Marginal Expected Shortfall

under asymptotic independence

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- **Literature:** Axiomatic approach (Chen et al. (2013)), CoVaR (Adrian and Brunnermeier (2011)), SRISK (Bownlees and Engle (2015)), SystRisk (Brunnermeier and Cheridito (2014)), Zhou (2010).

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- **Marginal Expected Shortfall:** Cai et al. (2015), Hua and Joe (2012), Hua and Joe (2011), Zhu and Li (2012). Assume asymptotically tail dependence!

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Some examples in Hua and Joe (2012).

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- We work under a specific paradigm: **regular variation**

Regular and hidden regular variation

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- Pareto distribution ($\alpha > 0$): $\overline{F}(x) = x^{-\alpha}$, $x \geq 1$.

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- Typical example for asymptotically tail independence.

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In general we take:

- $b(t) = (1/\mathbb{P}(\max(X, Y) > \cdot\,))^{\leftarrow}(t) \in \mathcal{RV}_{1/\alpha}$,
- $b_0(t) = (1/\mathbb{P}(\min(X, Y) > \cdot\,))^{\leftarrow}(t) \in \mathcal{RV}_{1/\alpha_0}$.

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- $\overline{C}$ Gumbel copula with parameter θ: $\alpha_0 = \alpha 2^{\theta^{-1}-1}$ and

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Suppose $\boldsymbol{Z} = (Z_1, Z_2), \boldsymbol{Y} = (Y_1, Y_2), \boldsymbol{V} = (V_1, V_2)$ are random vectors in $[0, \infty)^2$ such that $\boldsymbol{Z} = \boldsymbol{Y} + \boldsymbol{V}$.

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(C) $\boldsymbol{Y} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ where $\alpha \geq 1$.

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(D) $\boldsymbol{V} \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E})$ and does not possess asymptotic independence where $\alpha \leq \alpha_0 < 1 + \alpha$.

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$$\lim_{t \to \infty} \frac{P(\|\boldsymbol{V}\| > t)}{P(\|\boldsymbol{Y}\| > t)} = 0.$$

Then $\boldsymbol{Z} \in \mathcal{HRV}(\alpha, \alpha_0, b, b_0, \nu, \nu_0)$.

Asymptotic behavior of the MES

MAIN RESULT

THEOREM

Suppose $(X, Y) \geq \mathbf{0}$ with $(X, Y) \in \mathcal{HRV}(\alpha, \alpha_0, b, b_0, \nu, \nu_0)$, $\mathbb{E}|X| < \infty$, (X, y) are asymptotically tail independent and "some technical conditions holds".

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$$\lim_{p \downarrow 0} \underbrace{\frac{p b_0^{\leftarrow}(\mathrm{VaR}_{1-p}(Y))}{\mathrm{VaR}_{1-p}(Y)}}_{=:a(p)\to 0} \mathrm{MES}(p) = \int_0^\infty \nu_0((x, \infty) \times (1, \infty))\, dx.$$

Main result

Comparison of the asymptotically tail dependent and tail independent case: $\mathrm{MES}(p) = \mathbb{E}(X|Y > \mathrm{VaR}_{1-p}(Y))$

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- Tail dependent case (cf. Cai et al. (2015)):

$$\text{MES}^D(p) \sim \text{const. VaR}_{1-p}(X)$$

- Tail independent case (above theorem):

$$\text{MES}^I(p) \sim \text{const.} \underbrace{\frac{1}{p b_0^{\leftarrow}(\text{VaR}_{1-p}(Y))}}_{\leq C} \text{VaR}_{1-p}(Y).$$

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Estimation of the MES

ESTIMATION

- $(X_1, Y_1), \ldots, (X_n, Y_n)$ iid with $(X_1, Y_1) \in \mathcal{HRV}(\alpha, \alpha_0, b, b_0, \nu, \nu_0)$ and asymptotically independent.

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- **Empirical estimator**

$$\widehat{\text{MES}}_{E,n}(k/n) = \frac{1}{k} \sum_{i=1}^{n} X_i \mathbf{1}_{\{Y_i > Y_{(k:n)}\}}$$

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Theorem

Let $k = k(n)$ be a sequence of integers satisfying $k \to \infty$, $k/n \to 0$ and $b_0^{\leftarrow}(b_2(n/k))/n \to 0$ as $n \to \infty$. Then

$$\frac{\widehat{\text{MES}}_{E,n}(k/n)}{\text{MES}(k/n)} \xrightarrow{\mathbb{P}} 1.$$

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$$(X, Y) = B(X_1, X_3) + (1 - B)(X_2, X_2)$$

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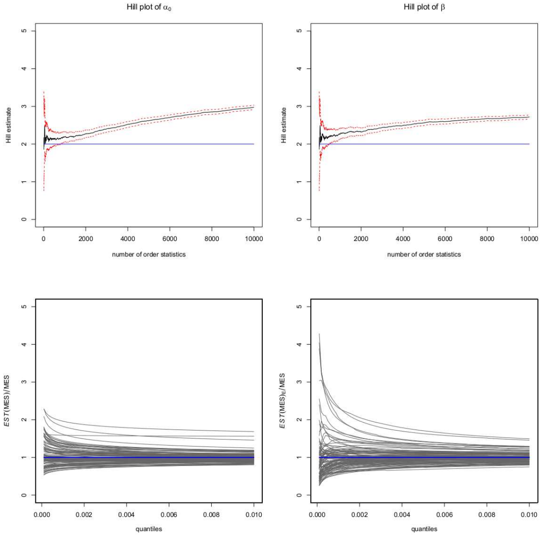
- Generate $n = 10000$ samples and fix $k = 100$.

EXAMPLES



Hill plot of α_0
Hill estimate
number of order statistics
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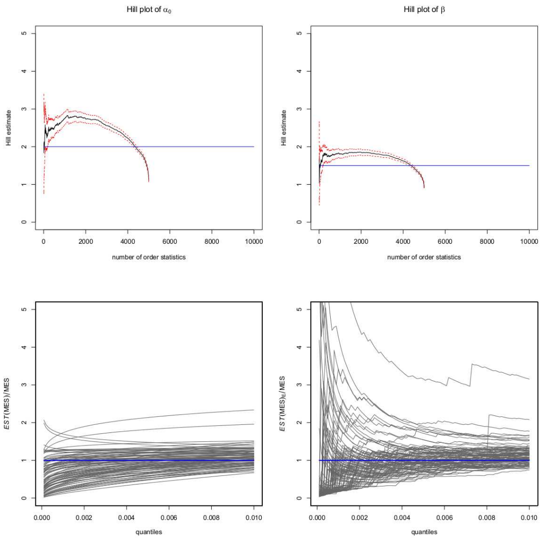

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CONCLUSION

- Asymptotic behavior of MES under asymptotic independence.
- A few important copula models come under this regime: Gaussian, EV copulas.
- Estimation procedure for MES under asymptotic independence.
- Similar results hold for the marginal mean excess

$$\mathbb{E}((X - \mathrm{VaR}_{1-p}(Y))_+ | Y > \mathrm{VaR}_{1-p}(Y)).$$

- A tractable method for measuring systemic risk.

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Thank you!