

Optimal incentives for a system of interacting Agents

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work in progress

joint with Dylan POSSAMAI

- Large economic literature on **contract design** in order to revisit equilibrium theory by incorporating **incentives** and **asymmetry of information** ;

*Customers know more about their tastes than firms,
firms know more about their costs than the government
and all agents take actions that are at least partly unobservable.*

B. Salanié, The economics of contracts

- Optimal contracting between a **Principal** and an **Agent** ;
- Mainly static or discrete time problems : *Spear & Srivastava, Salanié, Tirole, Laffont, Martimort, Radner* \implies Limited computations ;
- Extension to **continuous time models** : *Holmstrom & Milgrom, Sannikov, Cvitanic,...* ;
- More explicit solutions and strong connexion with the **theory of BSDEs**. ;
- Optimal contracting between a **Principal** and a **system of Agents in interactions...**

The classical Principal Agent problem

The Principal wishes to design a contract that :

- the Agent will accept, i.e. which provides him his reservation utility ;
- maximizes the expected utility of the Principal.

Three main cases depending on **how much information** is available to both the Principal and the Agent

1 Risk sharing or first best

Information is **symmetric** and the Principal and the Agent agree on how they share the risk

⇒ Principal chooses both the contract and the actions of the Agent

2 Moral hazard or second best

The actions of the Agent are **unobservable**

⇒ **First** solve the **Agent** problem for any contract, and **then** solve the **Principal** problem

3 Adverse selection of third best

The **type/characteristics** of the Agent is **unknown** for the Principal

⇒ The Principal offers a **menu of contracts**, one for each type of Agent, forcing the Agent to **reveal** himself.

Revisiting the Holmstrom and Milgrom moral hazard problem

- Controlled **output process** (such as a production good, or cash flows ...)

$$dX_t^a = a_t dt + \sigma dB_t$$

- The control a is s.t. $\mathcal{E}(\int_0^T a_t / \sigma dB_t)$ is unif. integrable;

- Weak formulation** : choice of a **probability** \mathbb{P}^a with Brownian Motion B^a

$$dX_t = a_t dt + \sigma dB_t^a$$

- The Agent observes B^a whereas **the Principal only observes X** ;
- The Agent chooses a **control** $(a_t)_t$ and receives a payment ξ at time T . He solves

$$\sup_a \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(\xi - \int_0^T k(a_t) dt \right) \right]$$

- The **Principal** chooses and pays the terminal payment ξ and solves

$$\sup_{\xi} \mathbb{E}^{\mathbb{P}^a} [U_P(X_T - \xi)]$$

- U_A and U_P are **exponential utility functions** with risk aversions R_A and R_P .

Solving the Agent's problem

- Consider a **given** \mathcal{F}_T^X -measurable payment **contract** ξ
- The problem of the Agent rewrites as follows

$$U_0^A = \sup_a \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(\xi - \int_0^T k(a_s) ds \right) \right]$$

- The dynamic version at time t is

$$U_t^A(a) = \operatorname{ess\,sup}_{a' \text{ on } [t, T]} J_t^{a, a'} \quad \text{where} \quad J_t^{a, a'} = \mathbb{E}^{\mathbb{P}^{a'}} \left[U_A \left(\xi - \int_t^T k(a'_s) ds \right) \mid \mathcal{F}_t^B \right]$$

- $e^{R_A \int_0^t k(a_s) ds} J_t^{a, a'}$ is a $\mathbb{P}^{a'}$ -martingale
- Introduce the log target process $Y^{a, a'} := \frac{-\ln(J_t^{a, a'})}{R_A}$
- Ito's formula and Girsanov Theorem imply that $Y^{a, a'}$ solves the **BSDE**

$$Y_t^{a, a'} = \xi + \int_t^T f(a', Z_s^{a, a'}) ds - \int_t^T Z_s^{a, a'} \sigma dB_s$$

with the family of drivers $f : (a', z) \mapsto -\frac{R_A}{2} \sigma^2 z^2 + a' z - k(a')$

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- $Y^{a, a'} := \frac{-\ln(J^{a, a'})}{R_A}$ solves the BSDE

$$Y_t^{a, a'} = \xi + \int_t^T f(a', Z_s^{a, a'}) ds - \int_t^T Z_s^{a, a'} \sigma dB_s$$

with the family of drivers $f : (a', z) \mapsto -\frac{R_A}{2} \sigma^2 z^2 + a' z - k(a')$

- We want to maximize $J^{a, a'}$ or similarly $Y^{a, a'}$ over a'
- **Comparison for BSDEs** \implies take $f^* := \sup_{a'} f(a', \cdot)$
- Optimal control a^* which maximizes the driver : $a^* = \bar{k}(z)$
- The class of **admissible contracts** relies on **integrability conditions on ξ** for the BSDE to be well posed

- For a given terminal payment ξ , the **log expected utility** Y of the Agent is

$$Y_t = \xi + \int_t^T f^*(Z_s^*) ds - \int_t^T Z_s^* \sigma dB_s$$

- The **participation constraint** of the Agent requires $Y_0 \geq y_0$
 \implies Consider terminal payment ξ of the form

$$\xi = y_0 + \int_t^T f^*(Z_s^*) ds - \int_t^T Z_s^* \sigma dB_s$$

- The problem of the Principal is

$$\sup_{\xi} \mathbb{E}^{P^{a^*(z^*)}} [U_P(X_T - \xi)]$$

- Maximizing over Z^* , it rewrites

$$\sup_{Z^*} \mathbb{E}^{P^{a^*(z^*)}} \left[\mathcal{E} \left(-R_P \int_0^T \sigma (1 - Z_s^*) dB_s^{a^*(z^*)} \right) U_P \left(\int_0^T \beta(Z_s^*) ds \right) \right]$$

with $\beta : z \mapsto a^*(z) + f^*(z) - za^*(z) - \frac{R_P}{2} \sigma^2 (1 - z)^2$

- Optimal Z is deterministic and given by a **constant pointwise sup** z^* of β

- The optimal contract is **linear** in the output process X

$$\xi_T^* = y_0 - Tf^*(z^*) + z^*(X_T - Ta^*(z^*))$$

- It provides exactly his **utility of reservation** to the Agent
- Example : quadratic cost $k : a \mapsto \frac{ka^2}{2}$

Optimal proportion of X in the contract

$$z^* = \frac{R_P + \frac{1}{k\sigma^2}}{R_A + R_P + \frac{1}{k\sigma^2}}$$

Higher than in the first best case :

$$\frac{R_P}{R_A + R_P}$$

Sannikov : infinite horizon and intertemporal payments

- For the Agent : similar approach via BSDEs with infinite horizons
- For the Principal : mixed control and optimal stopping problem
She looks for the optimal retirement stopping time, when she offers to the Agent its utility of continuation.
- Key idea : identify the continuation value of the Agent as a new state variable
⇒ Dynamically consistent problem, HJB equation...
- The Agent can get paid more than its reservation utility

Cvitanic et al : adverse selection via offering a menu of contracts

⇒ multidimensional HJB equation

Cvitanic, Possamai, Touzi : control of the volatility process

⇒ Path dependent PDEs

In general : optimal choice of the Agent via a stochastic maximum principle,

⇒ fully coupled BSDEs, for which well-posedness is not clear

A Principal wishes to hire N Agents

- A Principal requires to handle an N -dimensional output process X

$$dX_t = \Sigma_t dB_t \quad \text{with} \quad \Sigma \text{ bounded and invertible}$$

- She wishes to hire N Agents
- Each Agent will be assigned to one project but he can choose to impact (positively or negatively) any project.
- The control process for Agent j is a vector a^j .
 a^{ij} is the control used by Agent j in order to impact the project i .
- The controlled process X is given by

$$dX_t = b(t, a_t)dt + \Sigma_t dB_t^a$$

where B^a is a BM under \mathbb{P}^a defined by $\frac{d\mathbb{P}^a}{d\mathbb{P}} = \mathcal{E} \left(- \int_0^T b(s, a_s) \cdot \Sigma_s^{-1} dB_s \right)$

Assumption

The drift $b(t, \cdot)$ has linear growth and is C_b^1 .

Men do not desire to be rich, but to be richer than other men.
J.S. Mill

The **satisfaction** of each Agent comes from

- His terminal payment ξ^i ;
- **How well his project performed in comparison to the other Agents**

The Agent optimization problem is

$$\sup_{a^i} \mathbb{E}^{\mathbb{P}^a} \left[U_i^A \left(\xi^i + \gamma_i (X_T^i - \bar{X}_T^{-i}) - \int_0^T k^i(s, a_s^i) ds \right) \right]$$

where \bar{X}_T^{-i} represents the average output of all Agents except Agent i

- γ_i is a **competition index** for Agent i .
- Similar **relative performance concerns** as in e.g. *Espinosa-Touzi*.

- The Agent optimization problem is

$$\sup_{a^i} \mathbb{E}^{\mathbb{P}^a} \left[U_i^A \left(\xi^i + \gamma_i (X_T^i - \bar{X}_T^{-i}) - \int_0^T k^i(s, a_s^i) ds \right) \right]$$

Assumption

The *cost function* $k(t, \cdot)$ is C^1 , has each coordinates increasing and strictly convex. There exists $C > 0$ and $\ell \geq 2$ s.t.

$$\lim_{\|x\| \rightarrow +\infty} \frac{\|k(t, x)\|}{\|x\|} = +\infty, \quad \|k(t, x)\| \leq C \left(1 + \|x\|^\ell \right),$$

$$\text{and} \quad \|\nabla k(t, x)\| \leq C \left(1 + \|x\|^{\ell-1} \right).$$

- First best problem
 \implies The Principal chooses the payments and the actions of the Agents.
- Admissible contracts with **exponential moments** :

$$\mathcal{C}^{FB} := \left\{ \xi, \mathcal{F}_T\text{-measurable with } \xi \in M^\phi(\mathbb{R}^N) \right\},$$

where $M^\phi(\mathbb{R}^N)$ is the Morse-Transue space

Solving the first best problem : No moral hasard

- We denote

$$\bar{\gamma}^{-i} := \frac{1}{N-1} \sum_{j \neq i} \gamma_j \quad \text{and} \quad \frac{1}{\bar{R}_A} = \frac{1}{N} \sum_{i=1}^N \frac{1}{R_A^i}$$

Theorem

Given the reservation utilities (U_0^i) of the Agents, the optimal first best payment is as follows :

$$\begin{aligned} \xi_{FB}^i := & \frac{R_P \bar{R}_A}{R_A^i (\bar{R}_A + N R_P)} (\mathbf{1}_N + \gamma - \bar{\gamma}^-) \cdot X_T - \gamma_i (X_T^i - \bar{X}_T^{-i}) + \int_0^T k^i(s, (a^*)_s^i) ds \\ & + \frac{R_P \bar{R}_A}{R_A^i (\bar{R}_A + N R_P)} \int_0^T b(s, a_s^*) \cdot (\bar{\gamma}^- - \gamma - \mathbf{1}_N) ds - \frac{1}{R_A^i} \log(-U_0^i) \\ & + \frac{1}{2R_A^i} \left(\frac{R_P \bar{R}_A}{\bar{R}_A + N R_P} \right)^2 \int_0^T \|\Sigma_s(\bar{\gamma}^- - \gamma - \mathbf{1}_N)\|^2 ds. \end{aligned}$$

where the *optimal action* a_t^* is any *minimizer* of the map

$$a \mapsto (\bar{\gamma}^- - \gamma - \mathbf{1}_N) \cdot b(t, a) + \mathbf{1}_N \cdot k(t, a).$$

- The optimal contract is of the form :

$$(\xi^*)^i(a^*) = C_i + \frac{R_P \bar{R}_A}{R_A^i (\bar{R}_A + NR_P)} (\mathbf{1}_N + \gamma - \bar{\gamma}^-) \cdot X_T - \gamma_i (X_T^i - \bar{X}_T^{-i}),$$

for some constant C_i .

- Each Agent is penalized with the amount $-\gamma^i (X_T^i - \bar{X}_T^{-i})$, so as to suppress the appetite for competition of the Agents.
- Moreover, each Agent is paid a positive part of each projects, the percentage depending on the risk aversion of the Agent, and of the universal vector

$$\frac{R_P \bar{R}_A}{\bar{R}_A + NR_P} (\mathbf{1}_N + \gamma - \bar{\gamma}^-).$$

- If an Agent is particularly competitive, then any Agent will receive a large part of his project
- If an Agent is not very competitive, other Agents have incitation to reduce the value of his project as much as possible.

Linear drift and quadratic cost functions :

$$b(t, a) := \begin{pmatrix} a^{11} - a^{12} \\ a^{22} - a^{21} \end{pmatrix} \quad \text{and} \quad k(t, a) := \begin{pmatrix} \frac{k^{11}}{2} |a^{11}|^2 + \frac{k^{21}}{2} |a^{21}|^2 \\ \frac{k^{22}}{2} |a^{22}|^2 + \frac{k^{12}}{2} |a^{12}|^2 \end{pmatrix}$$

⇒ Optimal actions of the two Agents are

$$\text{Agent 1 :} \quad a^{11} = \frac{1 + \gamma_1 - \gamma_2}{k^{11}} \quad a^{21} = -\frac{1 + \gamma_2 - \gamma_1}{k^{21}}$$

$$\text{Agent 2 :} \quad a^{12} = \frac{1 + \gamma_2 - \gamma_1}{k^{12}} \quad a^{22} = -\frac{1 + \gamma_1 - \gamma_2}{k^{22}}$$

If Agent 1 is much more competitive than Agent 2 :

- Agent 1 will work towards his project and will also work to decrease the value of the project of Agent 2 ;
- Agent 2 will work to decrease the value of his own project and to increase the value of the project of Agent 1.

- Let consider Agents with **similar reservation utilities**.

What is the optimal competition scheme between Agents from the Principal viewpoint ?

- Maximizing the value function of the Principal boils down to minimizing

$$g : (\gamma_1, \gamma_2) \mapsto (1 + \gamma_1 - \gamma_2)^2 \alpha_1 + (1 + \gamma_2 - \gamma_1)^2 \alpha_2,$$

where

$$\alpha_1 := \frac{R_P \bar{R}_A}{\bar{R}_A + 2R_P} \sigma_1^2 - \left(\frac{1}{k^{11}} + \frac{1}{k^{22}} \right), \quad \alpha_2 := \frac{R_P \bar{R}_A}{\bar{R}_A + 2R_P} \sigma_2^2 - \left(\frac{1}{k^{12}} + \frac{1}{k^{21}} \right).$$

- For **low working costs**, $\alpha_1 + \alpha_2 \leq 0$

The Principal would like to hire Agents with $|\gamma_1 - \gamma_2| \rightarrow +\infty$

- For **high working costs**, $\alpha_1 + \alpha_2 > 0$,

The Principal wants to hire Agents with $\gamma_1 - \gamma_2 = \frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}$.

- More competitive Agents** must work on **less volatile projects**.

- We now turn to the **second best/moral hazard** problem ;
- **The Principal can not observe the actions of the Agents** and only controls the salary ξ that he offers ;
- Similar ideas as in the BSDE scheme of proof derived for the Principal - unique Agent case ;
- **Stackelberg equilibrium** between the **Principal** and the **system of Agents** ;
- **Nash equilibrium** between all **the interacting Agents**.

Definition

Given a contract ξ , a **Nash equilibrium** for the N Agents is an action $\mathbf{a}^*(\xi) \in \mathcal{A}$ such that for any $i = 1, \dots, N$, we have

$$\sup_{\mathbf{a} \in \mathcal{A}^i((\mathbf{a}^*)^{-i}(\xi))} U_0^i(\mathbf{a}, (\mathbf{a}^*)^{-i}(\xi), \xi^i) = U_0^i((\mathbf{a}^*)^i(\xi), (\mathbf{a}^*)^{-i}(\xi), \xi^i).$$

- Need a **rule** chosen by the system of Agents for **selecting collectively one Nash equilibrium** between possibly several.
- For instance, choice of the equilibrium with the **highest global utility of all Agents** :

$$\mathbf{x} \succeq \mathbf{y}, \quad \text{iff} \quad \sum_{i=1}^N U^i(x^i) \geq \sum_{i=1}^N U^i(y^i),$$

- If the system of Agents is **indifferent** between several equilibria, **the Principal chooses the best one for her**

Identifying the best reaction functions

- Let the **terminal payment** ξ be given
- Consider Agent i given the actions a^{-i} of the others

$$U_0^i(a^{-i}, \xi^i) := \sup_{a \in \mathcal{A}^i(a^{-i})} \mathbb{E}^{\mathbb{P}^{a \otimes_i a^{-i}}} \left[U_A^i \left(\xi^i + \gamma_i \left(X_T^i - \bar{X}_T^{-i} \right) - \int_0^T k^i(s, a_s) ds \right) \right].$$

- As in the unique Agent case, this leads to the consideration of **BSDEs** for the **log target process**.

$$Y_t^{i, a^{-i}, \xi^i} = \xi^i + \Gamma_i(X_T) + \int_t^T \tilde{f}^{i, a^{-i}}(s, Z_s^{i, a^{-i}, \xi^i}, a) ds - \int_t^T Z_s^{i, a^{-i}, \xi^i} \cdot \Sigma_s dW_s$$

$$\text{with } \tilde{f}^{i, a^{-i}}(t, \omega, z, a) := -\frac{R_A^i}{2} \|\Sigma(t)z\|^2 + b(t, a \otimes_i a_t^{-i}(\omega)) \cdot z - k^i(t, a_t).$$

- Consider the **maximal solution of the BSDE**, if it exists.
- Hoping for **comparison results** for the BSDE, let introduce

$$f^{i, a^{-i}}(t, \omega, z) := \sup_{a \in \mathcal{A}^i(a^{-i})} \tilde{f}^{i, a^{-i}}(t, \omega, z, a)$$

Theorem

There is a *one-to-one correspondence* between

- (i) a *Nash equilibrium* $a^*(\xi) \in \mathcal{A}$ such that for any $i = 1, \dots, N$, there exists some $p > 1$ such that

$$\left(\mathbb{E}^{\mathbb{P}^{a^*(\xi)}} \left[-e^{-R_A^i(\xi^i + \Gamma_i(X_T) - \int_0^T k^i(s, (a_s^*(\xi))^{\cdot i}) ds)} \middle| \mathcal{F}_t \right] \right)_{t \in [0, T]}$$

the *reverse Holder inequality* of order p for $\mathbb{P}^{a^*(\xi)}$.

- (ii) a *solution* (Y, Z) to the BSDE

$$Y_t^\xi = \xi + \Gamma(X_T) + \int_t^T f(s, Z_s^\xi, X_s) ds - \int_t^T Z_s^\xi \cdot \Sigma_s dW_s,$$

such that in addition $Z \in \mathbb{H}_{\text{BMO}}^2(\mathbb{P}, \mathcal{M}_N(\mathbb{R}))$.

The correspondence is given by, for any $i = 1, \dots, N$

$$(a_s^*(\xi))^{\cdot i} \in \operatorname{argmax}_{a \in \mathcal{A}^i((a^*)^{\cdot -i})} \left\{ \sum_{j=1}^N b^j(s, (a \otimes_i (a_s^*)^{\cdot -i}(s, Z_s))^{\cdot j}) Z_s^{ij} - k^i(s, a_s) \right\},$$

Finding a Nash equilibria reduces to **solving the multidimensional quadratic BSDE**.

- For small bounded terminal condition ξ , results of *Tevzadze*.
- counter examples in general, e.g. *Frei and Dos Reis*
- particular structure not valid in our setting : *Cheredito & Nam, Kramkov & Pulido, Hu & Tang, Jamneshan et al, Luo & Tangpi*

The Principal requires to **offer a terminal payment ξ which produces a Nash equilibrium for the system of Agents**.

$NA(\xi) := \{\text{Nash equilibria associated to } \xi \text{ satisfying the reverse Hölder cond.}\}$

$NAI(\xi) := \{a \in NA(\xi), a \succeq b, \text{ for any } b \in NA(\xi)\}.$

Admissible contracts for the second best problem :

$$C^{SB} := \left\{ \xi \in C^{FB}, NAI(\xi) \text{ is non-empty} \right\}.$$

Solving the Principal problem

- The Principal problem is

$$\sup_{\xi \in \mathcal{C}^{SB}} \sup_{a \in \text{NAI}(\xi)} \mathbb{E}^{\mathbb{P}^a} \left[-e^{-R_P(X_T - \xi) \cdot \mathbf{1}_N} - \sum_{i=1}^N \rho_i e^{-R_A^i (\xi^i + \gamma_i (X_T^i - \bar{X}_T^{-i}) - \int_0^T k^i(s, a_s^i) ds)} \right]$$

- The $(\rho_i)_i$ are the **Lagrange multipliers** of the **participation constraints**.
- For any $\xi \in \mathcal{C}^{SB}$, there exists a pair $(Y_0^\xi, Z^\xi) \in \mathbb{R}^N \times \mathbb{H}_{\text{BMO}}^2(\mathbb{P}, \mathcal{M}_N(\mathbb{R}))$ s.t.

$$\xi = Y_0^\xi - \Gamma(X_T) - \int_0^T f(s, Z_s^\xi, X_s) ds + \int_0^T Z_s^\xi \cdot \Sigma_s dW_s, \text{ a.s.} \quad (1)$$

- Optimization over **the process Z^ξ** .
- We know the **actions a^*** of the **system of Agents** in response to a payment $(Z_t^\xi)_t$.
- Optimal **deterministic** process **Z^*** .

Theorem

An *optimal contract* $\xi_{SB} \in \mathcal{C}^{SB}$ with reservation utilities $(U_0^i)_{1 \leq i \leq N}$, is given by

$$\begin{aligned} \xi_{SB}^i := & -\frac{1}{R_A^i} \log(-U_0^i) - \gamma_i (X_T^i - \bar{X}_T^{-i}) + \left(\int_0^T z_s^* dX_s \right)^i \\ & + \int_0^T k^i(s, (a_s^*((z_s^*)^{i,\cdot}))^{i,\cdot}) ds + \frac{R_A^i}{2} \int_0^T \|\Sigma(z_s^*)^{i,\cdot}\|^2 ds, \end{aligned}$$

where the matrix $a_s^*(z) \in \mathcal{M}_N(\mathbb{R})$ is defined by

$$(a_s^*(z))^{i,\cdot} \in \operatorname{argmax}_{a \in \mathcal{A}^i((a^*)^{i,\cdot,-i})} \left\{ b(t, a \otimes_i (a_s^*(z))^{i,\cdot,-i}) \cdot z^{i,\cdot} - k^i(t, a) \right\},$$

and where z_t^* is any *deterministic maximizer* of the map

$$\begin{aligned} z \mapsto & (\mathbf{1}_N + \gamma - \bar{\gamma}^-) \cdot b(t, a_t^*(z)) - k(t, a_t^*(z)) \cdot \mathbf{1}_N \\ & - \sum_{i=1}^N \frac{R_A^i}{2} \|\Sigma(t)z^{i,\cdot}\|^2 - \frac{R_P}{2} \|\Sigma(z^\top \mathbf{1}_N + \mathbf{1}_N + \gamma - \bar{\gamma}^-)\|^2. \end{aligned}$$

- z^* deterministic implies **existence and uniqueness** of solution for the optimal BSDE.
- For non-time dependent cost and drift function, the **optimal contract** is **linear** of the form

$$(\xi^*)^i(a^*) = C_i + z^* \cdot X_T - \gamma_i (X_T^i - \bar{X}_T^{-i}),$$

for some constant C_i .

- Each Agent gets his **utility of reservation**.
- Each Agent is paid a **fixed part** of each project.
- Each Agent is penalized with the amount $-\gamma^i(X_T^i - \bar{X}_T^{-i})$, so as to **suppress the appetite for competition** of the Agents.
- For the **linear drift/quadratic cost** case, we obtain **more explicit formulae** where in particular

$$a^*(z) = \begin{pmatrix} \frac{z^{11}}{k^{11}} & \frac{z^{12}}{k^{12}} \\ \frac{z^{21}}{k^{21}} & \frac{z^{22}}{k^{22}} \end{pmatrix}.$$

and z^* maximizes a 4-dimensional linear-quadratic function.

- More general **dynamics**/drift for the controlled output

$$dX_t = b(t, a_t, X_t)dt + \Sigma_t dB_t^a$$

- **Cost** function $k(t, a_t, X_t)$;

- More general performance concerns :

$$\gamma^i(X_T^i - \bar{X}_T^{-i}) \quad \text{replaced by} \quad \Gamma^i(X_T)$$

with any **linear growth function** Γ^i .

- Same resolution for the Agent problem ;
- **HJB characterization** for the Principal problem ;
- Recovers in particular the framework of *Goukasian & Wan* with **relative payments** concerns.

- Contracting between a Principal and a **system of interacting Agents** ;
- **Explicit** treatment of a **toy example** : linear contracts
- Clarification of the **BSDE methodology** for these problems ;
- **Mean field limit** : what happens when N goes to ∞ ?
- Consideration of **MacKean-Vlasov** output dynamics

$$dX_t = b(t, a_t, X_t, \mathcal{L}_t^X)dt + \Sigma_t dB_t^a$$