

AMBIGUOUS VOLATILITY, POSSIBILITY AND UTILITY IN CONTINUOUS TIME*

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January 19, 2012

Abstract

We formulate a model of utility for a continuous time framework that captures the decision-maker's concern with ambiguity or model uncertainty. The main novelty is in the range of model uncertainty that is accommodated. The probability measures entertained by the decision-maker are not assumed to be equivalent to a fixed reference measure and thus the model permits ambiguity about which scenarios are possible. Modeling ambiguity about volatility is a prime motivation and a major focus. Equilibrium relations for asset returns are derived in representative agent frameworks in both an Arrow-Debreu style economy and in a sequential Radner style economy.

Key words: ambiguity, uncertain volatility, option pricing, recursive utility, stochastic differential utility, G-expectation, G-Brownian motion, nonequivalent measures, uncertain possibility, quasisure analysis

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1. Introduction

1.1. Motivation and objectives

This paper formulates a model of utility for a continuous time framework that captures the decision-maker's concern with ambiguity or model uncertainty. Modeling ambiguity about volatility is a prime motivation and a major focus. Another focus is in the range of model uncertainty that is accommodated - the model accommodates also ambiguity about which scenarios are possible. At a technical level, this requires a significant departure from existing continuous time modeling of utility all of which is formulated within a probability space framework. The model is intended to provide a theoretical framework within which one could ultimately address the question "does ambiguity about volatility and possibility matter empirically?" In this paper, an application to financial economics takes the form of deriving equilibrium relations for asset returns in representative agent frameworks, in both an Arrow-Debreu style economy and in a sequential Radner style economy.

The model of utility is a continuous time version of multiple priors (or maxmin) utility formulated by Gilboa and Schmeidler [26] for a static setting.¹ Related continuous time models are provided by Chen and Epstein [7] and also Hansen, Sargent and coauthors (see Anderson et al. [1], for example).² The main value-added is that only the model proposed here captures ambiguity about volatility and about what is possible. In *all* existing literature on continuous time utility, ambiguity is modeled so as to retain the property that all priors are equivalent, that is, they agree which events are null. This universal restriction is driven not by an economic rationale but rather by the technical demands of continuous time modeling, specifically by the need to work within a probability space framework. Notably, in order to describe ambiguity authors invariably rely on Girsanov's theorem for changing measures. It provides a tractable characterization of alternative hypotheses about the true probability law, but it also limits alternative hypotheses to correspond to measures that are both mutually equivalent and that differ from one another only in what they imply about drift. This paper defines a more general framework within which one can model the utility of an individual who is not completely confident in any single probability law for volatility or in which future events are possible. This is done while maintaining a separation between

¹See the end of the introduction for comments on the potential use of alternatives to maxmin utility as the basis for continuous time modeling.

²The discrete time counterpart of the former is axiomatized in Epstein and Schneider [19].

risk aversion and intertemporal substitution as in Duffie and Epstein [16].

From an economics perspective, the assumption of equivalence seems far from innocuous. Informally, if her environment is complex, how could the decision-maker come to be certain of which scenarios regarding future asset prices and rates of return, for example, are possible? Moreover, in a continuous time setting ambiguity about volatility implies ambiguity about which scenarios are possible, and there is reason to model the former as we describe next.

A large literature has argued that stochastic time varying volatility is important for understanding features of asset returns, and particularly empirical regularities in derivative markets; see Eraker and Shaliastovich [22], Drechsler [14] and Bollerslev et al. [5], for recent examples of preference-based continuous time models; the latter argues extensively for the modeling advantages of the continuous time framework.³ In macroeconomic contexts, Bloom [4] and Fernandez-Villaverde and Guerron-Quintana [23] are recent studies that find evidence of stochastic time varying volatility and its effects on real variables. In all of these papers, evidence suggests that relevant volatilities follow complicated dynamics. The common modeling response is to postulate correspondingly complicated parametric laws of motion, including specification of the dynamics of the volatility of volatility. However, one might question whether agents in these models can learn these laws of motion precisely, and more generally, whether it is plausible to assume that agents become completely confident in any particular law of motion. In their review of the literature on volatility derivatives, Carr and Lee [6, pp. 324-5] raise this criticism of assuming a particular parametric process for the volatility of the underlying asset. The drawback they note is “the dependence of model value on the particular process used to model the short-term volatility.” They write that “the problem is particularly acute for volatility models because the quantity being modeled is not directly observable. Although an estimate for the initially unobserved state variable can be inferred from market prices of derivative securities, noise in the data generates noise in the estimate, *raising doubts that a modeler can correctly select any parametric stochastic process from the menu of consistent alternatives.*”

Thus we are led to develop a model of preference that accommodates ambiguity about volatility and in which the individual takes a stand only on bounds rather than on any particular parametric model of volatility dynamics. Maximization

³For example, they write (p. 5) that a continuous time formulation “has the distinct advantage of allowing for the calculation of internally consistent model implications across all sampling frequencies and return horizons.”

of this preference leads to decisions that are robust to misspecifications of the dynamics of volatility (as well as drift). Accordingly, we think of this aspect of our model as providing a way to *robustify stochastic volatility modeling*. See Example 3.3 for elaboration.

A possible objection to modeling ambiguity about volatility might take the form: “One can approximate the realized quadratic variation of a stock price (for example) arbitrarily well from frequent observations over any short time interval, and thus estimate the law of motion for its volatility extremely well. Consequently, ambiguity about volatility is implausible for a sophisticated agent.” However, even if one accepts the hypothesis that, contrary to the view of Carr and Lee, accurate estimation is possible, such an objection relies also on the assumption of a tight connection between the past and future that we relax. We are interested in situations where realized past volatility may not be a reliable predictor of volatility in the future. The rationale is that the stochastic environment is often too complex for a sophisticated individual to believe that her theory, whether of volatility or of other variables, captures all aspects. Being sophisticated, she is aware of the incompleteness of her theory. Accordingly, when planning ahead she believes there may be time-varying factors excluded by her theory that she understands poorly and that are difficult to identify statistically. Thus she perceives ambiguity when looking into the future. The amount of ambiguity may depend on past observations, and may be small for some histories, but it cannot be excluded *a priori*.

A similar objection might be raised to modeling ambiguity about possibility. It might take the form: “If distinct priors (or models) are not mutually absolutely continuous, then one can discriminate between them readily. Therefore, when studying nontransient ambiguity there is no loss in restricting priors to be equivalent.” The connection between past and future is again the core issue. Consider the individual at time t and her beliefs about the future. The source of ambiguity is her concern with locally time varying and poorly understood factors. This limits her confidence in predictions about the immediate future, or ‘next step’, to a degree that depends on history but that is not eliminated by the retrospective empirical discrimination between models.⁴ At a formal level, Epstein and Schneider [19] show that when backward induction reasoning is added to multiple priors utility, then the individual behaves *as if* the set of conditionals entertained at any

⁴See the discussion surrounding (1.3) for illustration of environments, limited in our view, where the remark on absolute continuity does have bite because of the tight connection assumed between past and future.

time t and state does not vary with marginal prior beliefs on time t measurable uncertainty. (They call this property rectangularity.) Thus looking back on past observations at t , even though the individual might be able to dismiss some priors or models as being inconsistent with the past, this is unimportant for prediction because the set of conditional beliefs about the future is unaffected.

In light of the continuous time setting, a natural application of the model is to financial economics. To demonstrate tractability, we apply the model to a representative agent endowment economy to study equilibrium asset returns in both Arrow-Debreu style and sequential Radner style frameworks (see Section 4). In particular, we describe how ambiguity about volatility and possibility are reflected in the implied version of the C-CAPM. As an illustration of the added explanatory power of the model, we show that it can rationalize the well documented feature of option prices whereby the Black-Scholes implied volatility exceeds the realized volatility of the underlying security. We also provide a counterpart for our setting of the fundamental arbitrage-free pricing rule that is usually described in terms of an equivalent martingale (or risk neutral) measure, thereby extending to the case of comprehensive ambiguity this cornerstone of asset pricing theory.

The paper proceeds as follows. In the remainder of the introduction, we provide an informal outline of the model and then cite more related literature. Section 2 briefly recalls the existing multiple-priors based recursive utility model. Section 3 is the heart of the paper and presents the model of utility, beginning with the construction of the set of priors and the definition of (nonlinear) conditional expectation. Subsection 3.5 describes how the nonequivalence of priors weakens the dynamic consistency properties of recursive models. Finally, Section 4 contains the application to asset returns. Most proofs are collected in appendices.

1.2. An informal outline of the model

Intuition for a special case of our model having ambiguity only about volatility can be derived from the following discrete-time trinomial model (see Levy et al. [37]). The special case is a generalization of Brownian motion that accommodates ambiguity about volatility. (The formal model is due to Peng [49], who calls it G-Brownian motion.⁵) The trinomial model ‘explains’ G-Brownian motion in the same way that the binomial tree provides intuition for Brownian motion.

Time varies over $\{0, h, 2h, \dots, (n-1)h, nh\}$, where $0 < h < 1$ scales the period length and n is a positive integer with $nh = T$. Uncertainty is driven by the

⁵See Appendix E below for an outline of key features.

colors of balls drawn from a sequence of urns. It is known that each urn contains 100 balls that are either red (R), yellow (Y) or green (G), where $R = G$ and $Y \leq 20$, and that the urns are constructed independently (informally speaking). Accordingly, they may very well differ in actual composition. A ball is drawn from each urn and the colors drawn determine the evolution of the state variable $B = (B_t)$ according to: $B_0 = 0$ and, for $t = h, \dots, nh$,

$$B_t - B_{t-h} = \begin{cases} h^{1/2} & \text{if } R_t \\ 0 & \text{if } Y_t \\ -h^{1/2} & \text{if } G_t \end{cases}$$

Thus one can be certain that B_t is a martingale, (that is, it is a martingale with respect to whichever probability measure describes the sequence of urns), but its one-step-ahead variance σ^2 depends on the number of yellow balls and thus is ambiguous - it equals ph , where we know only that $0 \leq 1 - p \leq 0.2$, or

$$\underline{\sigma}^2 = .8h \leq \sigma^2 = ph \leq h = \bar{\sigma}^2.$$

Because urns are perceived to be independent, past draws do not reveal anything about the future and leave ambiguity undiminished. This is an extreme case that is a feature of this example and is a counterpart of the assumption of i.i.d. increments in the familiar binomial tree.

This trinomial model converges weakly (see Dolinsky et al. [12]) to a continuous-time model on the interval $[0, T]$ in which the individual is certain that the driving process $B = (B_t)$ is a martingale, but its volatility is known only up to the interval $[\underline{\sigma}, \bar{\sigma}]$. To be more precise about the meaning of volatility, recall that the quadratic variation process of (B_t) is defined by

$$\langle B \rangle_t(\omega) = \lim_{\Delta t_k \rightarrow 0} \sum_{t_k \leq t} |B_{t_{k+1}}(\omega) - B_{t_k}(\omega)|^2 \quad (1.1)$$

where $0 = t_1 < \dots < t_n = t$ and $\Delta t_k = t_{k+1} - t_k$. (By Follmer [24] and Karandikar [34], the above limit exists almost surely for every measure that makes B a martingale, thus giving a universal stochastic process $\langle B \rangle$; because the individual is certain that B is a martingale, this limited universality is all we need.) Then the volatility (σ_t) of B is defined by

$$d\langle B \rangle_t = \sigma_t^2 dt.$$

Therefore, the interval constraint on volatility can be written also in the form

$$\underline{\sigma}^2 t \leq \langle B \rangle_t \leq \bar{\sigma}^2 t. \quad (1.2)$$

The preceding defines the stochastic environment. Consumption and other processes are defined accordingly (for example, they are required to be adapted to the natural filtration generated by B). We emphasize that our analysis is much more general than suggested by this outline. Importantly, the interval $[\underline{\sigma}, \bar{\sigma}]$ can be time and state varying, and the dependence on history of the interval at time t is unrestricted, thus *permitting any model of how ambiguity varies with observation (that is, learning) to be accommodated*. In addition, we admit multidimensional driving processes and also ambiguity about both drift and volatility.

The challenge in formulating the model is that it cannot be done within a probability space framework. Typically, the ambient framework is a probability space (Ω, P_0) , it is assumed that B is a Brownian motion under P_0 , and importantly, P_0 is used to define null events. Thus random variables and stochastic processes are defined only up to the P_0 -almost sure qualification and P_0 is an essential part of the definition of all formal domains. However, ambiguity about volatility implies that nullity (or possibility) cannot be defined by any single probability measure. This is illustrated starkly by the above example. Let B be a Brownian motion under P_0 and denote by $P^\underline{\sigma}$ and $P^\bar{\sigma}$ the probability distributions over continuous paths induced by the two processes $(\underline{\sigma}B_t)$ and $(\bar{\sigma}B_t)$. Given the ambiguity described by (1.2), $P^\underline{\sigma}$ and $P^\bar{\sigma}$ are two alternative hypotheses about the probability law driving uncertainty. It is apparent that they are mutually singular because

$$P^\underline{\sigma}(\{\langle B \rangle_T = \underline{\sigma}^2 T\}) = 1 = P^\bar{\sigma}(\{\langle B \rangle_T = \bar{\sigma}^2 T\}). \quad (1.3)$$

In particular, the priors representing alternative hypotheses about volatility are not equivalent. To overcome the resulting difficulty, we define appropriate domains of stochastic processes by using the entire set of probability measures \mathcal{P} induced by $(\sigma_t B_t)$ for some process (σ_t) satisfying the bounds. Statements about random variables are required to hold *quasisurely*, that is, P -almost surely for every P in \mathcal{P} .⁶

We caution against two possible misinterpretations of (1.3). The first is conceptual. If $P^\underline{\sigma}$ and $P^\bar{\sigma}$ were the only two hypotheses being considered, then ambiguity could be eliminated quickly because one can approximate volatility locally as in (1.1) and thus use observations on a short time interval to differentiate between the two hypotheses. This is possible because of the tight connection between past and future volatility imposed in each of $P^\underline{\sigma}$ and $P^\bar{\sigma}$. As discussed above, this is

⁶Quasisure stochastic analysis was developed by Denis and Martini [11]. See also Soner et al. [55] and Denis et al. [9] for elaboration on why a probability space framework is inadequate and for a comparison of quasisure analysis with related approaches.

not the kind of ambiguity we have in mind. The point of (1.3) is only to illustrate nonequivalence as simply as possible. Importantly, such nonequivalence of priors is a feature also of the more complex and interesting cases at which the model is directed.

A second more technical point is that (1.3) might suggest that a set \mathcal{P} modeling ambiguous volatility (but unambiguous drift) consists only of measures that are mutually singular. That is not the case as illustrated by Example 3.7.

1.3. Related literature

Besides those already mentioned, there are only a few relevant papers in the literature on continuous time utility theory. Denis and Kervarec [10] formulate multiple priors utility functions and study optimization in a continuous-time framework; they do not assume equivalence of measures but they restrict attention to the case where only terminal consumption matters and where all decisions are made at a single ex ante stage. Bion-Nadal and Kervarec [3] study risk measures (which can be viewed as a form of utility function) in the absence of certainty about what is possible.

There is a literature on the pricing of derivative securities when volatility is ambiguous. The problem was first studied by Lyons [41], Avellaneda et al. [2] and Lewicki and Avellaneda [38]; recent explorations include Denis and Martini [11], Cont [8] and Vorbrink [61].⁷ They employ no-arbitrage arguments to derive upper and lower bounds on security prices. Theorem 4.2 provides an analogous result for our endowment economy. However, it is to be noted that our result is the first to characterize these price bounds in terms of ‘state price densities,’ thus extending to the setting of comprehensive model uncertainty the classical characterization of arbitrage free prices. Another important difference is that our model of preference permits an equilibrium approach to pricing and consequently also sharper predictions about prices. This advantage of an equilibrium analysis is similar to that familiar in ambiguity-free incomplete market settings where no-arbitrage delivers multiple risk-neutral martingale measures whereas an equilibrium model can select among them. Finally, we study a Lucas-style endowment economy and thus take the endowment as the basic primitive, while the cited papers take the prices of primitive securities as given.

We mention one more potential application. Working in a discrete-time setting, Epstein and Schneider [20] point to ambiguous volatility as a way to model signals

⁷These papers often refer to uncertain volatility rather than to ambiguous volatility.

with ambiguous precision. This leads to a new way to measure information quality that has interesting implications for financial models (see also Illeditsch [33]). The utility framework that we provide should permit future explorations of this dimension of information quality in continuous time settings.

At a technical level, we exploit and extend recent advances in stochastic calculus that do not require a probability space framework. A seminal contribution is the notion of G-expectation due to Peng [47, 49, 50, 51], wherein a nonlinear expectations operator is defined by a set of undominated measures.⁸ Denis and Martini [11] develop quasi-sure stochastic analysis, although they do not address conditioning. The close connection between quasisure analysis and G-expectation analysis is revealed by Denis et al. [9] and further examined by Soner et al. [55].⁹ We combine elements of both approaches. Conditioning, or updating, is obviously a crucial ingredient in modeling dynamic preferences. In this respect we adapt the approach in Soner et al. [58] and Nutz [44] to conditioning undominated sets of measures.¹⁰ However, these analyses do not apply off-the-shelf because, for example, they permit ambiguity about volatility but not about drift. In particular, accommodating both kinds of ambiguity necessitates a novel construction of the set of priors.

We conclude this introduction with comments on the connection to literature in decision theory. While we adopt multiple priors utility as the basis for our continuous time model, one might wonder whether other static models of ambiguity aversion could be extended to capture ambiguous volatility in continuous time. However, for two models that have received considerable attention, the smooth ambiguity model of Klibanoff et al. [36] and multiplier utility, introduced into economics by Anderson et al. [1] and axiomatized by Strzalecki [60], there is reason to be skeptical. Skiadas [54] shows that the smooth model incorporated recursively into a continuous time Brownian framework is observationally indistinguishable from stochastic differential utility (Duffie and Epstein [16]) in the strong sense that they define the same preference order over consumption processes. A similar observational equivalence exists for multiplier utility (see Skiadas [52], for example). Thus while one might wish to use these models to reinterpret behavior as being due to ambiguity aversion rather than risk aversion, they do not expand

⁸A set of probability measures is undominated if there does not exist a probability measure P^* such that every measure in the set is absolutely continuous with respect to P^* .

⁹Related developments are provided by Denis et al. [9], Bion-Nadal et al. [3] and Soner et al. [55, 56, 57].

¹⁰Peng [47, 48] provides a related approach to conditioning.

the range of behavior that can be rationalized (see Epstein and Schneider [21] for further discussion). In both cases, it is ambiguity about drift alone that is at issue. However, because ambiguity about volatility seems (to us) to be more difficult to model, the potential of these alternatives to the approach adopted here is at worst not promising and at best an open question.

2. Recursive Utility under Equivalence

For perspective, we begin by outlining the Chen-Epstein model. Time t varies over $[0, T]$, where $T > 1$. Other primitives include a probability space $(\Omega, \mathcal{F}, P_0)$, a standard d -dimensional Brownian motion $B = (B_t)$ defined on $(\Omega, \mathcal{F}, P_0)$, and the Brownian filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. Consumption processes c take values in C , a convex subset of \mathbb{R}^ℓ . The objective is to formulate a utility function on a domain D of C -valued consumption processes.

If P_0 describes the individual's beliefs, then following Duffie and Epstein [16] utility may be defined by:¹¹

$$V_t^{P_0}(c) = E^{P_0} \left[\int_t^T f(c_s, V_s^{P_0}) ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (2.1)$$

Here $V_t^{P_0}$ gives the utility of the continuation $(c_s)_{s \geq t}$ and $V_0^{P_0}$ is the utility of the entire process c . The function f is a primitive of the specification, called an *aggregator*. The special case $f(c, v) = u(c) - \beta v$ delivers the standard expected utility specification

$$V_t(c) = E^{P_0} \left[\int_t^T e^{-\beta(s-t)} u(c_s) ds \mid \mathcal{F}_t \right]. \quad (2.2)$$

The use of more general aggregators permits a partial separation of risk aversion from intertemporal substitution.

To admit a concern with model uncertainty, Chen and Epstein replace the single measure P_0 by a set \mathcal{P}^Θ of measures equivalent to P_0 . This is done by specifying a suitable set of densities. For each well-behaved \mathbb{R}^d -valued process $\theta = (\theta_t)$, called a density generator, let

$$z_t^\theta \equiv \exp \left\{ -\frac{1}{2} \int_0^t \|\theta_s\|^2 ds - \int_0^t \theta_s^\top dB_s \right\}, \quad 0 \leq t \leq T,$$

¹¹Below we often suppress c and write V_t instead of $V_t(c)$. The dependence on the state ω is also frequently suppressed.

and let P^θ be the probability measure on (Ω, \mathcal{F}) with density z_T^θ , that is,

$$\frac{dP^\theta}{dP_0} = z_T^\theta; \text{ more generally, } \left. \frac{dP^\theta}{dP} \right|_{\mathcal{F}_t} = z_t^\theta \text{ for each } t. \quad (2.3)$$

Given a set Θ of density generators, the corresponding set of priors is

$$\mathcal{P}^\Theta = \{ P^\theta : \theta \in \Theta \text{ and } P^\theta \text{ is defined by (2.3)} \}. \quad (2.4)$$

By construction, all measures in \mathcal{P}^Θ are equivalent to P_0 . Because its role is only to define null events, any other member of \mathcal{P}^Θ could equally well serve as the reference measure.

Continuation utilities are defined by:

$$V_t = \inf_{P \in \mathcal{P}^\Theta} E^P \left[\int_t^T f(c_s, V_s) ds \mid \mathcal{F}_t \right]. \quad (2.5)$$

An important property of the utility process is dynamic consistency, which follows from the following recursivity: For every c in D ,

$$V_t = \min_{P \in \mathcal{P}^\Theta} E_P \left[\int_t^\tau f(c_s, V_s) ds + V_\tau \mid \mathcal{F}_t \right], \quad 0 \leq t < \tau \leq T. \quad (2.6)$$

Regarding interpretation, by the Girsanov theorem $B_t + \int_0^t \theta_s ds$ is a Brownian motion under P^θ . Thus as θ varies over Θ and P^θ varies over \mathcal{P}^Θ , alternative hypotheses about the drift of the driving process are defined. Accordingly, the infimum suggests that the utility functions V_t exhibit an aversion to ambiguity about the drift. Because B_t has variance-covariance matrix equal to the identity according to all measures in \mathcal{P}^Θ , there is no ambiguity about volatility. Neither is there any uncertainty about what is possible because P_0 defines which events are null.

3. Utility

The idea behind our construction of utility is as follows. The key is the specification of beliefs. We suppose that the individual believes that all uncertainty is driven by an \mathbb{R}^d -valued continuous process X whose realizations she will observe through time. She must form beliefs about this process; in other words, she needs to form beliefs over Ω , the set of all continuous trajectories. One possible assumption is that she is certain that $X = B$, where B is a Brownian motion under the

measure P_0 ; this yields the common specification of a “Brownian environment.” However, in general our agent is not certain that X has zero drift and/or unit variance (take $d = 1$ here). Thus she admits the possibility that $X = X^\theta$ where θ parametrizes an alternative specification of drift and volatility (see (3.1)). Each conceivable process X^θ induces a probability law, denoted P^θ , on the set of Ω of continuous trajectories. Because θ varies over a set Θ , a set \mathcal{P}^Θ of probability measures is induced. This is the set of priors used, as in the Gilboa-Schmeidler model, to define utility and guide choice between consumption processes. If all alternative hypotheses display unit variance, then ambiguity is limited to the drift as in the Chen-Epstein model, and measures in \mathcal{P}^Θ are pairwise equivalent. At the other extreme, if they all display zero drift, then ambiguity is limited to volatility and many measures in \mathcal{P}^Θ are mutually singular. The general model captures ambiguity about both drift and volatility.¹²

A key ingredient not mentioned thus far is conditioning. The absence of a reference measure poses a particular difficulty for updating because of the need to update beliefs conditional on events having zero probability according to some, but not all, priors. Our solution resembles that adopted in the theory of extensive form games, namely the use of conditional probability systems as primitives. It resembles also the approach in the discrete time model in Epstein and Schneider [19], where roughly, conditional beliefs about the next instant for every time and history are adopted as primitives and are pasted together by backward induction to deliver the ex ante set of priors.

3.1. Preliminaries

Time varies over $[0, T]$. Paths or trajectories of the driving process are assumed to be continuous and thus are modeled by elements of $C^d([0, T])$, the set of all \mathbb{R}^d -valued continuous functions on $[0, T]$, endowed with the sup norm. The generic path is $\omega = (\omega_t)_{t \in [0, T]}$, where we write ω_t instead of $\omega(t)$. All relevant paths begin at 0 and thus we define the canonical state space to be

$$\Omega = \{\omega = (\omega_t) \in C^d([0, T]) : \omega_0 = 0\}.$$

¹²Modeling ambiguity about volatility only would simplify the paper and might be thought to be advisable on the grounds of focussing on what is new. However, ambiguity about drift is at least as intuitive and has been shown to be useful in applied work; and it is not at all obvious how to combine the two forms of ambiguity in one model. Therefore, we present the general model. Indeed, an innovative part of our technical analysis relative to the cited math literature lies exactly in our modeling of ambiguity about *both* volatility *and* drift.

Define also the set of paths over the time interval $[0, t]$:

$${}^t\Omega = \{ {}^t\omega = ({}^t\omega_s) \in C^d([0, t]) : {}^t\omega_0 = 0 \}.$$

Identify ${}^t\Omega$ with a subspace of Ω by identifying any ${}^t\omega$ with the function on $[0, T]$ that is constant at level ${}^t\omega_t$ on $[t, T]$. We adopt the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ where \mathcal{F}_t is the Borel σ -field on ${}^t\Omega$. (Below, for any topological space we always adopt the Borel σ -field even where not mentioned explicitly.)

The coordinate process (B_t) , where $B_t(\omega) = \omega_t$, is denoted by B . Let P_0 be a probability measure on Ω such that B is Brownian motion under P_0 . It is a reference measure only in the mathematical sense of facilitating the description of the individual's set of priors; but the latter need not contain P_0 .

3.2. Drift and Volatility Hypotheses

The individual is not certain that B is the driving process. She entertains a range of alternative hypotheses $X^\theta = (X_t^\theta)$ parametrized by θ . Here $\theta_t = (\mu_t, \sigma_t)$ is an \mathcal{F} -progressively measurable process with values in $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ that describes a conceivable process for drift $\mu = (\mu_t)$ and for volatility $\sigma = (\sigma_t)$.¹³ The driving process X^θ solves the following stochastic differential equation (SDE) under P_0 :

$$dX_t^\theta = \mu_t(X^\theta)dt + \sigma_t(X^\theta)dB_t, \quad X_0^\theta = 0, \quad t \in [0, T]. \quad (3.1)$$

Denote by Θ^{SDE} the set of all processes θ that ensure a unique strong solution X^θ to the SDE.¹⁴

The primitive of the model is the process of correspondences (Θ_t) , where, for each t ,

$$\Theta_t : \Omega \rightsquigarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}.$$

It defines the set of admissible drift and volatility processes via

$$\Theta = \{ \theta \in \Theta^{SDE} : \theta_t(\omega) \in \Theta_t(\omega) \text{ for all } (t, \omega) \in [0, T] \times \Omega \}. \quad (3.2)$$

Defer technical details until the end of this subsection and proceed somewhat informally. Think of $\Theta_t(\omega)$ as the set of instantaneous drift and volatility pairs

¹³Write $\theta = (\mu, \sigma)$.

¹⁴By uniqueness we mean that $P_0(\{\sup_{0 \leq t \leq T} |X_t^\theta - X_t^{\theta'}| > 0\}) = 0$ for any other strong solution $(X_t^{\theta'})$. A Lipschitz condition and boundedness are sufficient for existence of a unique solution, but these properties are not necessary and a Lipschitz condition is not imposed.

thought to be conceivable at time t given the history corresponding to state ω . The dependence on history permits the model to accommodate learning. We emphasize that apart from technical regularity conditions, the form of history dependence is unrestricted and hence so is the nature of learning. Just as in the Chen-Epstein model, we provide a framework within which additional structure modeling learning can be added. Another noteworthy feature is that Θ , the set of admissible drift and volatility processes, consists of *all* (suitably measurable) selections from (Θ_t) . Thus processes θ in Θ may be much more erratic than might be suggested by the nature of the history dependence of Θ_t ; for example, even if the latter is independent of time and history, an admissible process θ can vary with time and history in complicated ways.

We illustrate the scope of the model through special cases or examples.

Example 3.1 (Ambiguous drift). *If $\Theta_t(\omega) \subset \mathbb{R}^d \times \{\sigma_t(\omega)\}$ for every t and ω , for some volatility process σ , then there is ambiguity only about drift. If $d = 1$, it is modeled by the random and time varying interval $[\underline{\mu}_t, \bar{\mu}_t]$. The regularity conditions below for (Θ_t) are satisfied if: $\sigma_t^2 \geq a > 0$ and $\bar{\mu}_t - \underline{\mu}_t > 0$ everywhere, and if $\bar{\mu}_t$ and $\underline{\mu}_t$ are continuous in ω uniformly in t .*

Example 3.2 (Ambiguous volatility). *If $\Theta_t(\omega) \subset \{\mu_t(\omega)\} \times \mathbb{R}^{d \times d}$ for every t and ω , for some drift process μ , then there is ambiguity only about volatility. If $d = 1$, it is modeled by the random and time varying interval $[\underline{\sigma}_t, \bar{\sigma}_t]$. The regularity conditions below for (Θ_t) are satisfied if: $\bar{\sigma}_t > \underline{\sigma}_t \geq a > 0$ everywhere, and if $\bar{\sigma}_t$ and $\underline{\sigma}_t$ are continuous in ω uniformly in t .*

A variation is important below. Allow $d \geq 1$ and let $\mu_t = 0$. Then (in a suitable sense) there is certainty that B is a martingale in spite of uncertainty about the true probability law. Volatility (σ_t) is a process of $d \times d$ matrices. Let the admissible volatility processes (σ_t) be those satisfying $\sigma_t \in \Gamma$, where Γ is any compact convex subset of $\mathbb{R}^{d \times d}$ such that, for all σ in Γ , $\sigma \sigma^\top \geq \hat{a}$ for some positive definite matrix \hat{a} .¹⁵ This specification formalizes a continuous time version of the trinomial model (Section 1.2) and is essentially equivalent to Peng's [49] notion of G -Brownian motion.¹⁶

¹⁵Given symmetric matrices A' and A , $A' \geq A$ if $A' - A$ is positive semidefinite.

¹⁶Peng provides generalizations of Itô's Lemma and Itô integration appropriate for G -Brownian motion that we exploit in Section 4 when considering asset pricing. (See Appendix E for more details.)

Example 3.3 (Robust stochastic volatility). *This is a special case of the preceding example but we describe it separately in order to highlight the connection of our model to the stochastic volatility literature. By a stochastic volatility model we mean the hypothesis that the driving process has zero drift and that its volatility is stochastic and is described by a single process (σ_t) satisfying regularity conditions of the sort given above. The specification of a single process for volatility indicates the individual's complete confidence in the implied dynamics. Suppose, however, that (σ_t^1) and (σ_t^2) describe two alternative stochastic volatility models that are put forth by expert econometricians;¹⁷ for instance, they might conform to the Hull and White [31] and Heston [28] parametric forms respectively. The models have comparable empirical credentials and are not easily distinguished empirically, but their implications for optimal choice (or for the pricing of derivative securities, which is the context for the earlier quote from Carr and Lee [6]) differ significantly. Faced with these two models, the individual might place probability $\frac{1}{2}$ on each being the true model. But why should she be certain that either one is true? Both (σ_t^1) and (σ_t^2) may fit data well to some approximation, but other approximating models may do as well. An intermediate model such as $(\frac{1}{2}\sigma_t^1 + \frac{1}{2}\sigma_t^2)$ is one alternative, but there are many others that “lie between” (σ_t^1) and (σ_t^2) and that plausibly should be taken into account. Accordingly, (assuming $d = 1$), let*

$$\underline{\sigma}_t(\omega) = \min\{\sigma_t^1(\omega), \sigma_t^2(\omega)\} \text{ and } \bar{\sigma}_t(\omega) = \max\{\sigma_t^1(\omega), \sigma_t^2(\omega)\},$$

and admit all volatility processes with values lying in the interval $[\underline{\sigma}_t(\omega), \bar{\sigma}_t(\omega)]$ for every ω . Given also the conservative nature of multiple priors utility, the individual will be led thereby to take decisions that are robust to (many) misspecifications of the dynamics of volatility. We emphasize that modeling such robustness requires moving outside a probability space framework.

When $d > 1$, one way to robustify is through adoption of (Θ_t) given by

$$\Theta_t(\omega) = \left\{ \sigma \in \mathbb{R}^{d \times d} : \sigma_t^1(\omega) (\sigma_t^1(\omega))^\top \leq \sigma \sigma^\top \leq \sigma_t^2(\omega) (\sigma_t^2(\omega))^\top \right\},$$

though other natural specifications exist in the multidimensional case.

Example 3.4 (Joint ambiguity). *The model is flexible in the way it relates ambiguity about drift and ambiguity about volatility. For example, a form of independence is modeled if (taking $d = 1$)*

$$\Theta_t(\omega) = [\underline{\mu}_t(\omega), \bar{\mu}_t(\omega)] \times [\underline{\sigma}_t(\omega), \bar{\sigma}_t(\omega)], \quad (3.3)$$

¹⁷There is an obvious extension to any finite number of models.

the Cartesian product of the intervals described in the preceding two examples.

An alternative hypothesis is that drift and volatility are thought to move together. This is captured by specifying, for example,

$$\Theta_t(\omega) = \{(\mu, \sigma) \in \mathbb{R}^2 : \mu = \mu_{\min} + z, \sigma^2 = \sigma_{\min}^2 + 2z/\gamma, 0 \leq z \leq \bar{z}_t(\omega)\}, \quad (3.4)$$

where μ_{\min} , σ_{\min}^2 and $\gamma > 0$ are fixed and known parameters. The regularity conditions for (Θ_t) are satisfied if \bar{z}_t is positive everywhere and continuous in ω uniformly in t . This specification is adapted from Epstein and Schneider [21].

Example 3.5 (Markovian ambiguity). Assume that (Θ_t) satisfies:

$$\omega'_t = \omega_t \implies \Theta_t(\omega') = \Theta_t(\omega).$$

Then ambiguity depends only on the current state and not on history. Note, however, that according to (3.2), the drift and volatility processes deemed possible are not necessarily Markovian - θ_t can depend on the complete history at any time. Thus the individual is not certain that the driving process is Markovian, but the set of processes that she considers possible at any given time is independent of history beyond the prevailing state.

Finally, we describe the technical regularity conditions imposed on (Θ_t) .

- (i) *Measurability*: The correspondence $(t, \omega) \mapsto \Theta_t(\omega)$ on $[0, s] \times \Omega$ is $\mathcal{B}([0, s]) \times \mathcal{F}_s$ -measurable for every $0 < s \leq T$.
- (ii) *Uniform Boundedness*: There is a compact subset \mathcal{K} in $\mathbb{R}^d \times \mathbb{R}^{d \times d}$ such that $\Theta_t : \Omega \rightsquigarrow \mathcal{K}$ each t .
- (iii) *Compact-Convex*: Each Θ_t is compact-valued and convex-valued.
- (iv) *Uniform Nondegeneracy*: There exists \hat{a} , a $d \times d$ real-valued positive definite matrix, such that for every t and ω , if $(\mu_t, \sigma_t) \in \Theta_t(\omega)$, then $\sigma_t \sigma_t^\top \geq \hat{a}$.
- (v) *Uniform Continuity*: The process (Θ_t) is uniformly continuous in the sense defined in Appendix A.
- (vi) *Uniform Interiority*: There exists $\delta > 0$ such that $ri^\delta \Theta_t(\omega) \neq \emptyset$ for all t and ω , where $ri^\delta \Theta_t(\omega)$ is the δ -relative interior of $\Theta_t(\omega)$. (For any $D \subseteq (\mathbb{R}^d \times \mathbb{R}^{d \times d})$ and $\delta > 0$, $ri^\delta D \equiv \{x \in D : (x + B_\delta(x)) \cap (\text{aff } D) \subset D\}$, where $\text{aff } D$ is the affine hull of D and $B_\delta(x)$ denotes the open ball of radius δ .)

- (vii) *Uniform Affine Hull*: The affine hulls of $\Theta_{t'}(\omega')$ and $\Theta_t(\omega)$ are the same for every (t', ω') and (t, ω) in $[0, T] \times \Omega$.

Conditions (i)-(iii) parallel assumptions made by Chen and Epstein [7]. A form of Nondegeneracy is standard in financial economics. The remaining conditions are adapted from Nutz [44] and are imposed in order to accommodate ambiguity in volatility. The major differences from Nutz' assumptions are in (vi) and (vii). Translated into our setting, he assumes that $\text{int}^\delta \Theta_t(\omega) \neq \emptyset$, where, for any D , $\text{int}^\delta \Theta_t(\omega) = \{x \in D : (x + B_\delta(x)) \subset D\}$. By weakening his requirement to deal with relative interiors, we are able to broaden the scope of the model in important ways (see the first three examples above). Because each $\Theta_t(\omega)$ is convex, if it also has nonempty interior then its affine hull is all of $\mathbb{R}^d \times \mathbb{R}^{d \times d}$. Then $\text{ri}^\delta \Theta_t(\omega) = \text{int}^\delta \Theta_t(\omega) \neq \emptyset$ and also (vii) is implied. In this sense, (vi)-(vii) are jointly weaker than assuming $\text{int}^\delta \Theta_t(\omega) \neq \emptyset$.

3.3. Priors, expectation and conditional expectation

We proceed to translate the set Θ of hypotheses about drift and volatility into a set of priors. Each θ induces (via P_0) a probability measure P^θ on (Ω, \mathcal{F}_T) given by

$$P^\theta(A) = P_0(\{\omega : X^\theta(\omega) \in A\}, A \in \mathcal{F}_T).$$

Therefore, we arrive at the set of priors \mathcal{P}^Θ given by

$$\mathcal{P}^\Theta = \{P^\theta : \theta \in \Theta\}. \quad (3.5)$$

(Note that P_0 lies in \mathcal{P}^Θ if and only if $(0, \mathbb{I}_{d \times d}) \in \Theta_t(\omega) \, dt \otimes dP_0\text{-a.e.}$, which need not be the case.) Fix Θ and denote the set of priors \mathcal{P}^Θ simply by \mathcal{P} .

Given \mathcal{P} , we define (nonlinear) expectation as follows. For a random variable ξ on (Ω, \mathcal{F}_T) , if $\sup_{P \in \mathcal{P}} E_P \xi < \infty$ define

$$\hat{E} \xi = \sup_{P \in \mathcal{P}} E_P \xi. \quad (3.6)$$

Because we will assume that the individual is concerned with worst-case scenarios, below we use the fact that

$$\inf_{P \in \mathcal{P}} E_P \xi = -\hat{E}[-\xi].$$

A naive approach to defining conditional expectation would be to use the standard conditional expectation $E_P[\xi \mid \mathcal{F}_t]$ for each P in \mathcal{P} and then to take the

(essential) supremum over \mathcal{P} . Such an approach immediately encounters a road-block due to the nonequivalence of priors. The conditional expectation $E_P[\xi \mid \mathcal{F}_t]$ is well defined only P -almost surely, while to be a meaningful object for analysis, a random variable must be defined quasisurely, that is, from the perspective of every measure in \mathcal{P} . In other words, and speaking very informally, conditional beliefs must be defined at every node deemed possible by some measure in \mathcal{P} .

To proceed, recall the construction of the set of priors through (3.1) and the set Θ of admissible drift and volatility processes. If $\theta = (\theta_s)$ is a conceivable scenario ex ante, then $(\theta_s(t\omega, \cdot))_{t \leq s \leq T}$ is seen by the individual ex ante as a conceivable continuation from time t along the history ω . We assume that then it is also a conceivable scenario ex post conditionally on (t, ω) , thus ruling out surprises or unanticipated changes in outlook. Accordingly, $X^{\theta, t, \omega} = (X_s^{\theta, t, \omega})_{t \leq s \leq T}$ is a conceivable conditional scenario for the driving process if it solves the following SDE under P_0 :

$$\begin{cases} dX_s^{\theta, t, \omega} = \mu_s(X_s^{\theta, t, \omega})ds + \sigma_s(X_s^{\theta, t, \omega})dB_s, & t \leq s \leq T \\ X_s^{\theta, t, \omega} = \omega_s, & 0 \leq s \leq t. \end{cases} \quad (3.7)$$

The solution $X^{\theta, t, \omega}$ induces a probability measure $P_t^{\theta, \omega} \in \Delta(\Omega)$, denoted simply by P_t^ω with θ suppressed when it is understood that $P = P^\theta$. For each P in \mathcal{P} , the measure $P_t^\omega \in \Delta(\Omega)$ is defined for every t and ω , and it is a version of the regular \mathcal{F}_t -conditional probability of P (see Lemma B.2).

The set of all such conditionals obtained as θ varies over Θ is denoted \mathcal{P}_t^ω , that is,

$$\mathcal{P}_t^\omega = \{P_t^\omega : P \in \mathcal{P}\}. \quad (3.8)$$

We take \mathcal{P}_t^ω to be the individual's set of priors conditional on (t, ω) .¹⁸

To elucidate this specification of conditioning, consider the case where there is ambiguity only about volatility as in Example 3.2. In the simplest specification, each σ_t is assumed to lie in the fixed interval $[\underline{\sigma}, \bar{\sigma}]$. Then an ex ante conceivable scenario for volatility is $\sigma_s = \underline{\sigma}$ for all time and states. Our assumption is that then at any t and conditional on any history prevailing then, volatility constant at $\underline{\sigma}$ remains a conceivable scenario for the time interval $[t, T]$. This makes sense if one is modeling an individual who perceives no connection between the past and

¹⁸The evident parallel with the earlier construction of the ex ante set \mathcal{P} can be expressed more formally because the construction via (3.7) can be expressed in terms of a process of correspondences $(\Theta_s^{t, \omega})_{t \leq s \leq T}$, $\Theta_s^{t, \omega} : C^d([t, T]) \rightsquigarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}$, satisfying counterparts of the regularity conditions (i)-(vii) on the time interval $[t, T]$.

future, as in the trinomial example. Note, however, that this is only an extreme special case of the model. In order to permit history to influence perceptions of the future, let each interval $[\underline{\sigma}_s, \bar{\sigma}_s]$ depend on the history at s . Then the volatility path constant at the level 1, for example, would be excluded ex ante unless $\underline{\sigma}_s \leq 1 \leq \bar{\sigma}_s$ globally. Suppose, however, that the process $(\sigma_s)_{0 \leq s \leq T}$ satisfies these interval constraints and thus is one ex ante hypothesis about volatility. Our key assumption is that then the continuation $(\sigma_s(t\omega, \cdot))_{t \leq s \leq T}$ is an admissible hypothesis conditional on (t, ω) . The rationale is that the influence of the past is already implicit in the ex ante view.

For a simple concrete example, let

$$(\underline{\sigma}_s)^2 = \underline{\epsilon} + \underline{\lambda} \cdot \langle B \rangle_s / s \quad \text{and} \quad (\bar{\sigma}_s)^2 = \bar{\epsilon} + \bar{\lambda} \cdot \langle B \rangle_s / s,$$

where $0 < \underline{\epsilon} < \bar{\epsilon}$, $0 < \underline{\lambda} < \bar{\lambda}$, and $\underline{\epsilon} + \underline{\lambda} < 1 < \bar{\epsilon} + \bar{\lambda}$. Along a path where $\langle B \rangle_t = t$ for all t , as in a Brownian motion, the (squared) volatility interval at each instant is $[\underline{\epsilon} + \underline{\lambda}, \bar{\epsilon} + \bar{\lambda}]$ which contains 1. Therefore, future volatility σ_s remaining constant at level 1 is an admissible conditional hypothesis everywhere along this trajectory. This is intuitive given the realized history. At the same time, other hypotheses for (σ_s) are also consistent with this history at points along the path. This is because of the individual's concern, discussed above, that the future could be different than the past. The effects of alternative histories on conditioning are simply parametrized. If, at a given t , $\langle B \rangle_t / t$ is very small (large), then the volatility interval at t is roughly $[\underline{\epsilon}, \bar{\epsilon}]$ (respectively $(\langle B \rangle_t / t)[\underline{\lambda}, \bar{\lambda}]$). In both cases, for suitable parameter values, history is thought to be inconsistent with a Brownian motion for B , or $\sigma_s = 1$, even locally for times s near t . Also at time 0 when anticipating the future, the individual rules out a Brownian motion locally after $t > 0$ in the event of future extremely small or large realizations for $\langle B \rangle_t / t$.

The sets of conditionals in (3.8) lead naturally to the following (nonlinear) conditional expectation on $UC_b(\Omega)$, the set of all bounded and uniformly continuous functions on Ω :

$$\hat{E}[\xi \mid \mathcal{F}_t](\omega) = \sup_{P \in \mathcal{P}_t^\omega} E_P \xi, \quad \text{for every } \xi \in UC_b(\Omega) \text{ and } (t, \omega) \in [0, T] \times \Omega. \quad (3.9)$$

Conditional expectation is defined thereby on $UC_b(\Omega)$, but this domain is not large enough for our purposes.¹⁹ For example, the conditional expectation of

¹⁹The definition is restricted to $UC_b(\Omega)$ in order to ensure the measurability of $\hat{E}[\xi \mid \mathcal{F}_t](\cdot)$

ξ need not be (uniformly) continuous in ω even if ξ is bounded and uniformly continuous, which is an obstacle to dealing with stochastic processes and recursive modeling. (Similarly, if one were to use the space $C_b(\Omega)$ of bounded continuous functions.) Thus we consider the larger domain $\widehat{L^2}(\Omega)$, the completion of $UC_b(\Omega)$ under the norm $\|\xi\| \equiv (\hat{E}[\|\xi\|^2])^{\frac{1}{2}}$.²⁰ Denis et al. [9] show that a random variable ξ defined on Ω lies in $\widehat{L^2}(\Omega)$ if and only if: (i) ξ is *quasicontinuous* - for every $\epsilon > 0$ there exists an open set $G \subset \Omega$ with $P(G) < \epsilon$ for every P in \mathcal{P} such that ξ is continuous on $\Omega \setminus G$; and (ii) ξ is uniformly integrable in the sense that $\lim_{n \rightarrow \infty} \sup_{P \in \mathcal{P}} E^P(\|\xi\|^2 \mathbf{1}_{\{\|\xi\| > n\}}) = 0$. This characterization is inspired by Lusin's Theorem for the classical case which implies that when $\mathcal{P} = \{P\}$, then $\widehat{L^2}(\Omega)$ reduces to the familiar space of P -squared integrable random variables. For general \mathcal{P} , $\widehat{L^2}(\Omega)$ is a proper subset of the set of measurable random variables ξ for which $\sup_{P \in \mathcal{P}} E^P(\|\xi\|^2) < \infty$.²¹ However, it is large in the sense of containing many discontinuous random variables; for example, $\widehat{L^2}(\Omega)$ contains every bounded and lower semicontinuous function on Ω (see the proof of Lemma B.8).

Another aspect of $\widehat{L^2}(\Omega)$ warrants emphasis. Two random variables ξ' and ξ are identified in $\widehat{L^2}(\Omega)$ if and only if $\|\xi' - \xi\| = 0$, which means that $\xi' = \xi$ almost surely with respect to P for *every* P in \mathcal{P} . In that case, say that the equality obtains *quasisurely* and write $\xi' = \xi$ *q.s.* Thus ξ' and ξ are distinguished whenever they differ with positive probability for *some* measure in \mathcal{P} . Accordingly, the space $\widehat{L^2}(\Omega)$ provides a more detailed picture of random variables than does any single measure in \mathcal{P} .

The next theorem (proven in Appendix B) shows that conditional expectation admits a suitably unique and well behaved extension from $UC_b(\Omega)$ to all of $\widehat{L^2}(\Omega)$. Accordingly, the sets \mathcal{P}_t^ω determine the conditional expectation for all random variables considered below.

Theorem 3.6 (Conditioning). *The mapping $\hat{E}[\cdot \mid \mathcal{F}_t]$ on $UC_b(\Omega)$ defined in (3.9) can be extended uniquely to a 1-Lipschitz continuous mapping $\hat{E}[\cdot \mid \mathcal{F}_t] : \widehat{L^2}(\Omega) \rightarrow \widehat{L^2}(\Omega)$, where 1-Lipschitz continuity means that*

$$\|\hat{E}[\xi' \mid \mathcal{F}_t] - \hat{E}[\xi \mid \mathcal{F}_t]\|_{\widehat{L^2}} \leq \|\xi' - \xi\|_{\widehat{L^2}} \quad \text{for all } \xi', \xi \in \widehat{L^2}(\Omega).$$

and proof of a suitable form of the law of iterated expectations. Indeed, Appendix B, specifically (B.2), shows that conditional expectation has a different representation when random variables outside $UC_b(\Omega)$ are considered.

²⁰It coincides with the completion of $C_b(\Omega)$; see [9].

²¹For example, if \mathcal{P} is the set of all Dirac measures with support in Ω , then $\widehat{L^2}(\Omega) = C_b(\Omega)$.

Moreover, the extension satisfies, for all ξ and η in $\widehat{L^2}(\Omega)$ and for all $t \in [0, T]$,

$$\hat{E}[\hat{E}[\xi \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \hat{E}[\xi \mid \mathcal{F}_s], \quad \text{for } 0 \leq s \leq t \leq T, \quad (3.10)$$

and:

- (i) If $\xi \geq \eta$, then $\hat{E}[\xi \mid \mathcal{F}_t] \geq \hat{E}[\eta \mid \mathcal{F}_t]$.
- (ii) If ξ is \mathcal{F}_t -measurable, then $\hat{E}[\xi \mid \mathcal{F}_t] = \xi$.
- (iii) $\hat{E}[\xi \mid \mathcal{F}_t] + \hat{E}[\eta \mid \mathcal{F}_t] \geq \hat{E}[\xi + \eta \mid \mathcal{F}_t]$ with equality if η is \mathcal{F}_t -measurable.
- (iv) $\hat{E}[\eta\xi \mid \mathcal{F}_t] = \eta^+ \hat{E}[\xi \mid \mathcal{F}_t] + \eta^- \hat{E}[-\xi \mid \mathcal{F}_t]$, if η is \mathcal{F}_t -measurable.

The Lipschitz property is familiar from the classical case of a single prior, where it is implied by Jensen's inequality (see the proof of Lemma B.8). The law of iterated expectations (3.10) is intimately tied to dynamic consistency of the preferences discussed below. The nonlinearity expressed in (iii) reflects the non-singleton nature of the set of priors. Other properties have clear interpretations.

The next example provides perspective on the meaning of “quasisure.”

Example 3.7 (Nonsingular Priors). Consider further Example 3.2, where there is ambiguity only about volatility, and where ambiguity is time and state invariant with $\sigma_t \in \Gamma = [1, 2]$. As pointed out in the introduction, the measures P^1 and P^2 corresponding to $\sigma_t^1 \equiv 1$ and $\sigma_t^2 \equiv 2$ respectively, are singular. However, the corresponding set of priors \mathcal{P} also contains nonsingular measures.²² For example, let $P = P^{(\sigma_t)}$ where

$$\sigma_t(\omega) = \begin{cases} 1, & t \leq \tau(\omega) \wedge T, \\ 2, & t \in (\tau(\omega) \wedge T, T], \end{cases}$$

and where τ is the passage time, $\tau(\omega) = \inf\{t \geq 0 : \omega_t = 1\}$. Define the events

$$A = \{\omega : \omega_t > 1 \text{ and } \langle B \rangle_t(\omega) = t, t \in [\frac{T}{2}, T]\},$$

and

$$C = \{\omega : \langle B \rangle_t(\omega) = t, t \in [0, T]\}.$$

Then $P(A) = 0$, $0 < P^1(A) < 1$, $P^1(C) = 1$ and $0 < P(C) < 1$. Thus P^1 and P are not singular, and neither is absolutely continuous with respect to the other.

Mutual singularity within \mathcal{P} would simplify quasisure analysis. For example, because the supports of P^1 and P^2 are disjoint, the equation “ $\xi' = \xi$ with

²²This is clear when there is ambiguity also about drift. The message here is that nonsingularity can exist even when there is ambiguity only about volatility.

probability 1 according to both P^1 and P^2 ” amounts to standard probability 1 statements on disjoint parts of the state space and thus is not far removed from a standard equation in random variables. However, singularity is not representative and the reader is urged not to be overly influenced by the example of P^1 and P^2 when thinking about equations of the form “ $\xi' = \xi$ q.s.”

3.4. The definition of utility

Because we turn to consideration of processes, we define $M^{2,0}(0, T)$, the class of processes η of the form

$$\eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) 1_{[t_i, t_{i+1})}(t),$$

where $\xi_i \in \widehat{L}^2({}^{t_i}\Omega)$, $0 \leq i \leq N-1$, and $0 = t_0 < \dots < t_N = T$.²³ Roughly, each such η is a step function in random variables from the spaces $\widehat{L}^2({}^{t_i}\Omega)$. For the usual technical reasons, we consider also suitable limits of such processes. Thus define $M^2(0, T)$ to be the completion of $M^{2,0}(0, T)$ under the norm

$$\| \eta \|_{M^2(0, T)} \equiv (\hat{E}[\int_0^T |\eta_t|^2 dt])^{\frac{1}{2}}.$$

Consumption at every time takes values in C , a convex subset of \mathbb{R}_+^d . Consumption processes $c = (c_t)$ lie in D , a subset of $M^2(0, T)$. For each c in D , we define a utility process $(V_t(c))$, where $V_t(c)$ is the utility of the continuation $(c_s)_{0 \leq s \leq t}$ and $V_0(c)$ is the utility of the entire process c . We often suppress the dependence on c and write simply (V_t) .

Let Θ and $\mathcal{P} = \mathcal{P}^\Theta$ be as above. Following Duffie and Epstein [16], the other primitive component of the specification of utility is the aggregator $f : C \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$. It is assumed to satisfy:

- (i) f is Borel measurable.
- (ii) Uniform Lipschitz for aggregator: There exists a positive constant K such that

$$|f(c, v') - f(c, v)| \leq K |v' - v|, \text{ for all } (c, v', v) \in C \times \mathbb{R}^2.$$

- (iii) $(f(c_t, v))_{0 \leq t \leq T} \in M^2(0, T)$ for each $v \in R$ and $c \in D$.

²³The space ${}^{t_i}\Omega$ was defined in Section 3.1.

We define V_t by

$$V_t = -\hat{E}\left[-\int_t^T f(c_s, V_s)ds \mid \mathcal{F}_t\right]. \quad (3.11)$$

Our main result follows (see Appendix C for a proof).

Theorem 3.8 (Utility). *Let (Θ_t) and f satisfy the above assumptions. Fix $c \in D$. Then:*

- (a) *There exists a unique process (V_t) in $M^2(0, T)$ solving (3.11).*
- (b) *The process (V_t) is the unique solution in $M^2(0, T)$ to $V_T = 0$ and*

$$V_t = -\hat{E}\left[-\int_t^\tau f(c_s, V_s)ds - V_\tau \mid \mathcal{F}_t\right], \quad 0 \leq t < \tau \leq T. \quad (3.12)$$

Part (a) proves that utility is well defined by (3.11). Recursivity is established in (b); its implications for dynamic consistency are discussed in the next section.

The most commonly used aggregator has the form

$$f(c_t, v) = u(c_t) - \beta v, \quad \beta \geq 0, \quad (3.13)$$

in which case utility admits the closed-form expression

$$V_t = -\hat{E}\left[-\int_t^T u(c_s)e^{-\beta s}ds \mid \mathcal{F}_t\right]. \quad (3.14)$$

More generally, closed-form expressions are rare. The following example illustrates the effect of volatility ambiguity.

Example 3.9 (Closed form). *Suppose that there is no ambiguity about the drift (Example 3.2), and that ambiguity about volatility is captured by the fixed interval $[\underline{\sigma}, \bar{\sigma}] \subset \mathbb{R}_{++}$. Consider consumption processes that are certain and constant, at level 0 for example, on the time interval $[0, 1)$, and that yield constant consumption on $[1, T]$ at a level that depends on the state ω_1 at time 1. Specifically, let*

$$c_t(\omega) = \psi(\omega_1), \quad \text{for } 1 \leq t \leq T,$$

where $\psi : \mathbb{R}^1 \rightarrow \mathbb{R}_+^1$. For simplicity, suppose further that u is linear. Then time 0 utility evaluated using (3.14), is, ignoring irrelevant constants,

$$V_0 = -\hat{E}[-\psi(\omega_1)].$$

If ψ is a convex function, then (see Levy et al. [37] and Peng [51])

$$V_0(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\underline{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy,$$

and if ψ is concave, then

$$V_0(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\bar{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy. \quad (3.15)$$

There is an intuitive interpretation for these formulae. Given risk neutrality, the individual cares only about the expected value of consumption at time 1. The issue is expectation according to which probability law? For simplicity, consider the following concrete specifications:

$$\psi_1(x) = |x - \kappa|, \text{ and } \psi_2(x) = -|x - \kappa|.$$

Then ψ_1 is convex and ψ_2 is concave. If we think of the the driving process as the price of a stock, then $\psi_1(\cdot)$ can be interpreted as a straddle - the sum of a European put and a European call option on the stock at the common strike price κ and expiration date 1. (We are ignoring nonnegativity constraints.) A straddle pays off if the stock price moves, whether up or down, and thus constitutes a bet on volatility. Accordingly, the worst case scenario is that the price process has the lowest possible volatility $\underline{\sigma}$. In that case, ω_1 is $N(0, \underline{\sigma}^2)$ and the indicated expected value of consumption follows. Similarly, $\psi_2(\cdot)$ describes the corresponding short position and amounts to a bet against volatility. Therefore, the relevant volatility for its worst case evaluation is the largest possible value $\bar{\sigma}$, consistent with the expression for utility given above.

When the function ψ is neither concave nor convex globally, closed-form expressions for utility are available only in extremely special and unrevealing cases; see Hu [29, Remark 15], for example. However, a generalization to d -dimensional processes is available and will be used below. Let $d \geq 1$ and consider G -Brownian motion - there is certainty that the driving process is a martingale and the volatility matrix σ_t in (3.1) is restricted to lie in the compact and convex set $\Gamma \subset \mathbb{R}^{d \times d}$ (satisfying the conditions described in Example 3.2). Consumption is as above except that, for some $a \in \mathbb{R}^d$,

$$c_t(\omega) = \psi(a^\top \omega_1), \text{ for } 1 \leq t \leq T.$$

Let $\underline{\sigma}$ be any solution to $\min_{\sigma \in \Gamma} \text{tr}(\sigma \sigma^\top a a^\top)$ and let $\bar{\sigma}$ be any solution to $\max_{\sigma \in \Gamma} \text{tr}(\sigma \sigma^\top a a^\top)$. If ψ is convex (concave), then the worst-case scenario is that $\sigma_t = \underline{\sigma}$ ($\bar{\sigma}$) for all t . Closed-form expressions for utility follow immediately.

3.5. Dynamic consistency

Dynamic consistency is an important property that is typically thought to be delivered by a recursive structure such as that in (3.12). However, uncertainty about what is possible complicates matters as we describe here.

The recursive structure (3.12) implies the following weak form of dynamic consistency: For any $0 < \tau < T$, and any two consumption processes c' and c that coincide on $[0, \tau]$,

$$[V_\tau(c') \geq V_\tau(c) \text{ quasisurely}] \implies V_0(c') \geq V_0(c).$$

Typically, (see Duffie and Epstein [16, p. 373] for example), dynamic consistency is defined so as to deal also with strict rankings, that is, if also $V_\tau(c') > V_\tau(c)$ on a “non-negligible” set of states, then $V_0(c') > V_0(c)$. This added requirement excludes the possibility that c' is chosen ex ante though it is indifferent to c , and yet it is not implemented fully because the individual switches to the conditionally strictly preferable c for some states at time τ . The issue is how to specify “non-negligible”. When there is certainty about what is possible, then all priors are equivalent and positive probability according to any single prior is the natural specification. However, when the measures in \mathcal{P} are not equivalent a similarly natural specification is unclear. A simple illustration of the consequence is given in the next example.

Example 3.10 (Weak Dynamic Consistency). Take $d = 1$. Let the endowment process e satisfy (under P_0)

$$d \log e_t = \sigma_t dB_t,$$

or equivalently,

$$de_t/e_t = \frac{1}{2}\sigma_t^2 dt + \sigma_t dB_t \quad P_0\text{-a.e.}$$

Here volatility is restricted only by $0 < \underline{\sigma} \leq \sigma_t \leq \bar{\sigma}$. Thus B is a G -Brownian motion relative to the corresponding set of priors \mathcal{P} . Utility is defined, for any consumption process c , by

$$V_0(c) = \inf_{P \in \mathcal{P}} E^P \left[\int_0^T e^{-\beta t} u(c_t) dt \right] = \inf_{P \in \mathcal{P}} \left[\int_0^T e^{-\beta t} E^P u(c_t) dt \right],$$

where

$$u(c_t) = (c_t)^\alpha / \alpha, \quad \alpha < 0.$$

Denote by P^* the prior in \mathcal{P} corresponding to $\sigma_t = \bar{\sigma}$ for all t ; that is, P^* is the measure on Ω induced by X^* ,

$$X_t^* = \bar{\sigma} B_t, \text{ for all } t \text{ and } \omega.$$

Then

$$V_0(e) = E^{P^*} \left[\int_0^T e^{-\beta s} u(e_t) dt \right].$$

(This is because $u(e_t) = \alpha^{-1} \exp(\alpha \log e_t)$ and because $\alpha < 0$ makes $x \mapsto e^{\alpha x} / \alpha$ concave, so that the argument in Example 3.9 can be adapted.)

Define the nonnegative continuous function φ on \mathbb{R} by

$$\varphi(x) = \begin{cases} 1 & x \leq \underline{\sigma}^2 \\ \frac{2x}{\underline{\sigma}^2 - \bar{\sigma}^2} - \frac{\underline{\sigma}^2 + \bar{\sigma}^2}{\underline{\sigma}^2 - \bar{\sigma}^2} & \underline{\sigma}^2 < x < \frac{\underline{\sigma}^2 + \bar{\sigma}^2}{2} \\ 0 & \frac{\underline{\sigma}^2 + \bar{\sigma}^2}{2} \leq x \end{cases}$$

Fix $\tau > 0$. Define the event N_τ by

$$N_\tau = \{\omega : \langle B \rangle_\tau = \underline{\sigma}^2 \tau\},$$

and the consumption process c by

$$c_t = \begin{cases} e_t & 0 \leq t \leq \tau \\ e_t + \varphi\left(\frac{\langle B \rangle_\tau}{\tau}\right) & \tau \leq t \leq T \end{cases}$$

Then $V_\tau(c) \geq V_\tau(e)$ quasisurely and a strict preference prevails on N_τ because $\varphi(\underline{\sigma}^2) = 1$ and $P^\sigma(N_\tau) = 1$. However, c is indifferent to e at time 0 because $\varphi\left(\frac{\langle B \rangle_\tau}{\tau}\right) = \varphi(\bar{\sigma}^2) = 0$ under P^* .²⁴

It is not completely clear how to model behavior in the absence of (strict) dynamic consistency. In the example, the individual would not be willing to bet on the occurrence of N_τ (low quadratic variation) for any stakes because for such a bet the worst case is that N_τ is null. Because she is not willing to bet on N_τ , it might seem plausible that she not consider N_τ when formulating a plan at time

²⁴One might suspect that a simpler example, with φ replaced by an indicator function, would work. However, indicator functions do not lie in $\widehat{L}^2(\Omega)$ and thus the consumption process constructed thereby may not lie in $M^2(0, T)$. In contrast, continuity of φ implies that c constructed above is in $M^2(0, T)$.

0. At a later stage, however, she presumably becomes aware of the conditional suboptimality of e and reoptimizes, but the details of such a model of dynamic behavior are unclear, particularly in a continuous time setting. Thus in the asset pricing application below we focus on dynamic behavior (and equilibria) where ex ante optimal plans are implemented for all time quasisurely. This requires that we examine behavior from conditional perspectives and not only ex ante. Accordingly, if the feasible set in the above example is $\{e, c\}$, the predicted choice would be c .

4. Asset Returns

We use the utility gradient approach of Duffie and Skiadas [18] (see also Skiadas [53] for a comprehensive overview of this approach) to characterize asset returns in a representative agent endowment economy where utility is as defined above though with some modifications. Both Arrow-Debreu style and Radner style sequential market structures are considered.

In the sequel, we limit ourselves to scalar consumption at every instant so that $C \subset \mathbb{R}_+$. At the same time we generalize utility to permit lumpy consumption at the terminal time. Thus we use the utility functions V_t given by

$$V_t(c, \xi) = -\hat{E}\left[-\int_t^T f(c_s, V_s(c, \xi))ds - u(\xi) \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T,$$

where (c, ξ) varies over a subset D of $M^2(0, T) \times \widehat{L^2}(\Omega)$. Here c denotes the absolutely continuous component of the consumption process and ξ is the lump of consumption at T . Our analysis of utility extends to this larger domain in a straightforward way.

When considering (c, ξ) , it is without loss of generality to restrict attention to versions of c for which

$$c_T = \xi.$$

With this normalization, we can abbreviate $(c, \xi) = (c, c_T)$ by c and identify c with an element of $D \subset M^2(0, T) \times \widehat{L^2}(\Omega)$.²⁵ Accordingly write $V_t(c, \xi)$ more simply as $V_t(c)$, where

$$V_t(c) = -\hat{E}\left[-\int_t^T f(c_s, V_s(c, \xi))ds - u(c_T) \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T. \quad (4.1)$$

²⁵Similarly, identify a (version of) an element of $M^2(0, T)$ with an element of $M^2(0, T) \times \widehat{L^2}(\Omega)$.

Throughout, if $Z = (Z_t)$ is a process, by the statement “ $Z_t \geq 0$ for every t *quasisurely* (*q.s.*)” we mean that for every t there exists $G_t \subset \Omega$ such that $Z_t(\omega) \geq 0$ for all $\omega \in G_t$ and $P(G_t) = 1$ for all P in \mathcal{P} .²⁶ If $Z_t = 0$ for every t *quasisurely*, then $Z = 0$ in $M^2(0, T)$ (because $\hat{E}[\int_0^T |Z_t|^2 dt] \leq \int_0^T \hat{E}[|Z_t|^2] dt$), but the converse is not valid in general. When Z is a random variable, $Z \geq 0$ *quasisurely* means that the inequality is valid *P-a.e.* for every P in \mathcal{P} .

4.1. Arrow-Debreu Equilibrium

We adapt the notion of Arrow-Debreu equilibrium to our setting (see Duffie [15], for example, for a textbook treatment in a continuous time framework). We assume a representative agent. She can purchase any consumption process *ex ante* subject only to a budget constraint.

Fix an endowment consumption process e in D . Fix also a “price density” process $\pi' = (\pi'_t) \in M^2(0, T)$. A consumption process c in D is taken to be feasible if

$$\sup_{P \in \mathcal{P}} E^P \left[\int_0^T \pi'_t (c_t - e_t) dt + \pi'_T (c_T - e_T) \right] \leq 0. \quad (4.2)$$

In the classical case $\mathcal{P} = \{P\}$, $\pi'_t(\omega)$ is the time 0 price, in P -density terms, for one unit of consumption delivered and consumed at (t, ω) .²⁷ In general, feasibility according to (4.2) means that c is affordable according to every scenario in \mathcal{P} , and π' is ‘universal’ in the sense of being a price density for each probability law in \mathcal{P} . The budget constraint reflects a comprehensive form of market completeness in permitting the purchase of consumption contingent on any event that is possible according to some prior in \mathcal{P} .

²⁶The following perspective may be helpful for nonspecialists in continuous time analysis. In the classical case of a probability space (Ω, \mathcal{F}, P) with filtration $\{\mathcal{F}_t\}$, if two processes X and Y satisfy “for each t , $X_t = Y_t$ *P-a.e.*”, then Y is called a *modification* of X . If $P(\{\omega : X_t = Y_t \ \forall t \in [0, T]\}) = 1$, then X and Y are said to be *indistinguishable*. These notions are equivalent when restricted to X and Y having a.s. right continuous sample paths, but the second is stronger in general. We point out, however, that the sense in which one constructs a Brownian motion exhibiting continuous sample paths is that a suitable modification exists (see the Kolmogorov-Centsov Theorem). A reference for the preceding is Karatzas and Shreve [35, pp. 2, 53].

²⁷Consistent with common notation, $\pi'_t(\omega)$ is often abbreviated π'_t with ω suppressed. Similarly for other processes, e.g. e_t depends on the state ω as does its conditional utility $V_t(e)$. State dependence should be understood throughout.

Define an *Arrow-Debreu equilibrium* to be a pair $(e, \pi') \in D \times M^2(0, T)$ such that $V_0(e) \geq V_0(c)$ for every c that is feasible in the sense of (4.2).

Under suitable conditions, an Arrow-Debreu equilibrium exists in our representative agent economy. Define $\pi^e = (\pi_t^e)$ by

$$\begin{aligned}\pi_t^e &= \exp\left(\int_0^t f_v(e_s, V_s(e)) ds\right) f_c(e_t, V_t(e)), \quad 0 \leq t < T, \\ \pi_T^e &= \exp\left(\int_0^T f_v(e_s, V_s(e)) ds\right) u_c(e_T).\end{aligned}\tag{4.3}$$

The next result shows that, under suitable conditions, (e, π^e) is an Arrow-Debreu equilibrium. Speaking informally, π^e describes marginal rates of substitution at e . Below we refer to π^e as a *supergradient at e* .

We need the following conditions for f and u .

(H1) f and u are continuously differentiable and concave.

(H2) There exists $\kappa > 0$ such that

$$\begin{aligned}\sup\{|f_c(c, V)|, |f(c, 0)|\} &< \kappa(1 + c) \quad \text{for all } (c, V) \in C \times \mathbb{R}^1, \\ |v_c(c)| &< \kappa(1 + c) \quad \text{for all } (c, V) \in C \times \mathbb{R}^1.\end{aligned}$$

In particular, condition (H2) implies that, for every $c \in M^2(0, T) \times \widehat{L^2}(\Omega)$,

$$u(c_T), u_c(c_T) \in \widehat{L^2}(\Omega) \text{ and } f(c_t, 0), f_c(c_t, V_t(c)) \in M^2(0, T).$$

Theorem 4.1 (Arrow-Debreu equilibrium). *Adopt the assumptions in Theorem 3.8, and assume conditions (H1)-(H2). Then (e, π^e) is an Arrow-Debreu equilibrium.*

Thus far we have assumed that markets are open only ex ante. The exclusion of markets ex post is without loss of generality in the classical model because, when preferences are dynamically consistent, such markets are redundant. Therefore, these markets are redundant not only under subjective expected utility maximization, but also when ambiguity is restricted to the drift as in Chen and Epstein [7]. However, the (strong) form of dynamic consistency on which this redundancy relies is not satisfied when there is ambiguity about volatility and hence uncertain possibility (Section 3.5). Thus consider briefly how the preceding changes when trading is allowed at every node along the tree.

Suppose that markets for continuation consumption processes are open at every time τ and quasisurely in every state ω . Define a *dynamic Arrow-Debreu*

equilibrium to be a pair $(e, \pi') \in D \times M^2(0, T)$ satisfying for every c : For each τ and quasisurely,

$$\begin{aligned} \widehat{E} \left[\int_{\tau}^T \pi'_t (c_t - e_t) dt + \pi'_T (c_T - e_T) \mid \mathcal{F}_{\tau} \right] &\leq 0 \implies \\ V_{\tau}(c) &\leq V_{\tau}(e). \end{aligned} \quad (4.4)$$

A dynamic equilibrium requires that e be conditionally weakly preferable to any alternative feasible process c , at any τ quasisurely, given that e has been followed on $[0, \tau)$. This is more demanding than the previous equilibrium notion with ex ante trade alone. However, the proof of Theorem 4.1 in Appendix D shows that (e, π^e) is an equilibrium also in this stronger sense.

4.2. Sequential security markets, no arbitrage and state prices

We consider security markets that permit sequential trading and explore what can be said about asset prices based on no-arbitrage or hedging arguments. It is well known, from Avellaneda et al. [2], for example, that ambiguous volatility implies market incompleteness and hence that perfect hedging is generally impossible. We derive superhedging and subhedging prices for any security. These form bounds for any equilibrium price if preference is suitably monotone. The main result is the existence of a state price process that can be used to describe upper- and lower-hedging prices. The result generalizes the familiar pricing formula using an equivalent martingale (or risk-neutral) measure that characterizes the absence of arbitrage when there is no ambiguity. State prices are often expressed via a density with respect to the reference measure. Such a representation is not possible here - the state price process must be ‘universal’ in the sense described in the preceding section.

The analysis is conducted under the assumption that B is a G -Brownian motion under \mathcal{P} (recall Example 3.2), and hence that ambiguity is confined to volatility. This allows us to exploit Peng’s generalizations of Itô’s Lemma and Itô integration; see Appendix E for brief descriptions.

First we introduce a process that will be used to define state prices. As noted earlier, by Follmer [24] and Karandikar [34] there exists a stochastic process $\langle B \rangle$ defined by the limit in (1.1) for every martingale measure, hence quasisurely. Following Soner et. al. [57], let

$$v_t = \overline{\lim}_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}), \quad (4.5)$$

where $\overline{\lim}$ is componentwise. In our context $v_t(\omega)$ takes values in $\mathbb{S}_d^{>0}$, the space of all $d \times d$ positive-definite matrices. Under P_0 , B is a Brownian motion and v_t equals the $d \times d$ identity matrix P_0 -a.e. More generally, if $P = P^{(\sigma_t)}$ is a prior in \mathcal{P} corresponding via the SDE (3.1) to (σ_t) , then

$$v_t = \sigma_t \sigma_t^\top \quad dt \times P^{(\sigma_t)}\text{-a.e.} \quad (4.6)$$

(Here is the proof: By Soner et al. [57, p. 4], $\langle B \rangle$ equals the quadratic variation of B $P^{(\sigma_t)}$ -a.e.; and by Oksendal [45, p. 56],

$$\left\langle \int_0^\cdot \sigma_s dB_s \right\rangle_t = \int_0^t \sigma_s \sigma_s^\top ds \quad P^{(\sigma_t)}\text{-a.e.}$$

Thus we have the $P^{(\sigma_t)}$ -a.e. equality in processes $\langle B \rangle = \left(\int_0^t \sigma_s \sigma_s^\top ds : 0 \leq t \leq T \right)$. Because $\langle B \rangle$ is absolutely continuous, its time derivative exists a.e. on $[0, T]$; indeed, the derivative at t is v_t . Evidently, $\frac{d}{dt} \int_0^t \sigma_s \sigma_s^\top ds = \sigma_t \sigma_t^\top$ for almost every t . Equation (4.6) follows.)

Consider now the following market environment. There is a single consumption good, a riskless asset with return r_t and d risky securities available in zero net supply. Returns R_t to the risky securities are given by

$$dR_t = b_t dt + s_t dB_t$$

where s_t is a $d \times d$ invertible volatility matrix. Unless stated otherwise, all equations and statements regarding processes are understood to hold for all times quasisurely. Define $\eta_t = s_t^{-1}(b_t - r_t 1)$, the market price of uncertainty (a more appropriate term here than market price of risk). It is assumed henceforth that (r_t) and $(v_t^{-1} \eta_t)$ are bounded processes in $M^2(0, T)$.²⁸

Define $\pi = (\pi_t)$ as the unique solution to

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t^\top v_t^{-1} dB_t, \quad \pi_0 = 1. \quad (4.7)$$

We refer to π as a *state price process*. Using a result by Peng [51, Ch. 5, Remark 1.3], we can express π explicitly in the following fashion paralleling the classical case:

$$\pi_t = \exp\left\{-\int_0^t r_s ds - \int_0^t \eta_s^\top v_s^{-1} dB_s - \frac{1}{2} \int_0^t \eta_s^\top v_s^{-1} \eta_s ds\right\}, \quad 0 \leq t \leq T. \quad (4.8)$$

²⁸The latter restriction will be confirmed below whenever η is taken to be endogenous.

Fix a dividend stream $(\delta, \delta_T) \in M^2(0, T) \times \widehat{L^2}(\Omega)$ and the time τ . Consider the following budget equation on $[\tau, T]$:

$$\begin{aligned} dY_t &= (r_t Y_t + \eta_t^\top \phi_t - \delta_t) dt + \phi_t^\top dB_t, \\ Y_\tau &= y, \end{aligned} \tag{4.9}$$

where B is G -Brownian motion under \mathcal{P} , y is initial wealth and ϕ is restricted to lie in $M^2(0, T)$. Denote the unique solution by $Y^{y, \phi, \tau}$.

Define the superhedging set

$$\mathcal{U}_\tau = \{y \geq 0 \mid \exists \phi \in M^2(0, T) \text{ s.t. } Y_T^{y, \phi, \tau} \geq \delta_T \text{ q.s.}\},$$

and the *superhedging price* $\bar{S}_\tau = \inf\{y \mid y \in \mathcal{U}_\tau\}$. Similarly define the subhedging set

$$\mathcal{L}_\tau = \{y \geq 0 \mid \exists \phi \in M^2(0, T) \text{ s.t. } Y_T^{-y, \phi, \tau} \geq -\delta_T \text{ q.s.}\}$$

and the *subhedging price* $\underline{S}_\tau = \sup\{y \mid y \in \mathcal{L}_\tau\}$.

Our characterization of superhedging and subhedging prices requires an additional arguably minor restriction on the security market. To express it, for any $\varepsilon > 0$, define

$$\widehat{L^{2+\varepsilon}}(\Omega) = \left\{ \xi \in \widehat{L^2}(\Omega) : \hat{E}[|\xi|^{2+\varepsilon}] < \infty \right\}.$$

The restriction is that π and δ satisfy

$$(\pi_T \delta_T + \int_0^T \pi_t \delta_t dt) \in \widehat{L^{2+\varepsilon}}(\Omega). \tag{4.10}$$

Theorem 4.2 (Hedging prices). *Fix a dividend stream $(\delta, \delta_T) \in M^2(0, T) \times \widehat{L^2}(\Omega)$. Suppose that r and $(v_t^{-1} \eta_t)$ are bounded processes in $M^2(0, T)$ and that (4.10) is satisfied. Then the superhedging and subhedging prices at any time τ are given by*

$$\bar{S}_\tau = \hat{E}\left[\int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt + \frac{\pi_T}{\pi_\tau} \delta_T \mid \mathcal{F}_\tau\right] \text{ q.s.}$$

and

$$\underline{S}_\tau = -\hat{E}\left[-\int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt - \frac{\pi_T}{\pi_\tau} \delta_T \mid \mathcal{F}_\tau\right] \text{ q.s.}$$

In the special case where the security δ can be perfectly hedged, that is, there exist y and ϕ such that $Y_T^{y,\phi,0} = \delta_T$, then

$$\bar{S}_0 = \underline{S}_0 = \hat{E}[\pi_T \delta_T + \int_0^T \pi_t \delta_t dt] = -\hat{E}[-\pi_T \delta_T - \int_0^T \pi_t \delta_t dt].$$

Because this asserts equality of the supremum and infimum of expected values of $\pi_T \delta_T + \int_0^T \pi_t \delta_t dt$ as the measures P vary over \mathcal{P} , it follows that $E^P[\pi_T \delta_T + \int_0^T \pi_t \delta_t dt]$ is constant for all such measures P , that is, the hedging price is unambiguous. In the further specialization where there is no ambiguity, P_0 is the single prior and one obtains pricing by an equivalent martingale measure whose density (on \mathcal{F}_t) with respect to P_0 is π_t .

We note that Vorbrink [61] obtains an analogous characterization of hedging prices under the assumption of G -Brownian motion. However, in place of our assumption (4.10), he assumes that $b_t = r_t$, so that the market price of uncertainty η_t vanishes and $\pi_t = \exp\{-\int_0^t r_s ds\}$.

4.3. Sequential equilibrium

In this section, we use state prices to study equilibrium in the economy with sequential security markets. We continue to assume that B is a G -Brownian motion under \mathcal{P} . We also specialize utility to conform to the standard aggregator. Thus utility has the following form (specializing (4.1)):

$$V_t(c) = -\hat{E}\left[-\int_t^T u(c_s) e^{-\beta(s-t)} ds - e^{-\beta(T-t)} u(c_T) \mid \mathcal{F}_t\right]. \quad (4.11)$$

The appropriate versions of (H1)-(H2) are assumed. In particular, u is concave and differentiable; assume also that $u_c > 0$. The supergradient process $\pi^e = (\pi_t^e)$ defined in (4.3) simplifies to

$$\pi_t^e = \exp(-\beta t) u_c(e_t), \quad 0 \leq t \leq T. \quad (4.12)$$

Securities are traded in order to finance deviations from the endowment process e . Denote the trading strategy by ψ_t , where ψ_{ti} is the proportion of wealth invested in risky security i . Then wealth Y_t evolves according to the equation

$$\begin{aligned} dY_t &= ([r_t + \psi_t^\top (b_t - r_t \mathbf{1})] Y_t - (c_t - e_t)) dt + Y_t \psi_t^\top s_t dB_t, \\ Y_0 &= 0, \quad c_T = e_T + Y_T \geq 0. \end{aligned}$$

The budget equation can be rewritten in the form

$$\begin{aligned} dY_t &= (r_t Y_t + \eta_t^\top \phi_t - (c_t - e_t))dt + \phi_t^\top dB_t, \\ Y_0 &= 0, c_T = e_T + Y_T \geq 0, \end{aligned} \quad (4.13)$$

where $\eta_t = s_t^{-1}(b_t - r_t 1)$ and $\phi_t = Y_t s_t^\top \psi_t$.

Refer to c as being *feasible* if $c \in D$ and there exists ϕ in $M^2(0, T)$ such that (4.13) is satisfied. More generally, for any $0 \leq \tau \leq T$, consider an individual with initial wealth Y_τ who trades securities and consumes on $[\tau, T]$. Say that c is *feasible on $[\tau, T]$ given initial wealth Y_τ* if the obvious conditions are satisfied.²⁹ Let $\Upsilon_\tau(Y_\tau)$ denote the set of all such feasible consumption processes.

State prices can be used to characterize feasible consumption plans as described next.

Theorem 4.3 (State prices). *Suppose that B is G -Brownian motion under \mathcal{P} . Define $\pi \in M^2(0, T)$ by (4.8) and let $0 \leq \tau < T$. If c is feasible on $[\tau, T]$ given initial wealth Y_τ , then*

$$\begin{aligned} Y_\tau &= \hat{E}\left[\int_\tau^T \frac{\pi_t}{\pi_\tau}(c_t - e_t)dt + \frac{\pi_T}{\pi_\tau}(c_T - e_T) \mid \mathcal{F}_\tau\right] \\ &= -\hat{E}\left[-\int_\tau^T \frac{\pi_t}{\pi_\tau}(c_t - e_t)dt - \frac{\pi_T}{\pi_\tau}(c_T - e_T) \mid \mathcal{F}_\tau\right]. \end{aligned} \quad (4.14)$$

The first equality alone would seem to be the counterpart for our setting of the familiar expected-present-discounted-value expression in the standard model. However, (4.14) states more through the second equality, which indicates (as in the discussion following Theorem 4.2) that in expected value terms any feasible consumption plan unambiguously exhausts initial wealth when consumption is priced using π . Such an interpretation is evident when $\tau = 0$; a similar interpretation is valid when $\tau > 0$ because of the description of conditioning in (B.2).³⁰

Say that $(e, (r_t, \eta_t))$ is a *sequential equilibrium* if for every c : For each τ and quasisurely,

$$c \in \Upsilon_\tau(0) \implies V_\tau(c) \leq V_\tau(e).$$

²⁹Though feasibility depends only on the continuation of c at τ , it is convenient simply to refer to feasibility of c .

³⁰The latter implies that, for any $X \in \widehat{L^2}(\Omega)$, $\hat{E}[X \mid \mathcal{F}_\tau] + \hat{E}[-X \mid \mathcal{F}_\tau] = 0$ if and only if, for every P in \mathcal{P} and every $P', P'' \in \mathcal{P}(t, P)$, $E_{P'}[X \mid \mathcal{F}_t] = E_{P''}[X \mid \mathcal{F}_t]$, P -a.e.

Thus equilibrium requires not only that the endowment e be optimal at time 0, but also that it remain optimal at any later time given that e has been followed to that point.

Theorems 4.3 and 4.1 suggest one approach to establishing existence of a sequential equilibrium, paralleling the Duffie and Skiadas [18] approach, which is to equate the state price and supergradient processes. The following corollary is immediate.

Corollary 4.4 (Sequential Equilibrium I). *Define $\pi^e, \pi \in M^2(0, T)$ by (4.12) and (4.8) respectively. Assume that for every t quasisurely,*

$$\pi_t^e / u_c(e_0) = \pi_t. \quad (4.15)$$

Then $(e, (r_t, \eta_t))$ is a sequential equilibrium.

Condition (4.15) relates consumption, the riskless rate and the market price of risk through the equation

$$\frac{du_c(e_t)}{u_c(e_t)} = (\beta - r_t) dt - \eta_t^\top v_t^{-1} dB_t.$$

To be more explicit, suppose that e satisfies, for all t quasisurely,

$$\frac{de_t}{e_t} = \mu_t^e dt + (s_t^e)^\top dB_t.$$

Then Ito's Lemma for G -Brownian motion implies that

$$b_t - r_t \mathbf{1} = s_t \eta_t = - \left(\frac{e_t u_{cc}(e_t)}{u_c(e_t)} \right) s_t v_t s_t^e, \quad (4.16)$$

which is a version of the C-CAPM for our setting.³¹ From the perspective of the measure P_0 according to which the coordinate process B is a Brownian motion, v_t is the identity matrix and one obtains the usual C-CAPM. However, this is only one perspective on the equation which refers to every prior in \mathcal{P} .

³¹Equality here means that the two processes $(b_t - r_t \mathbf{1})$ and $(-\left(\frac{e_t u_{cc}(e_t)}{u_c(e_t)}\right) s_t v_t s_t^e)$ are equal as processes in $M^2(0, T)$. Notice also that $v_t^{-1} \eta_t = -\left(\frac{e_t u_{cc}(e_t)}{u_c(e_t)}\right) s_t^e$, yielding a process in $M^2(0, T)$ and thus confirming our prior assumption on security markets.

4.4. Minimizing priors

Equation (4.15) is a sufficient condition for sequential equilibrium. Here we describe an alternative route to equilibrium that is applicable under an added assumption and yields an alternative form of C-CAPM.

The intuition for what follows is based on a well known consequence of the minimax theorem for multiple priors utility in abstract environments: For suitable optimization problems, if a prospect, say e , is feasible and if the set of priors contains a worst-case scenario P^* for e , then e is optimal if and only if it is optimal also for a Bayesian agent who uses the single prior P^* . Moreover, by a form of envelope theorem, P^* suffices to describe marginal rates of substitution at e and hence also supporting shadow prices. This suggests that there exist sufficient conditions for e to be part of an equilibrium in our setup that refer to P^* and less extensively to all other priors in \mathcal{P} . We proceed now to explore this direction.

As a first step, define $P^* \in \mathcal{P}$ to be a *minimizing measure* for e if

$$V_0(e) = E^{P^*}[\int_0^T u(e_t)e^{-\beta t}dt + e^{-\beta T}u(e_T)]. \quad (4.17)$$

As discussed when defining equilibrium, the fact that only weak dynamic consistency is satisfied requires that one take into account also conditional perspectives. Speaking informally, a minimizing measure P^* as above need not be minimizing conditionally at a later time because of the nonequivalence of priors and the uncertainty about what is possible. (Example 3.10 is readily adapted to illustrate this.) Thus to be relevant to equilibrium, a stronger notion of “minimizing” is required.

Recall that for any prior P in \mathcal{P} , P_τ^ω is the version of the regular conditional constructed via (3.7); importantly, it is well-defined for every (τ, ω) . Say that $P^* \in \mathcal{P}$ is a *dynamically minimizing measure* for e if, for all τ and quasi-surely,

$$V_\tau(e) = E^{(P^*)_\tau^\omega}[\int_\tau^T u(e_t)e^{-\beta(t-\tau)}dt - e^{-\beta(T-\tau)}u(e_T)]. \quad (4.18)$$

Next relax the quasisure equality (4.15) and assume instead: For every τ quasi-surely in ω ,

$$\pi_t^e/u_c(e_0) = \pi_t \text{ on } [\tau, T] \quad (P^*)_\tau^\omega\text{-a.e.} \quad (4.19)$$

Note that equality is assumed not only ex ante P^* -a.e. but also conditionally, even conditioning on events that are P^* -null but that are possible according to other priors in \mathcal{P} .

Now we can prove:

Theorem 4.5 (Sequential equilibrium II). *Let P^* be a dynamic minimizer for e and assume (4.19). Then $(e, (r_t, \eta_t))$ is a sequential equilibrium.*

Proof. The strategy is to argue that for any $c \in \Upsilon_\tau(0)$,

$$\begin{aligned}
V_\tau(c) - V_\tau(e) &= V_\tau(c) - E^{(P^*)^\omega_\tau} \left[\int_\tau^T u(e_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(e_T) \right] \\
&\stackrel{\dagger}{\leq} E^{(P^*)^\omega_\tau} \left[\int_\tau^T u(c_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(c_T) \right] \\
&\quad - E^{(P^*)^\omega_\tau} \left[\int_\tau^T u(e_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(e_T) \right] \\
&\leq E^{(P^*)^\omega_\tau} \left[\int_\tau^T \pi_t^e(c_t - e_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} \pi_T^e(c_T - e_T) \right]
\end{aligned}$$

$$\begin{aligned}
\text{by (4.19)} &= u_c(e_0) E^{(P^*)^\omega_\tau} \left[\int_\tau^T \pi_t(c_t - e_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} \pi_T(c_T - e_T) \right] \\
&\stackrel{\dagger\dagger}{\leq} u_c(e_0) \widehat{E} \left[\int_\tau^T \pi_t(c_t - e_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} \pi_T(c_T - e_T) \mid \mathcal{F}_\tau \right] \leq 0.
\end{aligned}$$

The inequalities marked \dagger and $\dagger\dagger$ are justified in Lemma D.2. Its proof exploits the particular version of the regular conditional that we have defined and demonstrates that the version is natural for our model. \blacksquare

Proceed as in the derivation of (4.16) to derive the following form of the C-CAPM: For every τ quasisurely,

$$b_t - r_t \mathbf{1} = - \left(\frac{e_t u_{cc}(e_t)}{u_c(e_t)} \right) s_t (\sigma_t^* \sigma_t^{*\top}) s_t^e \quad \text{on } [\tau, T] \quad (P^*)^\omega_\tau\text{-a.e.} \quad (4.20)$$

where $P^* = P^{(\sigma_t^*)}$ is induced by the process (σ_t^*) as in (3.1). We defer interpretation to the example below where the equation takes on a more concrete form. Here we note a connection between the two forms of C-CAPM that we have derived.

Lemma 4.6. *Assume that there exists a dynamic minimizer P^* for e . Then equation (4.16) implies (4.20).*

4.5. A final example

Theorem 4.5 begs the question whether or when dynamic minimizers exist. A difficulty is evident even when considering only ex ante minimizers as in (4.17). The set of priors \mathcal{P} is relatively compact in the topology induced by bounded continuous functions (this is a direct consequence of Gihman and Skorohod [25, Theorem 3.10]) but generally not compact and thus it need not include a minimizing measure.³² The situation is more complex for dynamic minimizers and we have no general answers at this point. But they exist in the following example. Its simplicity also helps to illustrate the effects of ambiguous volatility on asset returns.

We build on previous examples. Let $d \geq 1$. The endowment process e satisfies (under P_0)

$$d \log e_t = (s^e)^\top \sigma_t dB_t, \quad e_0 > 0 \text{ given}, \quad (4.21)$$

where s^e is constant and the volatility matrix σ_t is restricted only to lie in Γ , a compact and convex subset of $\mathbb{R}^{d \times d}$ as described in Example 3.2. Thus B is a G-Brownian motion relative to the corresponding set of priors \mathcal{P} . We assume that P_0 lies in \mathcal{P} . Utility is defined, for any consumption process c , by (4.11), where the felicity function u is³³

$$u(c_t) = (c_t)^\alpha / \alpha, \quad 0 \neq \alpha < 1.$$

There exists a dynamic minimizer for e that depends on the sign of α . Compute that

$$u(e_t) = \alpha^{-1} e_t^\alpha = \alpha^{-1} e_0^\alpha \exp \left\{ \alpha \int_0^t (s^e)^\top \sigma_s dB_s \right\}$$

Let $\underline{\sigma}$ be any solution to $\min_{\sigma \in \Gamma} \text{tr}(\sigma \sigma^\top s^e (s^e)^\top)$ and let $\bar{\sigma}$ be any solution to $\max_{\sigma \in \Gamma} \text{tr}(\sigma \sigma^\top s^e (s^e)^\top)$. (If $d = 1$, then Γ is a compact interval and $\underline{\sigma}$ and $\bar{\sigma}$ are its left and right endpoints.) Let P^* be the measure on Ω induced by X^* ,

$$X_t^* = \bar{\sigma}^\top B_t, \text{ for all } t \text{ and } \omega;$$

define P^{**} similarly using $\underline{\sigma}$ and X^{**} . Then, by a slight extension of the observation in Example 3.9, P^* is a dynamic minimizer for e if $\alpha < 0$ and P^{**} is a dynamic

³²We can show that a minimizing measure exists in the corresponding closure of \mathcal{P} .

³³Such power specifications violate some of the regularity conditions for aggregators and felicities assumed above. However, the assertions to follow can be verified.

minimizer for e if $\alpha > 0$.³⁴ We describe further implications assuming $\alpha < 0$; the corresponding statements for $\alpha > 0$ will be obvious to the reader.

Corresponding regular conditionals have a simple form. For example, following (3.7), for every (τ, ω) , $(P^*)_\tau^\omega$ is the measure on Ω induced by the SDE

$$\begin{cases} dX_t = \bar{\sigma} dB_t, & \tau \leq t \leq T \\ X_t = \omega_t, & 0 \leq t \leq \tau \end{cases}$$

Thus under $(P^*)_\tau^\omega$, $B_t - B_\tau$ is $N(0, \bar{\sigma}\bar{\sigma}^\top(t - \tau))$ for $\tau \leq t \leq T$.

The C-CAPM (4.20) takes the form (assuming $\alpha < 0$): For every τ quasisurely in ω ,

$$b_t - r_t \mathbf{1} = (1 - \alpha) s_t(\bar{\sigma}\bar{\sigma}^\top) s^e \quad \text{on } [\tau, T] \quad (P^*)_\tau^\omega\text{-a.e.}$$

For comparison purposes, it is convenient to express this equation partially in terms of P_0 . The measures P_0 and P^* differ only via the change of variables defined via the SDE (3.1). Therefore, we arrive at the following equilibrium condition: *For every τ quasisurely in ω ,*

$$\widehat{b}_t - \widehat{r}_t \mathbf{1} = (1 - \alpha) \widehat{s}_t(\bar{\sigma}\bar{\sigma}^\top) s^e \quad \text{on } [\tau, T] \quad (P_0)_\tau^\omega\text{-a.e.} \quad (4.22)$$

where $\widehat{b}_t = b_t(X^\bar{\sigma})$, $\widehat{r}_t = r_t(X^\bar{\sigma})$ and $\widehat{s}_t = s_t(X^\bar{\sigma})$, corresponding to the noted change of variables under which $B_t \mapsto X_t^\bar{\sigma} = \bar{\sigma}^\top B_t$. Note that the difference between random variables with and without hats is ultimately not important because they follow identical distributions under $(P_0)_\tau^\omega$ and $(P^*)_\tau^\omega$ respectively.

The impact of ambiguous volatility is most easily seen by comparing with the standard C-CAPM obtained assuming complete confidence in the single probability law P_0 which renders B a standard Brownian motion. Then the prediction for asset returns is

$$b_t - r_t \mathbf{1} = (1 - \alpha) s_t s^e \quad \text{on } [0, T] \quad P_0\text{-a.e.}$$

or equivalently: *For every τ and P_0 -almost surely in ω ,*

$$b_t - r_t \mathbf{1} = (1 - \alpha) s_t s^e \quad \text{on } [\tau, T] \quad (P_0)_\tau^\omega\text{-a.e.} \quad (4.23)$$

³⁴That the minimizing measure corresponds to constant volatility is a feature of this example. More generally, the minimizing measure in \mathcal{P} models stochastic volatility. It is interesting to note that in Example 3.3, when volatility is modeled by robustifying the Hull-White and Heston parametric forms, the minimizing measure does not lie in either parametric class. Rather it corresponds to pasting the two alternatives together endogenously, that is, in a way that depends on the endowment process and on α .

There are two differences between the latter and the equilibrium condition (4.22) for our model. First the “instantaneous covariance” between asset returns and consumption is modified from $s_t s^e$ to $\widehat{s}_t(\overline{\sigma}\overline{\sigma}^\top)s^e$ reflecting the lack of confidence in P_0 and the fact that $\overline{\sigma}$ is the worst-case volatility scenario for the representative agent. Such an effect, whereby ambiguity leads to standard equilibrium conditions except that the reference measure is replaced by the worst-case measure, is familiar from the literature and leads to a form of observational equivalence. The second difference is new. Condition (4.23) refers to the single measure P_0 only and events that are null under P_0 are irrelevant. In contrast, the condition (4.22) is required to hold quasisurely in ω because, as described in Example 3.10, dynamic consistency requires that possibility be judged according to all priors in \mathcal{P} .

Turn to a brief consideration of corresponding equilibrium prices. Fix a dividend stream $(\delta, \delta_T) \in M^2(0, T) \times \widehat{L}^2(\Omega)$ where the security is available in zero net supply. Then its equilibrium price $S^\delta = (S_\tau^\delta)$ is given by: For all τ quasisurely in ω ,

$$S_\tau^\delta = E^{(P^*)^\omega_\tau} \left[\int_\tau^T \frac{\pi_t^e}{\pi_\tau^e} \delta_t dt + \frac{\pi_T^e}{\pi_\tau^e} \delta_T \right], \quad (4.24)$$

which lies between the hedging bounds in Theorem 4.2 by (4.19).³⁵ In the context of the example with $\alpha < 0$, for all τ quasisurely in ω ,

$$S_\tau^\delta(\omega) = E^{(P^*)^\omega_\tau} \left[\int_\tau^T \exp(-\beta(t - \tau)) (e_t/e_\tau)^{\alpha-1} \delta_t dt + \exp(-\beta(T - \tau)) (e_T/e_\tau)^{\alpha-1} \delta_T \right].$$

If $\delta = e$, then elementary calculations yield the time τ price of the endowment stream in the form

$$S_\tau^e = A_\tau e_\tau = A_\tau \exp(\alpha(s^e)^\top B_\tau) \quad q.s.$$

where A_τ and B_τ are deterministic constants. Thus $\log(S_\tau^e/A_\tau) = \log e_\tau$ and the logarithm of (deflated) price is also a G-Brownian motion.

We can also price an option on the endowment. This is interesting because of the intuitive and studied connection between (ambiguous) volatility and options (recall the literature cited in the introduction). Thus let $\delta_t = 0$ for $0 \leq t < T$ and $\delta_T = \psi(S_T)$. Denote its price process by S^ψ . One can see from (4.24) that any such derivative is priced in equilibrium as though σ_t were constant at $\overline{\sigma}$ (or

³⁵The proof is analogous to that of Lemma D.2, particularly surrounding (D.5).

at $\underline{\sigma}$ if $\alpha > 0$). In particular, for a European call option where $\delta_T = (S_T^e - \kappa)^+$, its price at τ is $e_\tau \cdot S_\tau^{BS}((s^e)^\top \bar{\sigma}, T, \kappa)$, where the latter term denotes the Black-Scholes price at τ for a call option with strike price κ and expiry time T when the underlying security price process is geometric Brownian motion with volatility $(s^e)^\top \bar{\sigma}$. Thus the Black-Scholes implied variance is $\text{tr}(\bar{\sigma} \bar{\sigma}^\top s^e (s^e)^\top)$ which exceeds every conceivable realized variance $\text{tr}(\sigma \sigma^\top s^e (s^e)^\top)$, $\sigma \in \Gamma$, consistent with a documented empirical feature of option prices.

Note that more generally for options where terminal payoff is given by the function ψ , the use of $\underline{\sigma}$ versus $\bar{\sigma}$ for pricing does *not* depend on the convexity/concavity of ψ . This is in stark contrast to the partial equilibrium option pricing literature (Avellaneda et al. [2], for example) that takes the underlying security price process as a primitive. Here the worst-case volatility is determined by the endowment and the preference parameter α . Interpretation of the restriction $\alpha < 0$ (or $\alpha > 0$) is confounded by the dual role of α in the additive expected utility model. However, we can generalize the example by adopting the so-called Kreps-Porteus aggregator (Duffie and Epstein [16]) that permits a partial separation between risk aversion and intertemporal substitutability. Then the same characterization of the worst-case volatility is valid with $1 - \alpha$ interpretable as the measure of relative risk aversion.

A. Appendix: Uniform Continuity

Define the shifted canonical space by

$$\Omega^t \equiv \{\omega \in C^d([t, T]) \mid \omega_t = 0\}.$$

Denote by B^t the canonical process on Ω^t , (a shift of B), by P_0^t the probability measure on Ω^t such that B^t is a Brownian motion, and by $\mathcal{F}^t = \{\mathcal{F}_\tau^t\}_{t \leq \tau \leq T}$ the filtration generated by B^t .

Fix $0 \leq s \leq t \leq T$. For $\omega \in \Omega^s$ and $\tilde{\omega} \in \Omega^t$, the concatenation of ω and $\tilde{\omega}$ at t is the path

$$(\omega \otimes_t \tilde{\omega})_\tau \triangleq \omega_\tau 1_{[s, t)}(\tau) + (\omega_t + \tilde{\omega}_\tau) 1_{[t, T]}(\tau), \quad s \leq \tau \leq T.$$

Given an \mathcal{F}_T^s -measurable random variable ξ on Ω^s and $\omega \in \Omega^s$, define the shifted random variable $\xi^{t, \omega}$ on Ω^t by

$$\xi^{t, \omega}(\tilde{\omega}) \triangleq \xi(\omega \otimes_t \tilde{\omega}), \quad \tilde{\omega} \in \Omega^t.$$

For an \mathcal{F}^s -progressively measurable process $(X_\tau)_{s \leq \tau \leq T}$, the shifted process $(X_\tau^{t,\omega})_{t \leq \tau \leq T}$ is \mathcal{F}^t -progressively measurable.

Let $(\Theta_s)_{0 \leq s \leq T}$ be a process of correspondences as in Section 3.2. For each (t, ω) in $[0, T] \times \Omega$, define a new process of correspondences $(\Theta_s^{t,\omega})_{t \leq s \leq T}$ by:

$$\Theta_s^{t,\omega}(\tilde{\omega}) \triangleq \Theta_s(\omega \otimes_t \tilde{\omega}), \quad \tilde{\omega} \in \Omega^t.$$

Then

$$\Theta_s^{t,\omega} : \Omega^t \rightsquigarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}.$$

The new process inherits conditions (i)-(iv) and (vi). The same is true for (v), which we define next.

The following definition is adapted from Nutz [44, Defn. 3.2]. Say that (Θ_t) is *uniformly continuous* if for all $\delta > 0$ and $(t, \omega) \in [0, T] \times \Omega$ there exists $\epsilon(t, \omega, \delta) > 0$ such that if $\sup_{0 \leq s \leq t} |\omega_s - \omega'_s| \leq \epsilon$, then

$$ri^\delta \Theta_s^{t,\omega}(\tilde{\omega}) \subseteq ri^\epsilon \Theta_s^{t,\omega'}(\tilde{\omega}) \text{ for all } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

The process (Θ_t) , and hence also Θ , are fixed throughout the appendices. Thus we write \mathcal{P} instead of \mathcal{P}^Θ . Define $\mathcal{P}^0 \subset \mathcal{P}$ by

$$\mathcal{P}^0 \equiv \{P \in \mathcal{P} : \exists \delta > 0 \text{ } \theta_t(\omega) \in ri^\delta \Theta_t(\omega) \text{ for all } (t, \omega) \in [0, T] \times \Omega\}. \quad (\text{A.1})$$

B. Appendix: Conditioning

Theorem 3.6 is proven here.

In fact, we prove more than is stated in the theorem because we prove also the following representation for conditional expectation. For each $t \in [0, T]$ and $P \in \mathcal{P}$, define

$$\mathcal{P}(t, P) = \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_t\}. \quad (\text{B.1})$$

Then for each $\xi \in \widehat{L^2}(\Omega)$, $t \in [0, T]$ and $P \in \mathcal{P}$,³⁶

$$\hat{E}[\xi \mid \mathcal{F}_t] = \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E_{P'}[\xi \mid \mathcal{F}_t], \quad P\text{-a.e.} \quad (\text{B.2})$$

³⁶Because all measures in $\mathcal{P}(t, P)$ coincide with P on \mathcal{F}_t , essential supremum is defined as in the classical case (see He et al. [27, pp. 8-9], for example). Thus the right hand side of (B.2) is defined to be any random variable ξ^* satisfying: (i) ξ^* is \mathcal{F}_t -measurable, $E_{P'}[\xi \mid \mathcal{F}_t] \leq \xi^*$ P -a.e. and (ii) $\xi^* \leq \xi^{**}$ P -a.e. for any other random variable ξ^{**} satisfying (i).

Perspective on this representation follows from considering the special case where all measures in \mathcal{P} are equivalent. Fix a measure P_0 in \mathcal{P} . Then the condition (B.2) becomes

$$\hat{E}[\xi \mid \mathcal{F}_t] = \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E_{P'}[\xi \mid \mathcal{F}_t], \quad P_0\text{-a.e.}, \text{ for every } P \in \mathcal{P}.$$

Accordingly, the random variable on the right side is (up to P_0 -nullity) independent of P . Apply $\cup_{P \in \mathcal{P}} \mathcal{P}(t, P) = \mathcal{P}$ to conclude that³⁷

$$\hat{E}[\xi \mid \mathcal{F}_t] = \operatorname{ess\,sup}_{P' \in \mathcal{P}} E_{P'}[\xi \mid \mathcal{F}_t], \quad P_0\text{-a.s.}$$

In other words, conditioning amounts to applying the usual Bayesian conditioning to each measure in \mathcal{P} and taking the upper envelope of the resulting expectations. This coincides with the prior-by-prior Bayesian updating rule in the Chen-Epstein model (apart from the different convention there of formulating expectations using infima rather than suprema). When the set \mathcal{P} is undominated, different measures in \mathcal{P} typically provide different perspectives on any random variable. Accordingly, (B.2) describes $\hat{E}[\xi \mid \mathcal{F}_t]$ completely by describing how it appears when seen through the lens of every measure in \mathcal{P} .

Our proof adapts arguments from Nutz [44]. He constructs a time consistent sublinear expectation in a setting with ambiguity about volatility but not about drift. Because of this difference and because we use a different approach to construct the set of priors, his results do not apply directly.

For any probability measure P on the canonical space Ω , a corresponding *regular conditional probability* P_t^ω is defined to be any mapping $P_t^\omega : \Omega \times \mathcal{F}_T \rightarrow [0, 1]$ satisfying the following conditions:

- (i) for any ω , P_t^ω is a probability measure on (Ω, \mathcal{F}_T) .
- (ii) for any $A \in \mathcal{F}_T$, $\omega \mapsto P_t^\omega(A)$ is \mathcal{F}_t -measurable.
- (iii) for any $A \in \mathcal{F}_T$, $E^P[1_A \mid \mathcal{F}_t](\omega) = P_t^\omega(A)$, P -a.e.

Of course, P_t^ω is not defined uniquely by these properties. We will fix a version defined via (3.7) after proving in Lemma B.2 that P_t^ω defined there satisfies the conditions characterizing a regular conditional probability. This explains our use of the same notation P_t^ω in both instances.

³⁷The P_0 -null event can be chosen independently of P by He et al. [27, Theorem 1.3]: Let \mathcal{H} be a non-empty family of random variables on any probability space. Then the essential supremum exists and there is a countable number of elements (ξ_n) of \mathcal{H} such that $\operatorname{ess\,sup} \mathcal{H} = \vee_n \xi_n$.

If P is a probability on Ω^s and $\omega \in \Omega^s$, for any $A \in \mathcal{F}_T^t$ we define

$$P^{t,\omega}(A) \triangleq P_t^\omega(\omega \otimes_t A),$$

where $\omega \otimes_t A \triangleq \{\omega \otimes_t \tilde{\omega} \mid \tilde{\omega} \in A\}$.

For each $(t, \omega) \in [0, T] \times \Omega$, let

$$\theta_s(\tilde{\omega}) = (\mu_s(\tilde{\omega}), \sigma_s(\tilde{\omega})) \in ri^\delta \Theta_s^{t,\omega}(\tilde{\omega}) \text{ for all } (s, \tilde{\omega}) \in [t, T] \times \Omega^t,$$

where $\delta > 0$ is some constant. Let $X^{t,\theta} = (X_s^{t,\theta})$ be the solution of the following equation (under P_0^t)

$$dX_s^{t,\theta} = \mu_s(X^{t,\theta})ds + \sigma_s(X^{t,\theta})dB_s^t, \quad X_t^{t,\theta} = 0, \quad s \in [t, T].$$

Then $X^{t,\theta}$ and P_0^t induce a probability measure $P^{t,\theta}$ on Ω^t .

Remark 1. *For nonspecialists we emphasize the difference between the preceding SDE and (3.7). The former is defined on the time interval $[t, T]$, and is a shifted version of (3.1), while (3.7) is defined on the full interval $[0, T]$. This difference is reflected also in the difference between the induced measures: the shifted measure $P_0^t \in \Delta(\Omega^t)$ and the conditional measure $P_t^\omega \in \Delta(\Omega)$. Part of the analysis to follow concerns shifted SDE's, random variables and measures and their relation to unshifted conditional counterparts. (See also Appendix A.)*

Let $\mathcal{P}^0(t, \omega)$ be the collection of such induced measures $P^{t,\theta}$. Define $\deg(t, \omega, P^{t,\theta}) = \delta^*/2 > 0$, where δ^* is the supremum of all δ such that

$$\theta_s(\tilde{\omega}) \in ri^\delta \Theta_s^{t,\omega}(\tilde{\omega}) \text{ for all } s \text{ and } \tilde{\omega}.$$

Note that at time $t = 0$, $\mathcal{P}^0(0, \omega)$ does not depend on ω and it coincides with \mathcal{P}^0 .

For each $(t, \omega) \in [0, T] \times \Omega$, let $\mathcal{P}(t, \omega)$ be the collection of all induced measures $P^{t,\theta}$ such that

$$\theta_s(\tilde{\omega}) = (\mu_s(\tilde{\omega}), \sigma_s(\tilde{\omega})) \in \Theta_s^{t,\omega}(\tilde{\omega}) \text{ for all } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

Note that $\mathcal{P}(0, \omega) = \mathcal{P}$.

Now we investigate the relationship between $\mathcal{P}(t, \omega)$ and \mathcal{P}_t^ω . (Recall that for any $P = P^\theta$ in \mathcal{P} , the measure P_t^ω is defined via (3.7) and \mathcal{P}_t^ω is the set of all such measures as in (3.8).)

For any $\theta = (\mu, \sigma) \in \Theta$, $(t, \omega) \in [0, T] \times \Omega$, define the shifted process $\bar{\theta}$ by

$$\bar{\theta}_s(\tilde{\omega}) = (\bar{\mu}_s(\tilde{\omega}), \bar{\sigma}_s(\tilde{\omega})) \triangleq (\mu_s^{t,\omega}(\tilde{\omega}), \sigma_s^{t,\omega}(\tilde{\omega})) \text{ for } (s, \tilde{\omega}) \in [t, T] \times \Omega^t. \quad (\text{B.3})$$

Then $\bar{\theta}_s(\tilde{\omega}) \in \Theta_s^{t,\omega}(\tilde{\omega})$. Consider the equation

$$\begin{cases} d\bar{X}_s = \bar{\mu}_s(\bar{X}.)ds + \bar{\sigma}_s(\bar{X}.)dB_s^t, & s \in [t, T], \\ \bar{X}_t = 0. \end{cases} \quad (\text{B.4})$$

Under P_0^t , the solution \bar{X} induces a probability measure $P^{t,\bar{\theta}}$ on Ω^t . By the definition of $\mathcal{P}(t, \omega)$, $P^{t,\bar{\theta}} \in \mathcal{P}(t, \omega)$.

Lemma B.1. $\{(P')^{t,\omega} : P' \in \mathcal{P}_t^\omega\} = \mathcal{P}(t, \omega)$.

Proof. \subset : For any $\theta = (\mu, \sigma) \in \Theta$, $P = P^\theta$ and $(t, \omega) \in [0, T] \times \Omega$, we claim that

$$(P_t^\omega)^{t,\omega} = P^{t,\bar{\theta}}$$

where $P^{t,\bar{\theta}}$ is defined through (B.4). Because B has independent increments under P_0 , the shifted solution $(X_s^{t,\omega})_{t \leq s \leq T}$ of (3.7) has the same distribution as does $(\bar{X}_s)_{t \leq s \leq T}$. This proves the claim.

\supset : Prove that for any $P^{t,\bar{\theta}} \in \mathcal{P}(t, \omega)$, there exists $\theta \in \Theta$ such that (where $P = P^\theta$)

$$P_t^\omega \in \mathcal{P}_t^\omega \text{ and } (P_t^\omega)^{t,\omega} = P^{t,\bar{\theta}}.$$

Each $P^{t,\bar{\theta}}$ in $\mathcal{P}(t, \omega)$ is induced by the solution \bar{X} of (B.4), where $\bar{\theta}(\tilde{\omega})$, defined in (B.3), lies in $\Theta_s^{t,\omega}(\tilde{\omega})$ for every $\tilde{\omega} \in \Omega^t$. For any $\theta \in \Theta$ such that

$$\theta_s^{t,\omega}(\tilde{\omega}) = (\mu_s^{t,\omega}(\tilde{\omega}), \sigma_s^{t,\omega}(\tilde{\omega})) = (\bar{\mu}_s(\tilde{\omega}), \bar{\sigma}_s(\tilde{\omega})), \quad s \in [t, T],$$

consider the equation (3.7). Under P_0 , the solution X induces a probability P_t^ω on Ω . Because B has independent increments under P_0 , we know that $(P_t^\omega)^{t,\omega} = P^{t,\bar{\theta}}$. This completes the proof. \blacksquare

Lemma B.2. For any $\theta \in \Theta$ and $P = P^\theta$, $\{P_t^\omega : (t, \omega) \in [0, T] \times \Omega\}$ is a version of the regular conditional probability of P .

Proof. Firstly, for any $0 < t_1 < \dots < t_n \leq T$ and bounded, continuous functions φ and ψ , we prove that

$$E^P[\varphi(B_{t_1 \wedge t}, \dots, B_{t_n \wedge t})\psi(B_{t_1}, \dots, B_{t_n})] = E^P[\varphi(B_{t_1 \wedge t}, \dots, B_{t_n \wedge t})\psi_t] \quad (\text{B.5})$$

where $t \in [t_k, t_{k+1})$ and, for any $\hat{\omega} \in \Omega$,

$$\psi_t(\hat{\omega}) \triangleq E^{P^{t, \bar{\theta}}}[\psi(\hat{\omega}(t_1), \dots, \hat{\omega}(t_k), \hat{\omega}(t) + B_{t_{k+1}}^t, \dots, \hat{\omega}(t) + B_{t_n}^t)].$$

Note that $P^{t, \bar{\theta}}$ on Ω^t is induced by \bar{X} (see (B.3)) under P_0^t , and $P = P^\theta$ on Ω is induced by $X = X^\theta$ (see (3.1)) under P_0 . Then,

$$\begin{aligned} \psi_t(X(\omega)) &= \\ E^{P_0^t}[\psi(X_{t_1}(\omega), \dots, X_{t_k}(\omega), X_t(\omega) + \bar{X}_{t_{k+1}}(\tilde{\omega}), \dots, X_t(\omega) + \bar{X}_{t_n}(\tilde{\omega}))]. \end{aligned} \quad (\text{B.6})$$

Because B has independent increments under P_0 , the shifted regular conditional probability

$$P_0^{t, \omega} = P_0^t, \text{ } P_0\text{-a.e.}$$

Thus (B.6) holds under probability $P_0^{t, \omega}$.

Because $P_0^{t, \omega}$ is the shifted probability of $(P_0)_t^\omega$, we have

$$\begin{aligned} &\psi_t(X(\omega)) \\ &= E^{(P_0)_t^\omega}[\psi(X_{t_1}(\omega), \dots, X_{t_k}(\omega), X_{t_{k+1}}(\omega), \dots, X_{t_n}(\omega))] \\ &= E^{P_0}[\psi(X_{t_1}(\omega), \dots, X_{t_k}(\omega), X_{t_{k+1}}(\omega), \dots, X_{t_n}(\omega)) \mid \mathcal{F}_t](\omega), \text{ } P_0\text{-a.e.} \end{aligned}$$

Further, because P is induced by X and P_0 ,

$$\begin{aligned} &E^P[\varphi(B_{t_1 \wedge t}, \dots, B_{t_n \wedge t})\psi_t] \\ &= E^{P_0}[\varphi(X_{t_1 \wedge t}, \dots, X_{t_n \wedge t})\psi_t(X)] \\ &= E^{P_0}[\varphi(X_{t_1 \wedge t}, \dots, X_{t_n \wedge t})E^{P_0}[\psi(X_{t_1}, \dots, X_{t_n}) \mid \mathcal{F}_t]] \\ &= E^{P_0}[\varphi(X_{t_1 \wedge t}, \dots, X_{t_n \wedge t})\psi(X_{t_1}, \dots, X_{t_n})] \\ &= E^P[\varphi(B_{t_1 \wedge t}, \dots, B_{t_n \wedge t})\psi(B_{t_1}, \dots, B_{t_n})]. \end{aligned}$$

Secondly, note that (B.5) is true and φ and (t_1, \dots, t_n) are arbitrary. Then by the definition of the regular conditional probability, for P -a.e. $\hat{\omega} \in \Omega$ and $t \in [t_k, t_{k+1})$,

$$\psi_t(\hat{\omega}) = E^{\tilde{P}^{t, \hat{\omega}}}[\psi(\hat{\omega}(t_1), \dots, \hat{\omega}(t_k), \hat{\omega}(t) + B_{t_{k+1}}^t, \dots, \hat{\omega}(t) + B_{t_n}^t)], \quad (\text{B.7})$$

where $\tilde{P}^{t, \hat{\omega}}$ is the shift of the regular conditional probability of P given $(t, \hat{\omega}) \in [0, T] \times \Omega$.

By standard approximating arguments, there exists a set M such that $P(M) = 0$ and for any $\omega \notin M$, (B.7) holds for all continuous bounded function ψ and

(t_1, \dots, t_n) . This means that for $\omega \notin M$ and for all bounded \mathcal{F}_T^t -measurable random variables ξ

$$E^{\tilde{P}^{t,\omega}} \xi = E^{P^{t,\bar{\theta}}} \xi.$$

Then $\tilde{P}^{t,\omega} = P^{t,\bar{\theta}}$ P -a.e. By Lemma B.1, $\tilde{P}^{t,\omega} = P^{t,\bar{\theta}} = (P_t^\omega)^{t,\omega}$, P -a.e. Thus P_t^ω is a version of the regular conditional probability for P . \blacksquare

In the following, we always use P_t^ω defined by (3.7) as the fixed version of regular conditional probability for $P \in \mathcal{P}$. Thus

$$E^{P_t^\omega} \xi = E^P[\xi \mid \mathcal{F}_t](\omega), \quad P\text{-a.e.}$$

Because we will want to consider also dynamics beginning at arbitrary s , let $0 \leq s \leq T$, $\bar{\omega} \in \Omega$, and $P \in \mathcal{P}(s, \bar{\omega})$. Then given $(t, \omega) \in [s, T] \times \Omega^s$, we can fix a version of the regular conditional probability, also denoted P_t^ω (here a measure on Ω^s), which is constructed in a similar fashion via a counterpart of (3.7). Define

$$\mathcal{P}_t^\omega(s, \bar{\omega}) = \{P_t^\omega : P \in \mathcal{P}(s, \bar{\omega})\} \text{ and}$$

$$\mathcal{P}_t^{0,\omega}(s, \bar{\omega}) = \{P_t^\omega : P \in \mathcal{P}^0(s, \bar{\omega})\}.$$

In each case, the obvious counterpart of the result in Lemma B.1 is valid.

The remaining arguments are divided into three steps. First we prove that if $\xi \in UC_b(\Omega)$ and if the set \mathcal{P} is replaced by \mathcal{P}^0 defined in (A.1), then the counterparts of (B.2) and (3.10) are valid. Then we show that \mathcal{P}^0 (resp. $\mathcal{P}^0(t, \omega)$) is dense in \mathcal{P} (resp. $\mathcal{P}(t, \omega)$). In Step 3 the preceding is extended to apply to all ξ in the completion of $UC_b(\Omega)$.

Step 1

Given $\xi \in UC_b(\Omega)$, define

$$v_t^0(\omega) \triangleq \sup_{P \in \mathcal{P}^0(t, \omega)} E^P \xi^{t, \omega}, \quad (t, \omega) \in [0, T] \times \Omega. \quad (\text{B.8})$$

Lemma B.3. *Let $0 \leq s \leq t \leq T$ and $\bar{\omega} \in \Omega$. Given $\epsilon > 0$, there exist a sequence $(\hat{\omega}^i)_{i \geq 1}$ in Ω^s , an \mathcal{F}_t^s -measurable countable partition $(E^i)_{i \geq 1}$ of Ω^s , and a sequence $(P^i)_{i \geq 1}$ of probability measures on Ω^t such that*

- (i) $\|\omega - \hat{\omega}^i\|_{[s,t]} \equiv \sup_{s \leq \tau \leq t} |\omega_\tau - \hat{\omega}_\tau^i| \leq \epsilon$ for all $\omega \in E^i$;
- (ii) $P^i \in \mathcal{P}^0(t, \bar{\omega} \otimes_s \omega)$ for all $\omega \in E^i$ and $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0$;
- (iii) $v_t^0(\bar{\omega} \otimes_s \hat{\omega}^i) \leq E^{P^i}[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}^i}] + \epsilon$.

Proof. Given $\epsilon > 0$ and $\hat{\omega} \in \Omega^s$, by (B.8) there exists $P(\hat{\omega}) \in \mathcal{P}^0(t, \bar{\omega} \otimes_s \hat{\omega})$ such that

$$v_t^0(\bar{\omega} \otimes_s \hat{\omega}) \leq E^{P(\hat{\omega})}[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}}] + \epsilon.$$

Because (Θ_t) is uniformly continuous, there exists $\epsilon(\hat{\omega}) > 0$ such that

$$P(\hat{\omega}) \in \mathcal{P}^0(t, \bar{\omega} \otimes_s \omega') \text{ for all } \omega' \in B(\epsilon(\hat{\omega}), \hat{\omega}) \text{ and } \inf_{\omega' \in B(\epsilon(\hat{\omega}), \hat{\omega})} \deg(t, \bar{\omega} \otimes_s \omega', P(\hat{\omega})) > 0$$

where $B(\epsilon, \hat{\omega}) \triangleq \{\omega' \in \Omega^s \mid \|\omega' - \hat{\omega}\|_{[s,t]} < \epsilon\}$ is the open $\|\cdot\|_{[s,t]}$ ball. Then $\{B(\epsilon(\hat{\omega}), \hat{\omega}) \mid \hat{\omega} \in \Omega^s\}$ forms an open cover of Ω^s . There exists a countable subcover because Ω^s is separable. We denote the subcover by

$$B^i \triangleq B(\epsilon(\hat{\omega}^i), \hat{\omega}^i), \quad i = 1, 2, \dots$$

and define a partition of Ω^s by

$$E^1 \triangleq B^1, \quad E^{i+1} \triangleq B^{i+1} \setminus (E^1 \cup \dots \cup E^i), \quad i \geq 1.$$

Set $P^i \triangleq P(\hat{\omega}^i)$. Then (i)-(iii) are satisfied. ■

For any $A \in \mathcal{F}_T^s$, define

$$A^{t, \omega} = \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A\}.$$

Lemma B.4. Let $0 \leq s \leq t \leq T$, $\bar{\omega} \in \Omega$ and $P \in \mathcal{P}^0(s, \bar{\omega})$. Let $(E^i)_{0 \leq i \leq N}$ be a finite \mathcal{F}_t^s -measurable partition of Ω^s . For $1 \leq i \leq N$, assume that $P^i \in \mathcal{P}^0(t, \bar{\omega} \otimes_s \omega)$ for all $\omega \in E^i$ and that $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0$. Define \bar{P} by

$$\bar{P}(A) \triangleq P(A \cap E^0) + \sum_{i=1}^N E^P[P^i(A^{t, \omega}) 1_{E^i}(\omega)], \quad A \in \mathcal{F}_T^s.$$

Then: (i) $\bar{P} \in \mathcal{P}^0(s, \bar{\omega})$.

(ii) $\bar{P} = P$ on \mathcal{F}_t^s .

(iii) $\bar{P}^{t, \omega} = P^{t, \omega}$ P -a.e. $[\omega]$ on E^0 .

(iv) $\bar{P}^{t, \omega} = P^i$ P -a.e. $[\omega]$ on E^i , $1 \leq i \leq N$.

Proof. (i) Let θ (resp. θ^i) be the \mathcal{F}^s (resp. \mathcal{F}^t) -measurable process such that $P = P^\theta$ (resp. $P^i = P^{\theta^i}$). Define $\bar{\theta}$ by

$$\bar{\theta}_\tau(\omega) \triangleq \theta_\tau(\omega)1_{[s,t)}(\tau) + [\theta_\tau(\omega)1_{E^0}(\omega) + \sum_{i=1}^N \theta_\tau^i(\omega^t)1_{E^i}(\omega)]1_{[t,T]}(\tau) \quad (\text{B.9})$$

for $(\tau, \omega) \in [s, T] \times \Omega^s$. Then $P^{\bar{\theta}} \in \mathcal{P}^0(s, \bar{\omega})$ and $\bar{P} = P^{\bar{\theta}}$ on \mathcal{F}_T^s .

(ii) Let $A \in \mathcal{F}_t^s$. We prove $\bar{P}(A) = P(A)$. Note that for $\omega \in \Omega^s$, if $\omega \in A$, then $A^{t,\omega} = \Omega^t$; otherwise, $A^{t,\omega} = \emptyset$. Thus, $P^i(A^{t,\omega}) = 1_A(\omega)$ for $1 \leq i \leq N$, and

$$\bar{P}(A) = P(A \cap E^0) + \sum_{i=1}^N E^P[1_A(\omega)1_{E^i}(\omega)] = \sum_{i=0}^N E^P[A \cap E^i] = P(A).$$

(iii)-(iv) Recall the definition of P_t^ω by (3.7). Note that $\bar{P} = P^{\bar{\theta}}$ where $\bar{\theta}$ is defined by (B.9). Then it is easy to show that the shifted regular conditional probability $\bar{P}^{t,\omega}$ satisfies (iii)-(iv). \blacksquare

The technique used in proving Nutz [44, Theorem 4.5] can be adapted to prove the following dynamic programming principle.

Proposition B.5. Let $0 \leq s \leq t \leq T$, $\xi \in UC_b(\Omega)$ and define v_t^0 by (B.8). Then

$$v_s^0(\bar{\omega}) = \sup_{P' \in \mathcal{P}^0(s, \bar{\omega})} E^{P'}[(v_t^0)^{s, \bar{\omega}}] \text{ for all } \bar{\omega} \in \Omega, \quad (\text{B.10})$$

$$v_s^0 = \text{esssup}_{P' \in \mathcal{P}^0(s, P)} E^{P'}[v_t^0 \mid \mathcal{F}_s] \text{ } P\text{-a.e. for all } P \in \mathcal{P}^0, \quad (\text{B.11})$$

and

$$v_s^0 = \text{esssup}_{P' \in \mathcal{P}^0(s, P)} E^{P'}[\xi \mid \mathcal{F}_s] \text{ } P\text{-a.e. for all } P \in \mathcal{P}^0. \quad (\text{B.12})$$

Proof. *Proof of (B.10):* First prove \leq . Let $\bar{\omega} \in \Omega$ and $\omega \in \Omega^s$, by Lemma B.1,

$$\{(P')^{t,\omega} \mid P' \in \mathcal{P}_t^{0,\omega}(s, \bar{\omega})\} = \mathcal{P}^0(t, \bar{\omega} \otimes_s \omega).$$

For $P \in \mathcal{P}^0(s, \bar{\omega})$,

$$\begin{aligned} & E^{P^{t,\omega}}[(\xi^{s, \bar{\omega}})^{t, \omega}] = E^{P^{t,\omega}}[\xi^{t, \bar{\omega} \otimes_s \omega}] \\ & \leq \sup_{P' \in \mathcal{P}^0(t, \bar{\omega} \otimes_s \omega)} E^{P'}[\xi^{t, \bar{\omega} \otimes_s \omega}] \\ & = v_t^0(\bar{\omega} \otimes_s \omega). \end{aligned} \quad (\text{B.13})$$

Note that $v_t^0(\bar{\omega} \otimes_s \omega) = (v_t^0)^{s, \bar{\omega}}(\omega)$. Taking the expectation under P on both sides of (B.13) yields

$$E^P \xi^{s, \bar{\omega}} \leq E^P [(v_t^0)^{s, \bar{\omega}}].$$

The desired result holds by taking the supremum over $P \in \mathcal{P}^0(s, \bar{\omega})$.

Prove \geq . Let $\delta > 0$. Because ${}^t\Omega$ is a Polish space and $(v_t^0)^{s, \bar{\omega}}$ is \mathcal{F}_t^s -measurable, by Lusin's Theorem there exists a compact set $G \in \mathcal{F}_t^s$ with $P(G) > 1 - \delta$ and such that $(v_t^0)^{s, \bar{\omega}}$ is uniformly continuous on G .

Let $\epsilon > 0$. By Lemma B.3, there exist a sequence $(\hat{\omega}^i)_{i \geq 1}$ in G , an \mathcal{F}_t^s -measurable partition $(E^i)_{i \geq 1}$ of G , and a sequence $(P^i)_{i \geq 1}$ of probability measures such that

- (a) $\|\omega - \hat{\omega}^i\|_{[s, t]} \leq \epsilon$ for all $\omega \in E^i$;
- (b) $P^i \in \mathcal{P}^0(t, \bar{\omega} \otimes_s \omega)$ for all $\omega \in E^i$ and $\inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0$;
- (c) $v_t^0(\bar{\omega} \otimes_s \hat{\omega}^i) \leq E^{P^i}[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}^i}] + \epsilon$.

Let

$$A_N \triangleq E^1 \cup \dots \cup E^N, \quad N \geq 1.$$

For $P \in \mathcal{P}^0(s, \bar{\omega})$, define

$$\mathcal{P}^0(s, \bar{\omega}, t, P) \triangleq \{P' \in \mathcal{P}^0(s, \bar{\omega}) : P' = P \text{ on } \mathcal{F}_t^s\}.$$

Apply Lemma B.4 to the finite partition $\{E^1, \dots, E^N, A_N^c\}$ of Ω^s to obtain a measure $P_N \in \mathcal{P}^0(s, \bar{\omega})$ such that $P_N \in \mathcal{P}^0(s, \bar{\omega}, t, P)$ and

$$P_N^{t, \omega} = \begin{cases} P^{t, \omega} & \text{for } \omega \in A_N^c, \\ P^i & \text{for } \omega \in E^i, \quad 1 \leq i \leq N \end{cases} \quad (\text{B.14})$$

Because $(v_t^0)^{s, \bar{\omega}}$ and ξ are uniformly continuous on G , there exist moduli of continuity $\rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\cdot)$ and $\rho^{(\xi)}(\cdot)$ such that

$$\begin{aligned} & |(v_t^0)^{s, \bar{\omega}}(\omega) - (v_t^0)^{s, \bar{\omega}}(\omega')| \leq \rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\|\omega - \omega'\|_{[s, t]}), \\ & |\xi^{t, \bar{\omega} \otimes_s \omega} - \xi^{t, \bar{\omega} \otimes_s \omega'}| \leq \rho^{(\xi)}(\|\omega - \omega'\|_{[s, t]}). \end{aligned}$$

Let $\omega \in E^i$ for some $1 \leq i \leq N$. Then

$$\begin{aligned} & (v_t^0)^{s, \bar{\omega}}(\omega) \\ & \leq (v_t^0)^{s, \bar{\omega}}(\hat{\omega}^i) + \rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\epsilon) \\ & \leq E^{P^i}[\xi^{t, \bar{\omega} \otimes_s \hat{\omega}^i}] + \epsilon + \rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\epsilon) \\ & \leq E^{P^i}[\xi^{t, \bar{\omega} \otimes_s \omega}] + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\epsilon) \\ & = E^{P_N^{t, \omega}}[\xi^{t, \bar{\omega} \otimes_s \omega}] + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\epsilon) \\ & = E^{P_N}[\xi^{s, \bar{\omega}} | \mathcal{F}_t^s](\omega) + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{((v_t^0)^{s, \bar{\omega}}|G)}(\epsilon) \text{ for } P\text{-a.e. } \omega \in E^i. \end{aligned} \quad (\text{B.15})$$

These inequalities are due respectively to uniform continuity of v_t^0 , Lemma B.3(iii), uniform continuity of ξ , equation (B.14), and the fact that $P_N \in \mathcal{P}(t, P)$. Because $P = P_N$ on \mathcal{F}_t^s , taking the P -expectation on both sides yields

$$E^P[(v_t^0)^{s, \bar{\omega}} 1_{A_N}] \leq E^{P_N}[\xi^{s, \bar{\omega}} 1_{A_N}] + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{((v_t^0)^{s, \bar{\omega}} | G)}(\epsilon).$$

Note that $\xi \in UC_b(\Omega)$ and $P_N(G \setminus A_N) = P(G \setminus A_N) \rightarrow 0$ as $N \rightarrow \infty$. Let $N \rightarrow \infty$ and $\epsilon \rightarrow 0$ to obtain that

$$E^P[(v_t^0)^{s, \bar{\omega}} 1_G] \leq \sup_{P' \in \mathcal{P}^0(s, \bar{\omega}, t, P)} E^{P'}[\xi^{s, \bar{\omega}} 1_G].$$

Because $\delta > 0$ is arbitrary, similar arguments show that

$$E^P[(v_t^0)^{s, \bar{\omega}}] \leq \sup_{P' \in \mathcal{P}^0(s, \bar{\omega}, t, P)} E^{P'} \xi^{s, \bar{\omega}} \leq \sup_{P' \in \mathcal{P}^0(s, \bar{\omega})} E^{P'} \xi^{s, \bar{\omega}} = v_s^0(\bar{\omega}).$$

But $P \in \mathcal{P}^0(s, \bar{\omega})$ is arbitrary. This completes the proof of (B.10).

Proof of (B.11): Fix $P \in \mathcal{P}^0$. First we prove that

$$v_t^0 \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}^0(t, P)} E^{P'}[\xi \mid \mathcal{F}_t] \quad P\text{-a.e.} \quad (\text{B.16})$$

Argue as in the second part of the preceding proof, specialized to $s = 0$. Conclude that there exists $P_N \in \mathcal{P}^0(t, P)$ such that, as a counterpart of (B.15),

$$v_t^0(\omega) \leq E^{P_N}[\xi \mid \mathcal{F}_t](\omega) + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{(v_t^0 | G)}(\epsilon) \text{ for } P\text{-a.e. } \omega \in A_N.$$

Because $P = P_N$ on \mathcal{F}_t , as $N \rightarrow \infty$ and $\delta \rightarrow 0$, one obtains (B.16).

Now prove the inequality \leq in (B.11). For any $P' \in \mathcal{P}^0(s, P)$, we know that $(P')^{t, \omega} \in \mathcal{P}^0(t, \omega)$. From (B.10),

$$v_t^0(\omega) \geq E^{(P')^{t, \omega}}[\xi^{t, \omega}] = E^{P'}[\xi \mid \mathcal{F}_t](\omega) \quad P'\text{-a.e.}$$

Taking the conditional expectation on both sides yields $E^{P'}[\xi \mid \mathcal{F}_s] \leq E^{P'}[v_t^0 \mid \mathcal{F}_s] \quad P'\text{-a.e.}$, hence also $P\text{-a.e.}$. Thus

$$v_s^0 \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}^0(s, P)} E^{P'}[\xi \mid \mathcal{F}_s] \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}^0(s, P)} E^{P'}[v_t^0 \mid \mathcal{F}_s] \quad P\text{-a.e.}$$

Thirdly, we prove the converse direction holds in (B.11). For any $P' \in \mathcal{P}^0(s, P)$, by (B.10) we have

$$v_s^0(\omega) \geq E^{(P')^{s, \omega}}[(v_t^0)^{s, \omega}] = E^{P'}[v_t^0 \mid \mathcal{F}_s](\omega)$$

P' -a.e. on \mathcal{F}_s and hence P -a.e.

Equation (B.12) is implied by (B.11) because $v_T^0 = \xi$. ■

STEP 2

Refer to the topology induced on $\Delta(\Omega)$ by bounded continuous functions as the weak-convergence topology.

Lemma B.6. (a) \mathcal{P}^0 is dense in \mathcal{P} in the weak convergence topology.

(b) For each t and ω , $\mathcal{P}^0(t, \omega)$ is dense in $\mathcal{P}(t, \omega)$ in the weak convergence topology.

Proof. (a) Let $P^{\theta^0} \in \mathcal{P}^0$ and $P^\theta \in \mathcal{P}$, and define

$$\theta^\epsilon = \epsilon\theta^0 + (1 - \epsilon)\theta,$$

where $0 < \epsilon < 1$. By Uniform Interiority for (Θ_t) , there exists $\delta > 0$ such that $\theta_t^\epsilon(\omega) \in \text{ri}^\delta \Theta_t(\omega)$ for all t and ω . Thus $P^{\theta^\epsilon} \in \mathcal{P}^0$.

By the standard approximation of a stochastic differential equation (see Gihman and Skorohod [25, Thm 3.15]), as $\epsilon \rightarrow 0$ there exists a subsequence of X^{θ^ϵ} , which we still denote by X^{θ^ϵ} , such that

$$\sup_{0 \leq t \leq T} |X_t^{\theta^\epsilon} - X_t^\theta| \rightarrow 0 \text{ } P_0\text{-a.e.}$$

The Dominated Convergence Theorem implies that $P^{\theta^\epsilon} \rightarrow P^\theta$.

(b) The proof is similar. ■

For any $\theta = (\mu, \sigma) \in \Theta$, $(t, \omega) \in [0, T] \times \Omega$, define the shifted process $\bar{\theta}$, the process \bar{X} and the probability measure $P^{t, \bar{\theta}}$ exactly as in (B.3) and (B.4). As noted earlier, $P^{t, \bar{\theta}} \in \mathcal{P}(t, \omega)$.

Recall that for any P in \mathcal{P} , the measure P_t^ω is defined via (3.7); \mathcal{P}_t^ω is the set of all such measures (3.8). By construction,

$$P_t^\omega(\{\bar{\omega} \in \Omega : \bar{\omega}_s = \omega_s \text{ } s \in [0, t]\}) = 1.$$

We show shortly that P_t^ω is a version of the regular conditional probability for P .

Given $\xi \in UC_b(\Omega)$, define

$$v_t(\omega) \triangleq \sup_{P \in \mathcal{P}(t, \omega)} E^P \xi^{t, \omega}, \text{ } (t, \omega) \in [0, T] \times \Omega. \quad (\text{B.17})$$

Lemma B.7. For any $(t, \omega) \in [0, T] \times \Omega$ and $\xi \in UC_b(\Omega)$, we have

$$v_t(\omega) = v_t^0(\omega), \quad (\text{B.18})$$

$$v_t(\omega) = \sup_{P \in \mathcal{P}_t^\omega} E^P \xi, \quad (\text{B.19})$$

and

$$v_t = \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\xi \mid \mathcal{F}_t] \text{ } P\text{-a.e. for all } P \in \mathcal{P}. \quad (\text{B.20})$$

Furthermore, for any $0 \leq s \leq t \leq T$,

$$v_s(\omega) = \sup_{P' \in \mathcal{P}(s, \omega)} E^{P'}[(v_t)^{s, \omega}] \text{ for all } \omega \in \Omega, \quad (\text{B.21})$$

and

$$v_s = \operatorname{ess\,sup}_{P' \in \mathcal{P}(s, P)} E^{P'}[v_t \mid \mathcal{F}_s] \text{ } P\text{-a.e. for all } P \in \mathcal{P}. \quad (\text{B.22})$$

Proof. *Proof of (B.18):* It is implied by the fact that $\mathcal{P}^0(t, \omega)$ is dense in $\mathcal{P}(t, \omega)$.

Proof of (B.19): By the definition of $v_t(\omega)$, we know that

$$v_t(\omega) = \sup_{P \in \mathcal{P}(t, \omega)} E^P \xi^{t, \omega}.$$

By Lemma B.1,

$$\{(P')^{t, \omega} \mid P' \in \mathcal{P}_t^\omega\} = \mathcal{P}(t, \omega).$$

Thus

$$v_t(\omega) = \sup_{\hat{P} \in \mathcal{P}_t^\omega} E^{\hat{P}^{t, \omega}} \xi^{t, \omega} = \sup_{\hat{P} \in \mathcal{P}_t^\omega} E^{\hat{P}} \xi.$$

Proof of (B.20): Fix $P \in \mathcal{P}$. For any $P' \in \mathcal{P}(t, P)$, by Lemmas B.1 and B.2, $(P')^{t, \omega} \in \mathcal{P}(t, \omega)$. By the definition of $v_t(\omega)$,

$$v_t(\omega) \geq E^{(P')^{t, \omega}} \xi^{t, \omega} = E^{P'}[\xi \mid \mathcal{F}_t](\omega)$$

P' -a.e. on \mathcal{F}_t and hence P -a.e. Thus

$$v_t \geq \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\xi \mid \mathcal{F}_t] \text{ } P\text{-a.e.}$$

Now we prove the reverse inequality. By (B.18), $v_t(\omega) = v_t^0(\omega)$. Then, using the same technique as in the proof of Proposition B.5 for the special case $s = 0$, there exists $P_N \in \mathcal{P}(t, P)$ such that, as a counterpart of (B.15),

$$\begin{aligned} v_t(\omega) &\leq E^{P_N}[\xi \mid \mathcal{F}_t](\omega) + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{(v_t|G)}(\epsilon) \\ &\leq \sup_N E^{P_N}[\xi \mid \mathcal{F}_t](\omega) + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{(v_t|G)}(\epsilon) \end{aligned}$$

for P -a.e. $\omega \in A_N$. Let $N \rightarrow \infty$ to obtain that, for P -a.e. $\omega \in G$,

$$v_t(\omega) \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\xi \mid \mathcal{F}_t](\omega) + \rho^{(\xi)}(\epsilon) + \epsilon + \rho^{(v_t|G)}(\epsilon).$$

Let $\epsilon \rightarrow 0$ to derive

$$v_t \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\xi \mid \mathcal{F}_t], \text{ } P\text{-a.e. on } G.$$

Note that G depends on δ , but not on ϵ . Let $\delta \rightarrow 0$ and conclude that

$$v_t \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\xi \mid \mathcal{F}_t], \text{ } P\text{-a.e.}$$

Proof of (B.21) and (B.22): The former is due to (B.10) and the fact that $P^0(t, \omega)$ is dense in $P(t, \omega)$. The proof of (B.22) is similar to the proof of (B.12) in Proposition B.5. ■

Now for any $\xi \in UC_b(\Omega)$, we define conditional expectation by

$$\hat{E}[\xi \mid \mathcal{F}_t](\omega) \triangleq v_t(\omega).$$

STEP 3

Thus far we have defined $\hat{E}[\xi \mid \mathcal{F}_t](\omega)$, for all t and ω , for any $\xi \in UC_b(\Omega)$. Now extend the operator $\hat{E}[\cdot \mid \mathcal{F}_t]$ to the completion of $UC_b(\Omega)$.

Lemma B.8 (Extension). *The mapping $\hat{E}[\cdot \mid \mathcal{F}_t]$ on $UC_b(\Omega)$ can be extended uniquely to a 1-Lipschitz continuous mapping $\hat{E}[\cdot \mid \mathcal{F}_t] : \widehat{L^2}(\Omega) \rightarrow \widehat{L^2}(t\Omega)$.*

Proof. Define $\widetilde{L}^2(t\Omega)$ to be the space of \mathcal{F}_t -measurable random variables X satisfying

$$\|X\| \triangleq (\hat{E}[|X|^2])^{\frac{1}{2}} = \left(\sup_{P \in \mathcal{P}} E^P[|X|^2]\right)^{\frac{1}{2}} < \infty.$$

Obviously, $\widehat{L}^2(t\Omega) \subset \widetilde{L}^2(t\Omega)$.

(i) We prove that $\hat{E}[\cdot | \mathcal{F}_t]$ can be uniquely extended to a 1-Lipschitz continuous mapping

$$\hat{E}[\cdot | \mathcal{F}_t] : \widehat{L}^2(\Omega) \rightarrow \widetilde{L}^2(t\Omega).$$

For any ξ and η in $UC_b(\Omega)$,

$$|\hat{E}[\xi' | \mathcal{F}_t] - \hat{E}[\xi | \mathcal{F}_t]|^2 \leq \left(\hat{E}[|\xi' - \xi|^2 | \mathcal{F}_t]\right)^2 \leq \hat{E}[|\xi' - \xi|^2 | \mathcal{F}_t],$$

where the first inequality follows primarily from the subadditivity of $\hat{E}[\cdot | \mathcal{F}_t]$, and the second is implied by Jensen's inequality applied to each P in \mathcal{P} . Thus

$$\begin{aligned} & \|\hat{E}[\xi' | \mathcal{F}_t] - \hat{E}[\xi | \mathcal{F}_t]\| \\ &= \left(\hat{E}[|\hat{E}[\xi' | \mathcal{F}_t] - \hat{E}[\xi | \mathcal{F}_t]|^2]\right)^{1/2} \\ &\leq \left(\hat{E}[\hat{E}[|\xi' - \xi|^2 | \mathcal{F}_t]]\right)^{1/2} \\ &= \left(\hat{E}[|\xi' - \xi|^2]\right)^{1/2} = \|\xi' - \xi\|, \end{aligned}$$

where the second equality is due to the 'law of iterated expectations' for integrands in $UC_b(\Omega)$ proven in Lemma B.7.

As a consequence, $\hat{E}[\xi | \mathcal{F}_t] \in \widetilde{L}^2(t\Omega)$ for ξ and η in $UC_b(\Omega)$, and $\hat{E}[\cdot | \mathcal{F}_t]$ extends uniquely to a 1-Lipschitz continuous mapping from $\widehat{L}^2(\Omega)$ into $\widetilde{L}^2(t\Omega)$.

(ii) Now prove that $\hat{E}[\cdot | \mathcal{F}_t]$ maps $\widehat{L}^2(\Omega)$ into $\widehat{L}^2(t\Omega)$.

First we show that if $\xi \in UC_b(\Omega)$, then $\hat{E}[\xi | \mathcal{F}_t]$ is lower semicontinuous: Fix $\omega \in \Omega$. Since $\xi \in UC_b(\Omega)$, there exists a modulus of continuity $\rho^{(\xi)}$ such that for all $\omega' \in \Omega$ and $\tilde{\omega} \in \Omega^t$

$$\begin{aligned} & |\xi(\omega) - \xi(\omega')| \leq \rho^{(\xi)}(\|\omega - \omega'\|_{[0,T]}), \text{ and} \\ & |\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})| \leq \rho^{(\xi)}(\|\omega - \omega'\|_{[0,t]}). \end{aligned}$$

Consider a sequence (ω_n) such that $\|\omega - \omega_n\|_{[0,t]} \rightarrow 0$. For any $P \in \mathcal{P}(t, \omega)$, by the Uniform Continuity assumption for (Θ_t) , we know that $P \in \mathcal{P}(t, \omega_n)$ when n

is large enough. Thus

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} v_t(\omega_n) \\
&= \liminf_{n \rightarrow \infty} \sup_{P' \in \mathcal{P}(t, \omega_n)} E^{P'} \xi^{t, \omega_n} \\
&\geq \liminf_{n \rightarrow \infty} \left(\left(\sup_{P' \in \mathcal{P}(t, \omega_n)} E^{P'} \xi^{t, \omega} \right) - \rho^{(\xi)}(\|\omega - \omega_n\|_{[0, t]}) \right) \\
&= \sup_{P' \in \mathcal{P}(t, \omega_n)} E^{P'} \xi^{t, \omega} \geq E^P \xi^{t, \omega}.
\end{aligned}$$

Because $P \in \mathcal{P}(t, \omega)$ is arbitrary, this proves that $\liminf_{n \rightarrow \infty} v_t(\omega_n) \geq v_t(\omega)$, which is the asserted lower semicontinuity.

Next prove that any bounded lower semicontinuous function f on ${}^t\Omega$ is in $\widehat{L}^2({}^t\Omega)$: Because ${}^t\Omega$ is Polish, there exists a uniformly bounded sequence $f_n \in C_b({}^t\Omega)$ such that $f_n \uparrow f$ for all $\omega \in \Omega$. By Gihman and Skorohod [25, Theorem 3.10], \mathcal{P} is relatively compact in the weak convergence topology. Therefore, by Tietze's Extension Theorem (Mandelkern [42]), $C_b({}^t\Omega) \subset \widehat{L}^2({}^t\Omega)$, and by Denis et al. [9, Theorem 12], $\sup_{P \in \mathcal{P}} E^P(|f - f_n|^2) \rightarrow 0$. Thus $f \in \widehat{L}^2({}^t\Omega)$.

Combine these two results to deduce that $\hat{E}[\xi | \mathcal{F}_t] \in \widehat{L}^2({}^t\Omega)$ if $\xi \in UC_b(\Omega)$. From (i), $\{\hat{E}[\xi | \mathcal{F}_t] : \xi \in \widehat{L}^2(\Omega)\}$ is contained in the $\|\cdot\|$ -closure of $\{\hat{E}[\xi | \mathcal{F}_t] : \xi \in UC_b(\Omega)\}$. But $\{\hat{E}[\xi | \mathcal{F}_t] : \xi \in UC_b(\Omega)\}$ is contained in $\widehat{L}^2({}^t\Omega)$, which is complete under $\|\cdot\|$. This completes the proof. \blacksquare

Proof of (B.2): Fix $P \in \mathcal{P}$ and $X \in \widehat{L}^2(\Omega)$. Given $\epsilon > 0$, there exists $\bar{X} \in UC_b(\Omega)$ such that

$$\|\hat{E}[X | \mathcal{F}_t] - \hat{E}[\bar{X} | \mathcal{F}_t]\| \leq \|X - \bar{X}\| \leq \epsilon.$$

For any $P' \in \mathcal{P}(t, P)$,

$$\begin{aligned}
& E^{P'}[X | \mathcal{F}_t] - \hat{E}[X | \mathcal{F}_t] \\
&= E^{P'}[X - \bar{X} | \mathcal{F}_t] + (E^{P'}[\bar{X} | \mathcal{F}_t] - \hat{E}[\bar{X} | \mathcal{F}_t]) + (\hat{E}[\bar{X} | \mathcal{F}_t] - \hat{E}[X | \mathcal{F}_t]).
\end{aligned} \tag{B.23}$$

From Karatzas and Shreve [35, Theorem A.3], derive that there exists a sequence $P_n \in \mathcal{P}(t, P)$ such that

$$\text{ess sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\bar{X} | \mathcal{F}_t] = \lim_{n \rightarrow \infty} E^{P_n}[\bar{X} | \mathcal{F}_t] \quad P\text{-a.e.} \tag{B.24}$$

where P -a.e. the sequence on the right is increasing in n . Then by Lemma B.7,

$$\hat{E}[\bar{X} \mid \mathcal{F}_t] = \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[\bar{X} \mid \mathcal{F}_t] = \lim_{n \rightarrow \infty} E^{P_n}[\bar{X} \mid \mathcal{F}_t] \quad P\text{-a.e.} \quad (\text{B.25})$$

Denote $L^2(\Omega, \mathcal{F}_T, P)$ by $L^2(P)$. Compute $L^2(P)$ -norms on both sides of (B.23) to obtain, for every n ,

$$\begin{aligned} & \| E^{P_n}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t] \|_{L^2(P)} \\ & \leq \| X - \bar{X} \|_{L^2(P_n)} + \| E^{P_n}[\bar{X} \mid \mathcal{F}_t] - \hat{E}[\bar{X} \mid \mathcal{F}_t] \|_{L^2(P)} + \| \hat{E}[\bar{X} \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t] \|_{L^2(P)} \\ & \leq \| E^{P_n}[\bar{X} \mid \mathcal{F}_t] - \hat{E}[\bar{X} \mid \mathcal{F}_t] \|_{L^2(P)} + 2\epsilon \end{aligned}$$

By (B.25),

$$\limsup_{n \rightarrow \infty} \| E^{P_n}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t] \| \leq 2\epsilon.$$

Note that ϵ is arbitrary. Therefore, there exists a sequence $\hat{P}_n \in \mathcal{P}(t, P)$ such that $E^{\hat{P}_n}[X \mid \mathcal{F}_t] \rightarrow \hat{E}[X \mid \mathcal{F}_t]$, P -a.e., which implies that

$$\hat{E}[X \mid \mathcal{F}_t] \leq \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[X \mid \mathcal{F}_t]. \quad (\text{B.26})$$

As in (B.24), there exists a sequence $P'_n \in \mathcal{P}(t, P)$ such that

$$\operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[X \mid \mathcal{F}_t] = \lim_{n \rightarrow \infty} E^{P'_n}[X \mid \mathcal{F}_t] \quad P\text{-a.e.}$$

with the sequence on the right being increasing in n (P -a.e.). Set $A_n \triangleq \{E^{P'_n}[X \mid \mathcal{F}_t] \geq \hat{E}[X \mid \mathcal{F}_t]\}$. By (B.26), P -a.e.

$$0 \leq (E^{P'_n}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t])1_{A_n} \xrightarrow{n} \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t].$$

By (B.23) and (B.25), P -a.e.

$$E^{P'_n}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t] \leq E^{P'_n}[X - \bar{X} \mid \mathcal{F}_t] + (\hat{E}[\bar{X} \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t])$$

Take $L^2(P)$ -norms to derive

$$\begin{aligned} & \| \operatorname{ess\,sup}_{P' \in \mathcal{P}(t, P)} E^{P'}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t] \|_{L^2(P)} \\ & = \lim_{n \rightarrow \infty} \| (E^{P'_n}[X \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t])1_{A_n} \|_{L^2(P)} \\ & \leq \limsup_{n \rightarrow \infty} \| X - \bar{X} \|_{L^2(P'_n)} + \| \hat{E}[\bar{X} \mid \mathcal{F}_t] - \hat{E}[X \mid \mathcal{F}_t] \|_{L^2(P)} \\ & \leq 2\epsilon. \end{aligned}$$

This proves (B.2).

Proof of (3.10): It is sufficient to prove that, for $0 \leq s \leq t \leq T$, P -a.e.

$$\begin{aligned} & \operatorname{ess\,sup}_{P' \in \mathcal{P}(s, P)} E^{P'}[X \mid \mathcal{F}_s] \\ &= \operatorname{ess\,sup}_{P' \in \mathcal{P}(s, P)} E^{P'} \left[\operatorname{ess\,sup}_{P'' \in \mathcal{P}(t, P')} E^{P''}[X \mid \mathcal{F}_t] \mid \mathcal{F}_s \right] \end{aligned} \quad (\text{B.27})$$

The classical law of iterated expectations implies the inequality \leq in (B.27). Next prove the reverse inequality.

As in (B.24), there exists a sequence $P''_n \in \mathcal{P}(t, P')$ such that P' -a.e.

$$\lim_{n \rightarrow \infty} E^{P''_n}[X \mid \mathcal{F}_t] \uparrow \operatorname{ess\,sup}_{P'' \in \mathcal{P}(t, P')} E^{P''}[X \mid \mathcal{F}_t]$$

Then

$$\begin{aligned} & E^{P'} \left[\operatorname{ess\,sup}_{P'' \in \mathcal{P}(t, P')} E^{P''}[X \mid \mathcal{F}_t] \mid \mathcal{F}_s \right] \\ &= \lim_{n \rightarrow \infty} E^{P''_n}[X \mid \mathcal{F}_s] \\ &\leq \operatorname{ess\,sup}_{\tilde{P} \in \mathcal{P}(s, P)} E^{\tilde{P}}[X \mid \mathcal{F}_s] \quad P\text{-a.e.} \end{aligned}$$

This proves (3.10).

Proof properties (i)-(iv) is standard and is omitted. This completes the proof of Theorem 3.6. ■

C. Appendix: Proofs for Utility

Proof of Theorem 3.8: Part (a). Consider the following backward stochastic differential equation under \hat{E} :

$$V_t = \hat{E} \left[\xi + \int_t^T f(c_s, V_s) ds \mid \mathcal{F}_t \right], \quad t \in [0, T],$$

where $\xi \in \widehat{L^2}(\Omega_T)$ is terminal utility and $c \in D$. (Equation (3.11) is the special case where $\xi = 0$.) Given $V \in M^2(0, T)$, let

$$\Lambda_t(V) \equiv \hat{E} \left[\xi + \int_t^T f(c_s, V_s) ds \mid \mathcal{F}_t \right], \quad t \in [0, T].$$

We need the following regularity property of Λ .

Lemma C.1. Λ is a mapping from $M^2(0, T)$ to $M^2(0, T)$.

Proof. By assumption (ii) there exists a positive constant K such that

$$|f(c_s, V_s) - f(c_s, 0)| \leq K |V_s|, \quad s \in [0, T].$$

Because both V and $(f(c_s, 0))_{0 \leq s \leq T}$ are in $M^2(0, T)$, we have $(f(c_s, V_s))_{0 \leq s \leq T} \in M^2(0, T)$. Thus

$$\begin{aligned} & \hat{E}[|\xi + \int_t^T f(c_s, V_s) ds|^2] \\ & \leq 2\hat{E}[|\xi|^2] + (T-t) \int_t^T |f(c_s, V_s)|^2 ds \\ & \leq 2\hat{E}[|\xi|^2] + 2(T-t)\hat{E}[\int_t^T |f(c_s, V_s)|^2 ds] < \infty, \end{aligned}$$

which means that $(\xi + \int_t^T f(c_s, V_s) ds) \in \widehat{L}^2(\Omega)$. Argue further that

$$\begin{aligned} \hat{E}[|\Lambda_t(V)|^2] &= \hat{E}[(\hat{E}[\xi + \int_t^T f(c_s, V_s) ds | \mathcal{F}_t])^2] \\ &\leq \hat{E}[\hat{E}[|\xi + \int_t^T f(c_s, V_s) ds|^2 | \mathcal{F}_t]] \\ &= \hat{E}[|\xi + \int_t^T f(c_s, V_s) ds|^2] < \infty. \end{aligned}$$

Finally, $(\hat{E} \int_0^T |\Lambda_t|^2 dt)^{\frac{1}{2}} \leq (\int_0^T \hat{E}[|\Lambda_t|^2] dt)^{\frac{1}{2}} < \infty$ and $\Lambda \in M^2(0, T)$. ■

For any V and $V' \in M^2(0, T)$, by Theorem 3.6 the following approximation holds:

$$\begin{aligned} & |\Lambda_t(V) - \Lambda_t(V')|^2 \\ &= (\hat{E}[\int_t^T (f(c_s, V_s) - f(c_s, V'_s)) ds | \mathcal{F}_t])^2 \\ &\leq \hat{E}[(\int_t^T |f(c_s, V_s) - f(c_s, V'_s)| ds)^2 | \mathcal{F}_t] \\ &\leq (T-t)K^2 \hat{E}[\int_t^T |V_s - V'_s|^2 ds | \mathcal{F}_t] \\ &\leq L \hat{E}[\int_t^T |V_s - V'_s|^2 ds | \mathcal{F}_t], \end{aligned} \tag{C.1}$$

where K is the Lipschitz constant for the aggregator and $L = TK^2$. (The first inequality is due to the classical Jensen's inequality and (B.2).) Then for each

$r \in [0, T]$,

$$\begin{aligned}
& \hat{E}\left[\int_r^T |\Lambda_t(V) - \Lambda_t(V')|^2 dt\right] \\
& \leq L \hat{E}\left[\int_r^T \hat{E}\left[\int_t^T |V_s - V'_s|^2 ds \mid \mathcal{F}_t\right] dt\right] \\
& \leq L \int_r^T \hat{E}\left[\int_t^T |V_s - V'_s|^2 ds\right] dt \\
& \leq L(T-r) \hat{E}\left[\int_r^T |V_s - V'_s|^2 ds\right].
\end{aligned}$$

Set $\delta = \frac{1}{2L}$ and $r_1 = \max\{T - \delta, 0\}$. Then,

$$\hat{E}\left[\int_{r_1}^T |\Lambda_t(V) - \Lambda_t(V')|^2 dt\right] \leq \frac{1}{2} \hat{E}\left[\int_{r_1}^T |V_t - V'_t|^2 dt\right],$$

which implies that Λ is a contraction mapping from $M^2(r_1, T)$ to $M^2(r_1, T)$ and there exists a unique solution $(V_t) \in M^2(r_1, T)$ to the above BSDE. Because δ is independent of t , we can work backwards in time and apply a similar argument at each step to prove that there exists a unique solution $(V_t) \in M^2(0, T)$.

Part (b). Uniqueness of the solution is due to the contraction mapping property established in the proof of (a).

By Theorem 3.6,

$$\begin{aligned}
& -\hat{E}\left[-\int_t^\tau f(c_s, V_s) ds - V_\tau \mid \mathcal{F}_t\right] \\
& = -\hat{E}\left[-\int_t^\tau f(c_s, V_s) ds + \hat{E}\left[-\int_\tau^T f(c_s, V_s) ds \mid \mathcal{F}_\tau\right] \mid \mathcal{F}_t\right] \\
& = -\hat{E}\left[\hat{E}\left[-\int_t^T f(c_s, V_s) ds \mid \mathcal{F}_\tau\right] \mid \mathcal{F}_t\right] \\
& = -\hat{E}\left[-\int_t^T f(c_s, V_s) ds \mid \mathcal{F}_t\right] = V_t. \quad \blacksquare
\end{aligned}$$

D. Appendix: Proofs for Asset Returns

Proof of Theorem 4.1: We prove that (e, π^e) is a dynamic Arrow Debreu equilibrium.

Utility for the given consumption processes c and e are as in (4.1). Define δ_t implicitly by

$$f(c_t, V_t(c)) = f_c(e_t, V_t(e))(c_t - e_t) + f_v(e_t, V_t(e))(V_t(c) - V_t(e)) - \delta_t + f(e_t, V_t(e)),$$

for $0 \leq t < T$, and

$$u(c_T) = u_c(e_T)(c_T - e_T) - \delta_T + u(e_T).$$

Because f and u are concave, we have $\delta_t \geq 0$ on $[0, T]$.

Define, for $0 \leq t < T$,

$$\begin{aligned}\beta_t &= f_v(e_t, V_t(e)) \\ \gamma_t &= f_c(e_t, V_t(e))(c_t - e_t) + f(e_t, V_t(e)) - \beta_t V_t(e) - \delta_t \\ \gamma_T &= -u_c(e_T)e_T + u(e_T) - \delta_T \\ \zeta_t &= f(e_t, V_t(e)) - \beta_t V_t(e).\end{aligned}$$

Then

$$V_t(c) = -\hat{E}[-(u_c(e_T)c_T + \gamma_T) - \int_t^T (\beta_s V_s(c) + \gamma_s) ds \mid \mathcal{F}_t].$$

Because this is a linear backward stochastic differential equation, its solution has the form (by Hu and Ji [30])

$$V_t(c) = -\hat{E}[-(u_c(e_T)c_T + \gamma_T) \exp\{\int_t^T \beta_s ds\} - \int_t^T \gamma_s \exp\{\int_t^s \beta_{s'} ds'\} ds \mid \mathcal{F}_t].$$

Similarly for e , we have

$$V_t(e) = -\hat{E}[-u(e_T) - \int_t^T (\beta_s V_s(e) + \zeta_s) ds \mid \mathcal{F}_t],$$

and (by Hu and Ji [30]),

$$V_t(e) = -\hat{E}[-u(e_T) \exp\{\int_t^T \beta_s ds\} - \int_t^T \zeta_s \exp\{\int_t^s \beta_{s'} ds'\} ds \mid \mathcal{F}_t].$$

Apply the subadditivity of $\hat{E}[\cdot \mid \mathcal{F}_\tau]$ and the nonnegativity of δ_t to obtain

$$\begin{aligned}\exp\{\int_0^\tau \beta_s ds\} (V_\tau(c) - V_\tau(e)) &= \\ -\hat{E}[-(u_c(e_T)c_T + \gamma_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \gamma_t \exp\{\int_0^t \beta_s ds\} dt \mid \mathcal{F}_\tau] \\ -\{-\hat{E}[-u(e_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \zeta_t \exp\{\int_0^t \beta_s ds\} dt \mid \mathcal{F}_\tau]\} &= \end{aligned}$$

$$\begin{aligned}
& \hat{E}[-u(e_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \zeta_t \exp\{\int_0^t \beta_s ds\} dt \mid \mathcal{F}_\tau] \\
& - \hat{E}[-(u_c(e_T)c_T + \gamma_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \gamma_t \exp\{\int_0^t \beta_s ds\} dt \mid \mathcal{F}_\tau] \leq \\
& \hat{E}[-u(e_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \zeta_t \exp\{\int_0^t \beta_s ds\} dt \\
& - (-(u_c(e_T)c_T + \gamma_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \gamma_t \exp\{\int_0^t \beta_s ds\} dt \mid \mathcal{F}_\tau] = \\
& \hat{E}[(u_c(e_T)c_T + \gamma_T) \exp\{\int_0^T \beta_s ds\} + \int_\tau^T \gamma_t \exp\{\int_0^t \beta_s ds\} dt \\
& - u(e_T) \exp\{\int_0^T \beta_s ds\} - \int_\tau^T \zeta_t \exp\{\int_0^t \beta_s ds\} dt \mid \mathcal{F}_\tau] = \\
& \hat{E}[\exp\{\int_0^T \beta_s ds\} u_c(e_T)(c_T - e_T) + \int_\tau^T \exp\{\int_0^t \beta_s ds\} f_c(e_t, V_t(e))(c_t - e_t) dt \\
& - \exp\{\int_0^T \beta_s ds\} \delta_T - \int_\tau^T \exp\{\int_0^t \beta_s ds\} \delta_t dt \mid \mathcal{F}_\tau] \leq \\
& \hat{E}[\exp\{\int_0^T \beta_s ds\} u_c(e_T)(c_T - e_T) + \int_\tau^T \exp\{\int_0^t \beta_s ds\} f_c(e_t, V_t(e))(c_t - e_t) dt \mid \mathcal{F}_\tau] \\
& = \hat{E} \left[\pi_T^e (c_T - e_T) + \int_\tau^T \pi_t^e (c_t - e_t) dt \mid \mathcal{F}_\tau \right].
\end{aligned}$$

This completes the proof. ■

Lemma D.1. *Adopt the assumptions in Theorem 4.2. Consider the following BSDE driven by G -Brownian motion:*

$$\begin{aligned}
d\tilde{Y}_t &= (r_t \tilde{Y}_t + \eta_t^\top \tilde{\phi}_t - \delta_t) dt - dK_t + \tilde{\phi}_t^\top dB_t, \\
\tilde{Y}_T &= \delta_T.
\end{aligned}$$

Denote by $I(0, T)$ the space of all continuous nondecreasing processes $(K_t)_{0 \leq t \leq T}$ with $K_0 = 0$ and $K_T \in \widehat{L^2}(\Omega)$. Then there exists a unique triple

$$(\tilde{Y}_t, \tilde{\phi}_t, K_t) \in M^2(0, T) \times M^2(0, T) \times I(0, T),$$

satisfying the BSDE such that $K_0 = 0$ and where $-K_t$ is a G -martingale.

Proof. Apply Ito's formula to $\pi_t \tilde{Y}_t$ to derive

$$\begin{aligned}
& d(\pi_t \tilde{Y}_t) \\
&= \pi_t d\tilde{Y}_t + \tilde{Y}_t d\pi_t - \langle \pi_t \tilde{\phi}_t^\top, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle \\
&= (\pi_t \tilde{\phi}_t - \pi_t \tilde{Y}_t \eta_t^\top v_t^{-1}) dB_t - \pi_t \delta_t dt - \pi_t dK_t + [\pi_t \tilde{\phi}_t^\top \eta_t dt - \langle \pi_t \tilde{\phi}_t^\top, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle] \\
&= (\pi_t \tilde{\phi}_t - \pi_t \tilde{Y}_t \eta_t^\top v_t^{-1}) dB_t - \pi_t \delta_t dt - \pi_t dK_t.
\end{aligned}$$

Integrate on both sides to obtain

$$\pi_T \delta_T + \int_\tau^T \pi_t \delta_t dt = \pi_\tau \tilde{Y}_\tau - \int_\tau^T \pi_t dK_t + \int_\tau^T (\pi_t \tilde{\phi}_t^\top - \pi_t \tilde{Y}_t \eta_t^\top v_t^{-1}) dB_t. \quad (\text{D.1})$$

Let

$$M_\tau = \hat{E}[\pi_T \delta_T + \int_0^T \pi_t \delta_t dt \mid \mathcal{F}_\tau].$$

Then (M_τ) is a G -martingale. By the martingale representation theorem (see Appendix E), there exists a unique pair $(Z_t, \bar{K}_t) \in M^2(0, T) \times I(0, T)$ such that

$$M_\tau = \hat{E}[\pi_T \delta_T + \int_0^T \pi_t \delta_t dt] + \int_0^\tau Z_t dB_t - \bar{K}_\tau,$$

and such that $-\bar{K}_t$ is a G -martingale. This can be rewritten as

$$\begin{aligned}
M_\tau &= M_T - \int_\tau^T Z_t dB_t + \bar{K}_T - \bar{K}_\tau \\
&= \pi_T \delta_T + \int_0^T \pi_t \delta_t dt - \int_\tau^T Z_t dB_t + \bar{K}_T - \bar{K}_\tau.
\end{aligned}$$

Thus $(\tilde{Y}_t, \tilde{\phi}_t, K_t)$ is the desired solution where

$$\begin{aligned}
\tilde{Y}_\tau &= \frac{M_\tau}{\pi_\tau} - \int_0^\tau \frac{\pi_t}{\pi_\tau} \delta_t dt, \\
\tilde{\phi}_\tau^\top &= \frac{Z_\tau}{\pi_\tau} + \tilde{Y}_\tau \eta_\tau^\top v_\tau^{-1}, \text{ and } K_\tau = \int_0^\tau \frac{1}{\pi_t} d\bar{K}_t. \blacksquare
\end{aligned}$$

Proof of Theorem 4.2: We prove only the claim re superhedging. Proof of the other claim is similar.

Step 1: Prove that for any $y \in \mathcal{U}_\tau$,

$$y \geq \hat{E}\left[\int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt + \frac{\pi_T}{\pi_\tau} \delta_T \mid \mathcal{F}_\tau\right] \text{ q.s.}$$

If $y \in \mathcal{U}_\tau$, there exists ϕ such that $Y_T^{y,\phi,\tau} \geq \delta_T$. Apply (the G-Brownian version of) Itô's formula to $\pi_t Y_T^{y,\phi,\tau}$ to derive

$$\begin{aligned} & d(\pi_t Y_T^{y,\phi,\tau}) \\ &= \pi_t dY_t^{y,\phi,\tau} + Y_t^{y,\phi,\tau} d\pi_t - \langle \pi_t \phi_t^\top, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle \\ &= (\pi_t \phi_t - \pi_t Y_t^{y,\phi,\tau} \eta_t^\top v_t^{-1}) dB_t - \pi_t \delta_t dt + [\pi_t \phi_t^\top \eta_t dt - \langle \pi_t \phi_t^\top, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle]. \end{aligned}$$

Integration on both sides yields

$$\pi_T Y_T^{y,\phi,\tau} + \int_\tau^T \pi_t \delta_t dt = \pi_\tau y + \int_\tau^T (\pi_t \phi_t^\top - \pi_t Y_t^{y,\phi,\tau} \eta_t^\top v_t^{-1}) dB_t,$$

and taking conditional expectations yields

$$\begin{aligned} y &= \hat{E}\left[\frac{\pi_T}{\pi_\tau} Y_T^{y,\phi,\tau} + \int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt \mid \mathcal{F}_\tau\right] \\ &\geq \hat{E}\left[\frac{\pi_T}{\pi_\tau} \delta_T + \int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt \mid \mathcal{F}_\tau\right]. \end{aligned}$$

Step 2: There exists $\hat{y} \in \mathcal{U}_\tau$ and $\hat{\phi}$ such that $Y_T^{\hat{y},\hat{\phi},\tau} \geq \delta_T$ and

$$\hat{y} = \hat{E}\left[\int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt + \frac{\pi_T}{\pi_\tau} \delta_T \mid \mathcal{F}_\tau\right] \text{ q.s.}$$

Apply the preceding lemma. Rewrite equation (D.1) as

$$\pi_T \delta_T + \int_\tau^T \pi_t \delta_t dt = \pi_\tau \tilde{Y}_\tau + \int_\tau^T (\pi_t \tilde{\phi}_t^\top - \pi_t \tilde{Y}_t \eta_t^\top v_t^{-1}) dB_t - \int_\tau^T \pi_t dK_t. \quad (\text{D.2})$$

Because π_t is positive and $-K_t$ is a G -martingale, $\hat{E}[-\int_\tau^T \pi_t dK_t \mid \mathcal{F}_\tau] = 0$. Thus,

$$\hat{E}\left[\pi_T \delta_T + \int_\tau^T \pi_t \delta_t dt \mid \mathcal{F}_\tau\right] = \pi_\tau \tilde{Y}_\tau \text{ q.s.}$$

Finally, define $\hat{y} = \tilde{Y}_\tau$ and $\hat{\phi} = \tilde{\phi}$. Then $\hat{y} \in \mathcal{U}_\tau$ and

$$\hat{y} = \hat{E}\left[\int_\tau^T \frac{\pi_t}{\pi_\tau} \delta_t dt + \frac{\pi_T}{\pi_\tau} \delta_T \mid \mathcal{F}_\tau\right] \quad q.s.$$

This completes the proof of Theorem 4.2. ■

Proof of Theorem 4.3: Apply Itô's formula for G -Brownian motion to derive³⁸

$$\begin{aligned} & d(\pi_t Y_t) \\ &= \pi_t dY_t + Y_t d\pi_t - \langle \pi_t \phi_t^\top, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle \\ &= (\pi_t \phi_t - \pi_t Y_t \eta_t^\top v_t^{-1}) dB_t - \pi_t (c_t - e_t) dt + [\pi_t \phi_t^\top \eta_t dt - \langle \pi_t \phi_t^\top, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle]. \end{aligned} \tag{D.3}$$

Note that for any $a = (a_t) \in M^2(0, T)$,

$$\int_\tau^T a_t v_t^{-1} d\langle B \rangle_t = \int_\tau^T a_t dt, \quad q.s.$$

and therefore

$$\int_\tau^T \langle \pi_t \phi_t, \eta_t^\top v_t^{-1} d\langle B \rangle_t \rangle = \int_\tau^T \pi_t \phi_t^\top \eta_t dt.$$

Accordingly, integration on both sides of (D.3) yields,

$$\pi_T Y_T + \int_\tau^T \pi_t (c_t - e_t) dt = \pi_\tau Y_\tau + \int_\tau^T (\pi_t \phi_t^\top - \pi_t Y_t \eta_t^\top v_t^{-1}) dB_t.$$

Take conditional expectations to obtain

$$\hat{E}[\pi_T Y_T + \int_\tau^T \pi_t (c_t - e_t) dt \mid \mathcal{F}_\tau] = \pi_\tau Y_\tau + \hat{E}[\int_\tau^T (\pi_t \phi_t^\top - \pi_t Y_t \eta_t^\top v_t^{-1}) dB_t \mid \mathcal{F}_\tau].$$

Because B being G -Brownian motion implies that B is a martingale under every prior in \mathcal{P} , we have

$$0 = \hat{E}[\int_\tau^T (\pi_t \phi_t^\top - \pi_t Y_t \eta_t^\top v_t^{-1}) dB_t \mid \mathcal{F}_\tau] = \hat{E}[-\int_\tau^T (\pi_t \phi_t^\top - \pi_t Y_t \eta_t^\top v_t^{-1}) dB_t \mid \mathcal{F}_\tau],$$

which gives the desired result. ■

The next lemma completes the proof of Theorem 4.5.

³⁸For any d -dimensional (column) vectors x and y , we use $\langle x^\top, y^\top \rangle$ occasionally as alternative notation for the inner product $x^\top y$.

Lemma D.2. *Under the assumptions of Theorem 4.5, for every τ quasisurely*

$$V_\tau(c) \leq E^{(P^*)^\omega_\tau} \left[\int_\tau^T u(c_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(c_T) \right], \quad (\text{D.4})$$

and

$$\begin{aligned} & E^{(P^*)^\omega_\tau} \left[\int_\tau^T \pi_t(c_t - e_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} \pi_T(c_T - e_T) \right] \\ & \leq \widehat{E} \left[\int_\tau^T \pi_t(c_t - e_t) e^{-\beta(t-\tau)} ds + e^{-\beta(T-\tau)} \pi_T(c_T - e_T) \mid \mathcal{F}_\tau \right]. \end{aligned}$$

Proof. We prove the first inequality. The second is proven similarly.

We claim that for any $P \in \mathcal{P}$ and τ , there exists $\bar{P} \in \mathcal{P}$ such that

$$\bar{P} = P \text{ on } \mathcal{F}_\tau \text{ and } \bar{P}_t^\omega = (P^*)^\omega_\tau \text{ for all } (t, \omega) \in [\tau, T] \times \Omega. \quad (\text{D.5})$$

This follows from the contruction of priors in \mathcal{P} via the SDE (3.1). Let P^* and P be induced by θ^* and θ respectively and define $\bar{\theta} \in \Theta$ by

$$\bar{\theta}_t = \begin{cases} \theta_t & 0 \leq t \leq \tau \\ \theta_t^* & \tau < t \leq T. \end{cases}$$

Then $\bar{P} = P^{\bar{\theta}}$ satisfies (D.5). In particular, $\bar{P} \in \mathcal{P}(t, P)$, the set defined in (B.1).

By (B.2), for every $P \in \mathcal{P}$,

$$V_\tau(c) = \operatorname{ess\,inf}_{P' \in \mathcal{P}(\tau, P)} E_{P'} \left[\int_\tau^T u(c_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(c_T) \mid \mathcal{F}_\tau \right], \quad P\text{-a.e.}$$

Further, we have $P\text{-a.e.}$

$$\begin{aligned} V_\tau(c) & \leq E^{\bar{P}} \left[\int_\tau^T u(c_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(c_T) \mid \mathcal{F}_\tau \right] \\ & = E^{\bar{P}^\omega_\tau} \left[\int_\tau^T u(c_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(c_T) \right] \\ & = E^{(P^*)^\omega_\tau} \left[\int_\tau^T u(c_t) e^{-\beta(t-\tau)} dt + e^{-\beta(T-\tau)} u(c_T) \right] \end{aligned}$$

where the inequality is due to $\bar{P} \in \mathcal{P}(t, P)$ and the first equality follows from $\bar{P} = P$ on \mathcal{F}_τ and properties of regular conditionals (see Yong and Zhou [62, Propns. 1.9, 1.10]). Moreover, the preceding is true for any $P \in \mathcal{P}$. \blacksquare

Proof of Lemma 4.6: Assume (4.16) and derive (4.20).

Let $P^* = P^{(\sigma_t^*)}$, the measure induced by σ_t^* via the SDE (3.1). Fix $t \in [\tau, T]$ and let

$$N = \{\omega : b_t - r_t \mathbf{1} = - \left(\frac{e_t u_{cc}(e_t)}{u_c(e_t)} \right) s_t (\sigma_t^* \sigma_t^{*\top}) s_t^e, \quad (P^*)_{\tau}^{\omega} \text{-a.e.}\}.$$

We need to prove that $P(N) = 1$ for every $P \in \mathcal{P}$.

For any given $P \in \mathcal{P}$, $P = P^{\theta}$, we construct a new $\bar{P} \in \mathcal{P}$ with parameter $\bar{\theta}$ given by

$$\bar{\theta}_s = \begin{cases} \theta_s, & 0 \leq s \leq \tau \\ \theta_s^*, & \tau \leq s \leq T. \end{cases}$$

As in the proof of Lemma D.2, we have that

$$P = \bar{P} \text{ on } \mathcal{F}_{\tau} \text{ and } \bar{P}_{\tau}^{\omega} = (P^*)_{\tau}^{\omega}.$$

Let $Z_t = b_t - r_t \mathbf{1}$ and $Z'_t = - \left(\frac{e_t u_{cc}(e_t)}{u_c(e_t)} \right) s_t (\sigma_t^* \sigma_t^{*\top}) s_t^e$. Then

$$\begin{aligned} 0 &= E^{\bar{P}}[|Z_t - Z'_t|^2] = \\ &= E^{\bar{P}}[E^{\bar{P}}[|Z_t - Z'_t|^2 | \mathcal{F}_{\tau}]] = E^{\bar{P}}[E^{\bar{P}_{\tau}^{\omega}}[|Z_t - Z'_t|^2]] \\ &= E^{\bar{P}}[E^{(P^*)_{\tau}^{\omega}}[|Z_t - Z'_t|^2]] = E^P[E^{(P^*)_{\tau}^{\omega}}[|Z_t - Z'_t|^2]], \end{aligned}$$

where the first equality is by (4.16). Thus

$$E^P[E^{(P^*)_{\tau}^{\omega}}[|Z_t - Z'_t|^2]] = 0,$$

which implies (4.20). ■

E. Appendix: G-Brownian Motion

Peng [49] introduced G -Brownian motion using PDE's (specifically, a nonlinear heat equation). A characterization of G -Brownian motion in terms of quasisure analysis is given in Denis et al. [9] and Soner et al. [55]. For the convenience of the reader, in this appendix we outline some key elements of the theory of G -Brownian motion in terms of the specifics of our model.

By G -Brownian motion we shall mean the specification of the set of priors \mathcal{P} introduced in Example 3.2, where $\Theta_t = \{\{0\} \times \Gamma\}$ and Γ is a compact convex subset of $\mathbb{R}^{d \times d}$. We also require that, for all $\sigma \in \Gamma$, $\sigma \sigma^{\top} \geq \hat{a}$ for some positive definite

matrix \widehat{a} , a condition not required in Peng's general theory. The spaces $\widehat{L}^2(\Omega)$ and $M^2(0, T)$ are defined as in the text relative to \mathcal{P} , as are the expectation operator $\widehat{E}[\cdot]$ and its corresponding conditionals. Our expectation operator $\widehat{E}[\cdot]$ is called G -expectation by Peng, and his notion of conditional expectation corresponds to ours in this case. The d -dimensional coordinate process B is a G -Brownian motion under \mathcal{P} .

Itô Integral and Quadratic Variation Process: For each $\eta \in M^2(0, T)$, we can consider the usual Itô integral $\int_0^T \eta_t^\top dB_t$, which lies in $\widehat{L}^2(\Omega)$. Each $P \in \mathcal{P}$ provides a different perspective on the integral; a comprehensive view requires that one consider all priors. The quadratic variation process $\langle B \rangle$ also agrees with the usual quadratic variation process quasiregularly. In Section 4.2 we defined a universal process v (via (4.5)) and proved that

$$\langle B \rangle = \left(\int_0^t v_s ds : 0 \leq t \leq T \right) \text{ q.s.}$$

The following properties are satisfied for any $\lambda, \eta \in M^2(0, T)$, $X \in \widehat{L}^2(\Omega_T)$ and constant α :

$$\begin{aligned} \widehat{E}[B_t] &= 0, \quad \widehat{E}[\int_0^T \eta_t^\top dB_t] = 0, \\ \widehat{E}[(\int_0^T \eta_t^\top dB_t)^2] &= \widehat{E}[\int_0^T \eta_t^\top v_t \eta_t dt], \\ \int_0^T (\alpha \eta_t^\top + \lambda_t^\top) dB_t &= \alpha \int_0^T \eta_t^\top dB_t + \int_0^T \lambda_t^\top dB_t \text{ q.s.} \\ \widehat{E}[X + \int_s^T \eta_t^\top dB_t \mid \mathcal{F}_s] &= \widehat{E}[X \mid \mathcal{F}_s] + \widehat{E}[\int_s^T \eta_t^\top dB_t \mid \mathcal{F}_s] = \widehat{E}[X \mid \mathcal{F}_s] \text{ q.s.} \end{aligned}$$

For the one dimensional case ($\Gamma = [\underline{\sigma}, \bar{\sigma}]$, $\underline{\sigma} > 0$), we have

$$\begin{aligned} \underline{\sigma}^2 t &\leq \widehat{E}[(B_t)^2] \leq \bar{\sigma}^2 t, \\ \underline{\sigma}^2 \widehat{E}[\int_0^T \eta_t^2 dt] &\leq \widehat{E}[(\int_0^T \eta_t dB_t)^2] \leq \bar{\sigma}^2 \widehat{E}[\int_0^T \eta_t^2 dt]. \end{aligned}$$

Itô's Formula: Consider³⁹

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \gamma_s dB_s$$

where α and γ are in $M^2(0, T; \mathbb{R}^d)$ and $M^2(0, T; \mathbb{R}^{d \times d})$ respectively. (Define $M^2(0, T; \mathbb{R}^{\ell \times k})$ similarly to $M^2(0, T)$ for $\mathbb{R}^{\ell \times k}$ -valued processes.) We adapt Itô's

³⁹All equations should be understood to hold quasiregularly.

formula from Li and Peng [39, Theorem 5.4] or Soner et al. [56, Propn. 6.7] and rewrite it in our context. Let $0 \leq \tau \leq t \leq T$; define $v = (v^{ij})$ by (4.5). Then, for any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with continuous second order derivatives, we have

$$f(X_t) - f(X_\tau) = \int_\tau^t (f_x(X_s))^\top \gamma_s dB_s + \int_\tau^t (f_x(X_s))^\top \alpha_s ds + \frac{1}{2} \int_\tau^t \text{tr}[\gamma_s^\top f_{xx}(X_s) v_s \gamma_s] ds.$$

Consider the special case $f(x_1, x_2) = x_1 x_2$ and

$$X_t^i = X_0^i + \int_0^t \alpha_s^i ds + \int_0^t \gamma_s^i dB_s, \quad i = 1, 2,$$

where $\alpha^i \in M^2(0, T)$ and $\gamma^i \in M^2(0, T; \mathbb{R}^d)$, $i = 1, 2$. Then

$$X_t^1 X_t^2 - X_\tau^1 X_\tau^2 = \int_\tau^t X_s^1 dX_s^2 + \int_\tau^t X_s^2 dX_s^1 + \int_\tau^t \gamma_s^1 v_s (\gamma_s^2)^\top ds.$$

Formal rules: As in the classical Itô formula, if $dX_t = \alpha_t dt + \gamma_t dB_t$, then we can compute $(dX_t)^2 = (dX_t) \cdot (dX_t)$ by the following formal rules:⁴⁰

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = v_t dt.$$

Martingale Representation Theorem: An \mathcal{F} -progressively measurable $\widehat{L}^2(\Omega)$ -valued process M is called a G -martingale if and only if for any $0 \leq \tau < t$, $M_\tau = \hat{E}[M_t | \mathcal{F}_\tau]$. We adapt the martingale representation theorem from Song [59] and Soner et al. [55]. For any $\xi \in \widehat{L}^{2+\varepsilon}(\Omega)$ and $\varepsilon > 0$, if $M_t = \hat{E}[\xi | \mathcal{F}_t]$, $t \in [0, T]$, then we have the following unique decomposition:

$$M_t = M_0 + \int_0^t Z_s dB_s - K_t, \quad q.s.$$

where $Z \in M^2(0, T)$ and K is a continuous nondecreasing process with $K_0 = 0$, $K_T \in \widehat{L}^2(\Omega)$ and where $-K$ is a G -martingale.

⁴⁰Similar rules are given in Peng [50, 51].

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