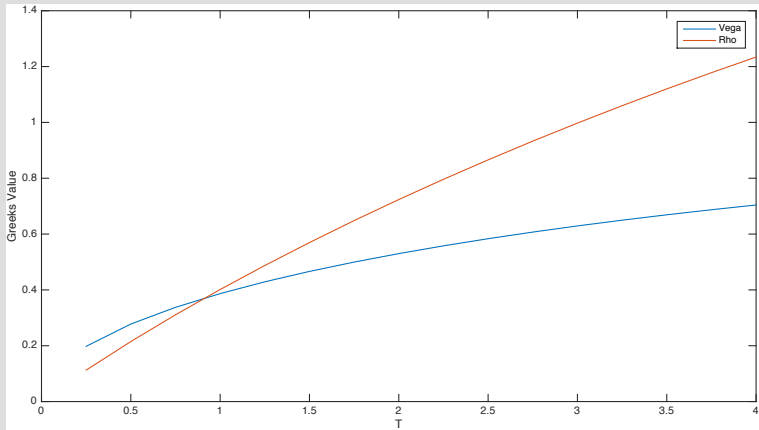
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- The rho risk can be shown to scale as T.
- **For longer maturities interest rate risk dominates volatility risk**.
- Need for a model which combinates stochastic volatility and stochastic interest rates.

Rho Vs Vega risk

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multi-Heston model

A Benchmark Approach to FX

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- allows for stochastic interest rates and volatilities.

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- Real world valuation formula for pricing (see [Platen and Heath, 2010]) and benchmarked risk minimization for hedging (see [Du and Platen, 2014])

Multi-Heston approach to FX: De Col, Gnoatto, Grasselli (2013)

1. **Classic Framework: assume existence of a risk neutral measure**
2. introduce an artificial currency S^0
3. take S^0 as (universal) numéraire for the N currencies
4. $S^{0,i}(t) =$ value of one unit of the currency i in terms of a universal numéraire
5. $S^{0,i}(t)$ evolves with multi-Heston dynamics.

Model dynamics

$S^{0,i}(t)$ has a multi-variate Heston stochastic volatility dynamics

$$\frac{dS_t^{0,i}}{S_t^{0,i}} = (r^0 - r^i)dt - (\mathbf{a}^i)^\top Diag(\sqrt{\mathbf{V}_t})d\mathbf{Z}_t^{Q^i}, \quad i = 1, ..., N;$$

$$dV_k(t) = \kappa_k^{Q^i}(\theta_k^{Q^i} - V_k(t))dt + \xi_k\sqrt{V_k}dW_k^{Q^i}(t), \quad k = 1, ..., d;$$

$d\langle Z_k, W_h \rangle_t = \rho_k \delta_{k,h} dt,$
$d\langle Z_k, Z_h \rangle_t = d\langle W_k, W_h \rangle_t = \delta_{k,h} dt, \ k, h = 1, ..., d$
Interpretation : the dynamics of the universal numéraire denominated in the i-th currency are driven by a linear projection of the variance factor $V(t)$ along a direction parametrized by a^i

Multi-Factor Heston Based dynamics

Exchange rates are ratios of the universal numéraire denominated in the corresponding currencies:

$$S^{i,j}(t) := S^{0,j}(t)/S^{0,i}(t)$$

$$\frac{dS^{i,j}(t)}{S^{i,j}(t)} = (r^i - r^j)dt + (\mathbf{a}^i - \mathbf{a}^j)^\top Diag(\sqrt{\mathbf{V}(t)})d\mathbf{Z}^{\mathbb{Q}^i}(t), \ i,j = 1, ..., N;$$

$$dV_k(t) = \kappa_k^{\mathbb{Q}^i}(\theta_k^{\mathbb{Q}^i} - V_k(t))dt + \xi_k\sqrt{V_k}dW_k^{\mathbb{Q}^i}(t), \ k = 1, ..., d;$$

Financial/analytic properties

1. Easy to extend to $N > 3$;
2. The covariation matrix among rates is positive semidefinite (bounds for correlations)
3. Heston Based dynamics: shares Heston features (Fx option pricing via Fourier transforms);
4. Triangular relation: The dynamics of all exchange rates share the same functional form

$$dS^{i,j}(t) \overset{f}{=} d(S^{i,l}(t)S^{l,j}(t))$$

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Nice calibration results but..

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6. What is more: relevant consequences in practice??

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- In the multi-Heston framework of De Col et al. (2013) all local martingales are true martingales.
- The multi-3/2 framework of Baldeaux et al. (2013) allows for both situations.

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- analytically tractable and possible to be calibrated?
- a model that does not require the existence of a risk neutral measure?

Choice of the risk premium and choice of the pricing model

Starting point: asset price dynamics

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then the dynamics of the GOP becomes

$$\frac{dD_t}{D_t} = rdt + <\pi_t, \pi_t dt + d\mathbf{Z}_t>$$

$\pi_t = $ *MARKET PRICE OF RISK*

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Viceversa: choice of π implies certain dynamics for S!

Choice of the risk premium and choice of the pricing model 2

In the Black-Scholes model,

$$\pi = \frac{\mu - r}{\sqrt{V}} = \frac{\mu - r}{\sigma}$$

In the Heston model, typical choice

$$\pi_t = \frac{\mu_t - r}{\sqrt{V_t}} = \lambda \sqrt{V_t}$$

Affine risk premium:

- affine model
- same Feller under both measures

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}(dW_t + \rho\lambda\sqrt{V_t})$$

Choice of the risk premium and choice of the pricing model 3

Other specification: the essentially affine risk premium

$$\pi_t = \frac{\mu_t - r}{\sqrt{V_t}} = \frac{\lambda_1 V_t + \lambda_2}{\sqrt{V_t}}$$

- still affine model
- if $\lambda_2 \neq 0$ different Feller !!

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}(dW_t + \rho\frac{\lambda_1 V_t + \lambda_2}{\sqrt{V_t}})$$

A Unified Approach for the Multi Heston and 3/2 Models

- De Col, Gnoatto, Grasselli (2013)

$$\pi = \lambda_1 \sqrt{V_t}$$

- Baldeaux, Grasselli, Platen (2015)

$$\pi = \frac{\lambda_2}{\sqrt{V_t}}$$

- Gnoatto, Grasselli, Platen (2015): essentially affine!

$$\pi = \lambda_1 \sqrt{V_t} + \frac{\lambda_2}{\sqrt{V_t}}$$

$$\frac{dD_t}{D_t} = rdt + \left(\lambda_1 \sqrt{V_t} + \frac{\lambda_2}{\sqrt{V_t}}\right)\left(dZ_t + \left(\lambda_1 \sqrt{V_t} + \frac{\lambda_2}{\sqrt{V_t}}\right)dt\right)$$

Modelling approach

- Given the mutual relationship between market price of risk and FX dynamics, we change our perspective and start by postulating the dynamics of the **growth optimal portfolio** under different **currency denominations**.

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- Given the mutual relationship between market price of risk and FX dynamics, we change our perspective and start by postulating the dynamics of the **growth optimal portfolio** under different **currency denominations**. This is the approach which was mimicked by De Col et al. (2013) by introducing the idea of an artificial currency (see above) in a pure risk-neutral setting.

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- Following Heath and Platen (2006), all exchange rates are determined as

$$S^{i,j}(t) = \frac{D^i(t)}{D^j(t)} \tag{1}$$

for $0 \leq t \leq \bar{T}$ and $i, j = 1, \ldots, N$. $\rightarrow$ Model is fully symmetric!

GOP dynamics

Dynamics of the GOP denominated in the i-th currency becomes

$$
\begin{aligned}
\frac{dD^i(t)}{D^i(t)} =\ & r^i dt + (\mathbf{a}^i)^\top Diag(\mathbf{V}(t))\mathbf{a}^i dt \\
& + (\mathbf{b}^i)^\top Diag^{-1}(\mathbf{V}(t))\mathbf{b}^i dt + 2(\mathbf{a}^i)^\top \mathbf{b}^i dt \\
& + (\mathbf{a}^i)^\top Diag^{1/2}(\mathbf{V}(t))d\mathbf{Z}(t) + (\mathbf{b}^i)^\top Diag^{-1/2}(\mathbf{V}(t))d\mathbf{Z}(t).
\end{aligned}
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Common stochastic factor $\mathbf{V}(t) \in \mathbb{R}^d$:

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dV_k(t) = \kappa_k(\theta_k - V_k(t))dt + \sigma_k V_k(t)^{1/2}dW_k(t), \quad k = 1, .., d,
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where $d < W_k, Z_k >= \rho_k dt$.

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where $d < W_k, Z_k >= \rho_k dt$. Dynamics of $S^{i,j} \rightarrow$ product rule.

EXAMPLE: The 4/2 model

In the case $d = 1$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma V(t)^{1/2}dW(t),$$

$$\frac{dD^i(t)}{D^i(t)} = (r^i + \pi(t)^\top \pi(t))dt + \left(a^i\sqrt{V(t)} + \frac{b^i}{\sqrt{V(t)}}\right)dZ(t),$$

where the i-th market price of risk $\pi^i(t)$ is the scalar process given by:

$$\pi^i(t) = a^i\sqrt{V(t)} + \frac{b^i}{\sqrt{V(t)}}.$$

$b^i = 0 \;\rightarrow\;$ Heston dynamics
$a^i = 0 \;\rightarrow\;$ 3/2 model

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- For each i s.t. we have a strict local martingale then risk neutral pricing is not justified.
- We study the $\mathbb{P}$ expectations

$$\mathbb{E}\left[\hat{B}^i(t)\right] = \hat{B}_0^i \prod_{k=1}^{d} \mathbb{E}\left[\xi_k^i(t)\right],$$

for

$$\xi_k^i(t) := \exp\left\{-\rho_k \int_0^t (a_k^i V_k^{1/2}(s) + b_k^i V_k^{-1/2}(s)) dW_k(s)\right. \tag{2}$$
$$\left. -\frac{1}{2}\rho_k^2 \int_0^t (a_k^i V_k^{1/2}(s) + b_k^i V_k^{-1/2}(s))^2 ds\right\}.$$

- The putative change of measure involves

$$d\tilde{W}_k(t) = dW_k(t) + \rho_k(a_k^i V_k^{1/2}(s) + b_k^i V_k^{-1/2}(s))dt,$$

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- The putative change of measure involves

$$d\tilde{W}_k(t) = dW_k(t) + \rho_k(a_k^i V_k^{1/2}(s) + b_k^i V_k^{-1/2}(s))dt,$$

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- Therefore, the process V has a different behavior at 0 under the two probability measures provided that

$$\sigma_k^2 \leq 2\kappa_k\theta_k \leq \sigma_k^2 + 2\rho_k\sigma_k b_k^i. \tag{3}$$

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$$\mathbb{E}\left[X_t^{-\alpha} e^{-\lambda X_t - \mu \int_0^t X_s \, ds - \nu \int_0^t \frac{ds}{X_s}}\right]$$

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- We extend the result to a multidimensional setting with stochastic interest rates of the form

$$r^i(t) = h^i + \langle \boldsymbol{H}^i, \boldsymbol{V}(t) \rangle + \langle \boldsymbol{G}^i, \boldsymbol{V}^{-1}(t) \rangle$$

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  pricing measures.
- Low maturities $\rightarrow$ Vega risk dominates $\rightarrow$ deterministic interest rates
  for simplicity.

Joint Calibrations
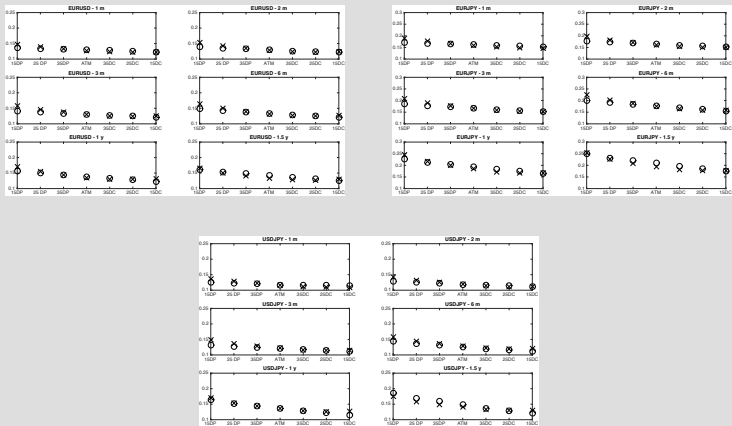
Measure	Test	02.23.2015	03.23.2015	04.22.2015	05.22.2015	06.22.2015
$\mathbb{P}$	$2\kappa_k\theta_k - \sigma_k^2$	0.0806	0.0490	0.0484	0.0488	0.0489
		0.3375	0.0787	0.0799	0.0753	0.0766
$\mathbb{Q}^{usd}$	$2\kappa_k\theta_k - \left(\sigma_k^2 + 2\rho_k\sigma_k b_k^{usd}\right)$	-0.0189	0.0135	-0.0007	-0.0027	0.0010
		0.1967	-0.0350	-0.0690	-0.0745	-0.0767
$\mathbb{Q}^{eur}$	$2\kappa_k\theta_k - \left(\sigma_k^2 + 2\rho_k\sigma_k b_k^{eur}\right)$	0.0159	0.0134	-0.0056	-0.0132	-0.0148
		0.3471	0.0338	0.0320	0.0215	0.0262
$\mathbb{Q}^{jpy}$	$2\kappa_k\theta_k - \left(\sigma_k^2 + 2\rho_k\sigma_k b_k^{jpy}\right)$	0.0221	0.0089	0.0031	0.0017	0.0017
		0.2547	0.0172	0.0097	0.0005	0.0087
$\mathbb{P}$						
$\mathbb{Q}^{usd}$		NO	NO	NO	NO	NO
$\mathbb{Q}^{eur}$	Risk neutral pricing possible	YES	YES	NO	NO	NO
$\mathbb{Q}^{jpy}$		YES	YES	YES	YES	YES

Tabelle: This table reports the Feller test under different (putative) measures. The value of the test is obtained from the calibrated parameters on multiple trading dates.

Implications

- Market seems to suggest the presence of **regime switches** between pricing approaches.
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- Toy pricing example: valuation of a zero-coupon bond in the absence of a martingale measure under deterministic interest rates. $\rightarrow$ significant pricing impact.

- Hedging performed by resorting on benchmarked risk minimization. We attain the final payoff using a non-self financing trading strategy with a lower initial value.

Thank you for your attention!

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